Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube

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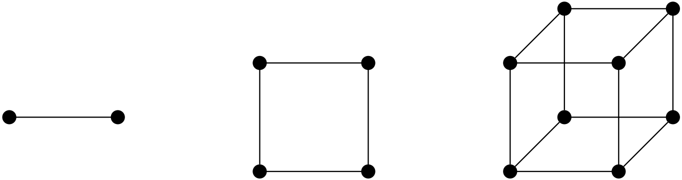
Hypercube

- $\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)



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- $e(G) := |E(G)|$
- $\mathrm{ex}_{\mathcal{Q}}(n, F) :=$ the maximum number of edges of a F-free subgraph of $\mathcal{Q}_n$
- $\pi_{\mathcal{Q}}(F) = \lim_{n\to\infty} \frac{\mathrm{ex}_{\mathcal{Q}}(n, F)}{e(\mathcal{Q}_n)}$

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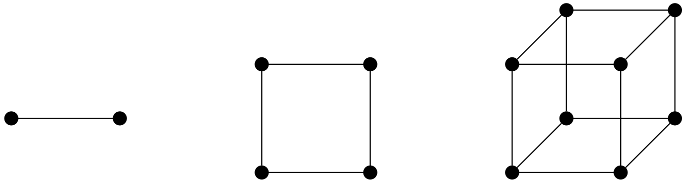

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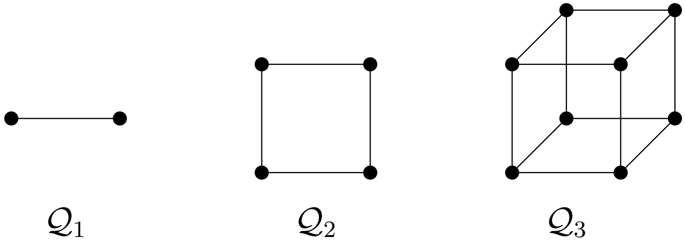

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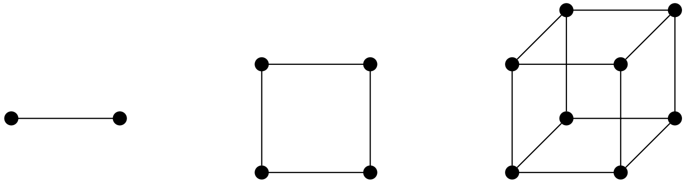

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$\pi_{\mathcal{Q}}(C_4) = 1/2$, $\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for* $t > 2$

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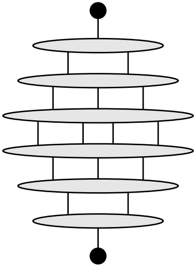
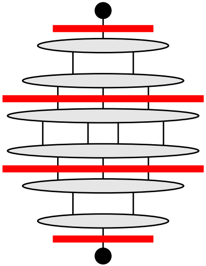


$\mathcal{Q}_7$



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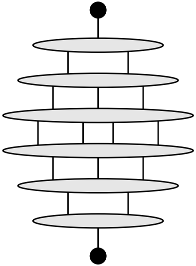

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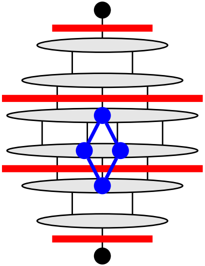

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$\pi_{\mathcal{Q}}(n, C_{2t}) = 0$ *for even* $t \geq 4$.

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$\pi_{\mathcal{Q}}(C_{2t}) = 0$ *for odd* $t \geq 7$.

if $\pi_{\mathcal{Q}}(C_{10}) = 0$ is still open.

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$ex_{\mathcal{Q}}(n, C_4) \geq \frac{1}{2}(1 + \frac{1}{\sqrt{n}})e(\mathcal{Q}_n)$ (valid when n is a power of 4)

THEOREM (CHUNG [1992])

$\pi_{\mathcal{Q}}(C_4) \leq 0.62284$.

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Theorem (Balogh–Hu–L–Liu, ind. Baber [2012+])

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Flag Algebras

Definition

$p(H, G)$: the probability that a random $|V(H)|$-set U in $V(G)$ induces $G[U]$ isomorphic to H.

Razborov [2007] developed flag algebras. Let $\mathcal{G}$ be the family of graphs forbidding some structures, then flag algebras can be used to bound

$$\lim_{G \in \mathcal{G}, |V(G)| \to \infty} p(H, G).$$

Results using Flag Algebras

Theorem (Hladký–Král'–Norine [2009])

Every n-vertex digraph with minimum outdegree at least $0.3465n$ *contains a triangle.*

Theorem (Hatami–Hladký–Král'–Norine–Razborov [2011], Grzesik [2011])

The number of C_5*s in a triangle-free graph of order n is at most* $(n/5)^5$.

Theorem (Falgas-Ravry–Vaughan [2011])

$\pi(K_4^-, C_5, F_{3,2}) = 12/49, \pi(K_4^-, F_{3,2}) = 5/18.$

$F_{3,2} : \{123, 145, 245, 345\}, C_5 : \{123, 234, 345, 451, 512\}.$

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Proof by an Example

Example

$\pi_{\mathcal{Q}}(C_4) \leq 2/3$

Bound infinite problem by a finite piece.

Definition

$\mathcal{H}_n$: the family of spanning subgraphs of $\mathcal{Q}_n$ not containing C_4.

Let $H \in \mathcal{H}_s, G \in \mathcal{H}_n, s < n$, $p(H, G)$ is the probability that a random s-hypercube in G induces H.

$\rho(G) = e(G)/e(\mathcal{Q}_n)$.

$$\rho(G) = \sum_{H \in \mathcal{H}_s} \rho(H) p(H, G)$$

$$\rho(G) \leq \max_{H \in \mathcal{H}_s} \rho(H)$$

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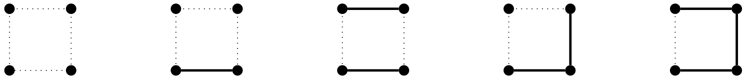

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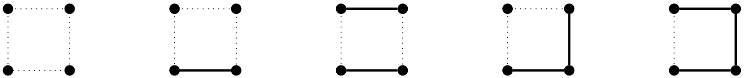

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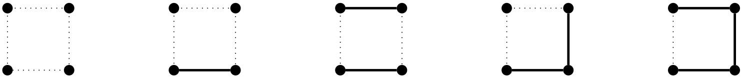

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If $0 \leq \sum_i c_{H_i} p(H_i, G)$, then

$$\rho(G) \leq \sum_i \left(\rho(H_i) + c_{H_i}\right) p(H_i, G)$$

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c_{H_i} might be negative

Example Continued: Flags

Definition

Flags: F=$(H,\theta), H \in \mathcal{H}_s, \theta : [2^k] \to V(H)$ is injective, $H[Im(\theta)] \in \mathcal{H}_k$.

Let $G \in \mathcal{H}_n, \theta : [1] \to V(G)$.

Definition

$p(F_i, \theta, G)$: the probability that a random 1-cube U in G subject to $Im(\theta) \subset U$ satisfies $(U, \theta) = F_i$.
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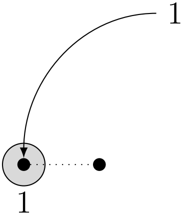
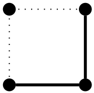


F_1



H_4

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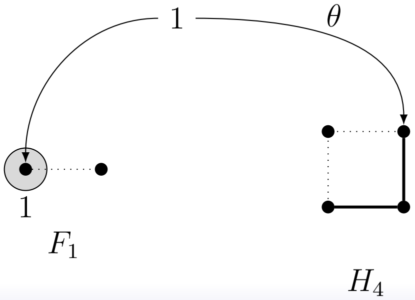

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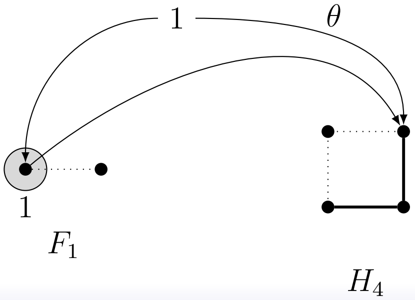

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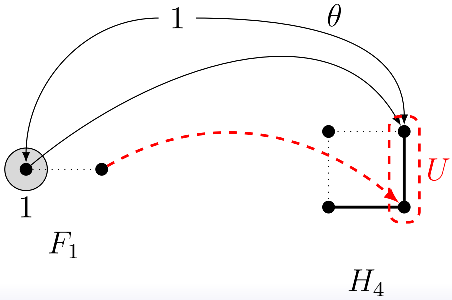

Example Continued: $p(F_i, \theta, G)$

Let $G \in \mathcal{H}_n, \theta : [1] \to V(G)$.

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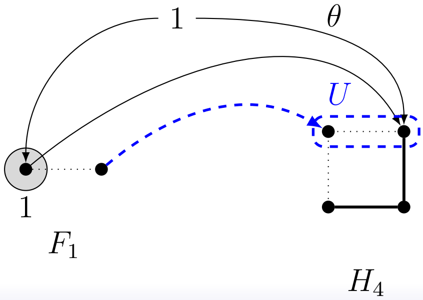

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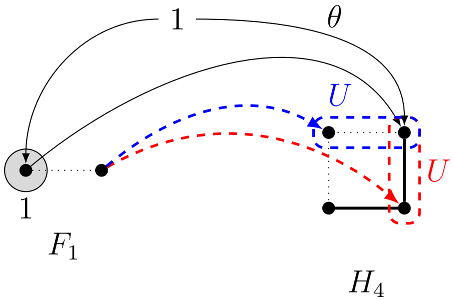

$p(F_1, \theta, H_4) = 1/2$

Proof Continued: Flags

Let $M = (m_{ij})$ be a positive semidefinite 2-by-2 matrix, define $\mathbf{p}_\theta = \{p(F_1, \theta; G), p(F_2, \theta; G)\}$, then

$$0 \leq \mathbb{E}_\theta[\mathbf{p}_\theta M \mathbf{p}_\theta^T]$$

Proof Continued: Flags

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Lemma

$$p(F_i, \theta, G) p(F_j, \theta, G) = p(F_i, F_j, \theta, G) + o(1),$$

$o(1) \to 0$ *as* $n \to \infty$.

Proof Continued: Flags

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$$\mathbb{E}_\theta[p(F_i, F_j, \theta; G)] = \sum_{H \in \mathcal{H}_2} \mathbb{E}_\theta[p(F_i, F_j, \theta; H)] p(H, G)$$

Proof Continued: Flags

Let $M = (m_{ij})$ be a positive semidefinite 2-by-2 matrix, define $\mathbf{p}_\theta = \{p(F_1, \theta; G), p(F_2, \theta; G)\}$, then

$$
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&= \sum_{1 \le i,j \le 2} \sum_{H \in \mathcal{H}_2} m_{ij} \mathbb{E}_\theta[p(F_i, F_j, \theta; H)] p(H, G) + o(1)
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Proof Continued: Flags

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Let

$$c_H(M) = \sum_{1 \le i,j \le 2} m_{ij} \mathbb{E}_\theta[p(F_i, F_j, \theta; H)],$$

then

$$0 \le \sum_{H \in \mathcal{H}_2} c_H(M) p(H, G) + o(1).$$

So

$$\rho(G) \le \sum_{H \in \mathcal{H}_2} (\rho(H) + c_H(M))\, p(H, G)$$

$$\pi_{\mathcal{Q}}(C_4) \le \max_{H \in \mathcal{H}_2} (\rho(H) + c_H(M))$$

Proof Continued: Flags

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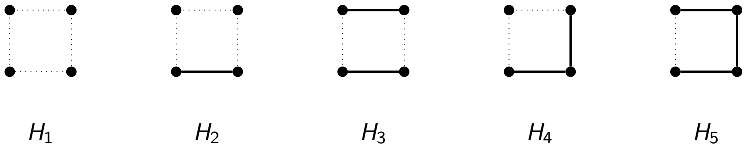
Computing $\mathbb{E}_\theta[p(F_i, F_j, \theta; H)]$



	H_1	H_2	H_3	H_4	H_5
F_1, F_1	1	1/2	0	1/4	0
F_1, F_2	0	1/4	1/2	1/4	1/4
F_2, F_2	0	0	0	1/4	1/2

Optimizing M

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\begin{aligned}
\rho(H_1) + c_{H_1} &= 0 + m_{11} \\
\rho(H_2) + c_{H_2} &= 1/4 + m_{11}/2 + m_{12}/2 \\
\rho(H_3) + c_{H_3} &= 1/2 + m_{12} \\
\rho(H_4) + c_{H_4} &= 1/2 + m_{11}/4 + m_{12}/2 + m_{22}/4 \\
\rho(H_5) + c_{H_5} &= 3/4 + m_{12}/2 + m_{22}/2
\end{aligned}$$

$$\pi_{\mathcal{Q}}(C_4) \leq \max_i (\rho(H_i) + c_{H_i})$$

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\end{aligned}$$

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Solution

Take

$$M = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 1/6 \end{pmatrix},$$

then

$$\max_i (\rho(H_i) + c_{H_i}) = 2/3$$

Results

Theorem (Balogh–Hu–L–Liu, ind. Baber [2012+])
$\pi_{\mathcal{Q}}(C_4) \leq 0.6068.$

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$\pi_{\mathcal{Q}}(C_6) \leq 0.3755.$

By using $\mathcal{H}_3$ and bigger flags.

Almost surely can be improved by waiting.

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Thank you for your attention!