Splitting a tournament into two subtournaments with given minimum outdegree

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Abstract

A \((k_1, k_2)\)-outdegree-splitting of a digraph \(D\) is a partition \((V_1, V_2)\) of its vertex set such that \(D[V_1]\) and \(D[V_2]\) have minimum outdegree at least \(k_1\) and \(k_2\), respectively. We show that there exists a minimum function \(f_T\) such that every tournament of minimum outdegree at least \(f_T(k_1, k_2)\) has a \((k_1, k_2)\)-outdegree-splitting, and \(f_T(k_1, k_2) \leq k_2^2/2 + 3k_1/2 + k_2 + 1\). We also show a polynomial-time algorithm that finds a \((k_1, k_2)\)-outdegree-splitting of a tournament if one exists, and returns ‘no’ otherwise. We give better bound on \(f_T\) and faster algorithms when \(k_1 = 1\).

1 Introduction

Let \(D\) be a digraph. For a vertex \(v \in V(D)\) the outdegree of \(v\), denoted by \(d^+_D(v)\), is the number of arcs directed away from \(v\). The minimum outdegree over all vertices of \(D\) is denoted by \(\delta^+(D)\). We drop \(D\) in \(d^+_D(v)\) and \(\delta^+(D)\) if it is clear from the context.

A \((k_1, k_2)\)-outdegree-splitting of a digraph \(D\) is a partition \((V_1, V_2)\) of its vertex set such that \(D[V_1]\) and \(D[V_2]\) have minimum outdegree at least \(k_1\) and \(k_2\), respectively. A digraph admitting a \((k_1, k_2)\)-outdegree-splitting is said to be \((k_1, k_2)\)-outdegree-splittable.

Problem 1 (Alon [1]). Is there a function \(f\) such that every digraph with minimum outdegree \(f(k_1, k_2)\) has a \((k_1, k_2)\)-outdegree-splitting?

The existence of the corresponding function \(f\) for the undirected analogue is easy and has been observed by many authors. Stiebitz [12] even proved the following tight result: if the minimum degree of an undirected graph \(G\) is \(d_1 + d_2 + \cdots + d_k\), where each \(d_i\) is a non-negative integer, then the vertex set of \(G\) can be partitioned into \(k\) pairwise disjoint sets \(V_1, \ldots, V_k\), so that for all \(i\), the induced subgraph on \(V_i\) has minimum degree at least \(d_i\). This is clearly tight, as shown by an appropriate complete graph.

Problem 1 is equivalent to the following:

Problem 2. Is there a function \(f'(k_1, k_2)\) such that every digraph with minimum outdegree \(f'(k_1, k_2)\) has two disjoint (induced) subdigraphs, one of them with minimum outdegree \(k_1\) and the other with minimum outdegree \(k_2\)?

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This follows from the following proposition.

**Proposition 3.** Let $D$ be a digraph with minimum outdegree at least $k_1 + k_2 - 1$. If $D$ contains two disjoint subdigraphs $D_1$ and $D_2$ such that $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$, then $D$ has a $(k_1, k_2)$-outdegree-splitting.

**Proof.** Consider two disjoint digraphs $D_1$ and $D_2$ with $\delta^+(D_1) = k_1$ and $\delta^+(D_2) = k_2$ such that $V(D_1) \cup V(D_2)$ is maximum. Suppose for a contradiction that $S = V(D) \setminus (V(D_1) \cup V(D_2))$ is not empty. Then every vertex $s \in S$ has at most $k_1 - 1$ outneighbours in $D_1$ otherwise $D_1 + s$ and $D_2$ contradict the maximality of $D_1$ and $D_2$. Hence every vertex of $S$ has at least $k_1 + k_2 - 1 - (k_1 - 1) = k_2$ outneighbours in $D - D_1$. It follows that $D - D_1$ has minimum degree $k_2$. So $D_1$ and $D - D_1$ contradicts the maximality of $D_1$ and $D_2$. \hfill $\square$

**Corollary 4.** $f(k_1, k_2) \leq \max\{f'(k_1, k_2), k_1 + k_2 - 1\}$.

This implies in particular that $f(1, 1) = f'(1, 1) = 3$. Indeed Thomassen [13] showed that every digraph of minimum outdegree at least 3 has two disjoint cycles. (In this paper, paths and cycles are always directed.)

This is a special case of Bermond-Thomassen Conjecture [3]:

**Conjecture 5** (Bermond and Thomassen [3]). Every digraph with $\delta^+ \geq 2k - 1$ contains $k$ disjoint cycles.

Note that Alon [11] proved that if $\delta^+ \geq 64k$ there are $k$ disjoint cycles.

A **tournament** is a digraph such that for every two distinct vertices $u, v$ there is exactly one arc with ends $\{u, v\}$ (so, either the arc $uv$ or the arc $vu$ but not both).

In this paper, we settle Problem [11] for tournaments.

**Theorem 6.** Every tournament of minimum outdegree at least $k_1^2/2 + 3k_1/2 + k_2 + 1$ has a $(k_1, k_2)$-outdegree-splitting.

To prove Theorem 6 we shall prove the following theorem.

**Theorem 7.** Every tournament with minimum outdegree at least $k$ has a subtournament with minimum outdegree $k$ and order at most $k^2/2 + 3k/2 + 1$.

We can then easily derive Theorem 6.

**Proof of Theorem 7** Let $T$ be a tournament of minimum outdegree at least $k^2/2 + 3k_1/2 + k_2 + 1$. By Theorem 7 there exists a subtournament $T_1$ with minimum outdegree at least $k_1$ and order at most $k_1^2/2 + 3k_1/2 + 1$. Let $T_2 = T - T_1$. Then $\delta^+(T_2) \geq \delta^+(T) - |V(T_1)| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a $(k_1, k_2)$-outdegree-splitting. \hfill $\square$

In fact, we prove a more general statement than Theorem 7 (Theorem 17). This enables us to prove the following generalization of Theorem 6.

**Theorem 8.** Let $m = \max\{k_1^2/2 + 3k_1/2 + k_2 + u_1 + 1, k_1 + u_2\}$, let $T$ be a tournament of minimum outdegree at least $m$ and let $U_1$ and $U_2$ be two disjoint subsets of $V(T)$ of cardinality $u_1$ and $u_2$ respectively. Then there is a $(k_1, k_2)$-outdegree-splitting $(V_1, V_2)$ of $T$ such that $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.

The bound of Theorem 6 is certainly not tight. Theorem 6 asserts that every tournament with minimum outdegree 4 has a $(1, 1)$-outdegree-splitting, but we know that having outdegree 3 is sufficient.
Problem 9. What is the minimum integer $f_T(k_1, k_2)$ such that every tournament with minimum outdegree at least $f_T(k_1, k_2)$ has a $(k_1, k_2)$-outdegree-splitting?

Theorem 6 implies that $f_T(1, k) \leq k + 3$. We describe examples implying $f_T(1, k) \geq k + 2$, and we conjecture that this lower bound is the exact value.

Conjecture 10. For any positive integer $k$, $f_T(1, k) = k + 1$.

In Section 4, we establish this conjecture for $k \in \{2, 3, 4\}$, that is, we prove $f_T(1, 2) = 4$, $f_T(1, 3) = 5$, and $f_T(1, 4) = 6$.

Next we consider problems of deciding whether a digraph admits a $(k_1, k_2)$-outdegree-splitting.

**Input:** A digraph $D$ and two positive integers $k_1$ and $k_2$.

**Question:** Does $D$ admit a $(k_1, k_2)$-outdegree-splitting?

Particular cases of this problem are when $k_1$ and $k_2$ are fixed integers and not part of the input. Hence for every fixed $k_1, k_2$, we have the following problem.

$$(k_1, k_2)$$-OUTDEGREE-SPLITTING

**Input:** A digraph $D$.

**Question:** Does $D$ admit a $(k_1, k_2)$-outdegree-splitting?

**Theorem 11.** $(1, 1)$-OUTDEGREE-SPLITTING is polynomial-time solvable.

**Proof.** Let us describe a polynomial-time algorithm solving $(1, 1)$-OUTDEGREE-SPLITTING.

If the input digraph $D$ has a vertex with outdegree 0, then the answer is ‘no’ because this vertex has outdegree 0 in any subdigraph of $D$ containing it. Henceforth we may assume that $\delta^+(D) \geq 1$.

It is well-known that a digraph with outdegree at least 1 contains a cycle. Therefore, Proposition 5 implies that a digraph with minimum outdegree at least 1 admits a $(1, 1)$-outdegree-splitting if and only if it contains two disjoint cycles. Thus it is enough to decide whether $D$ contains two disjoint cycles.

But deciding whether a digraph contains two disjoint cycles can be done in polynomial time as shown by McCuaig [7]. (See also [9].) \qed

In Section 5, we consider the restriction of these problems to tournaments.

**TOURNAMENT OUTDEGREE-SPLITTING**

**Input:** A tournament $T$ and two positive integers $k_1$ and $k_2$.

**Question:** Does $T$ admit a $(k_1, k_2)$-outdegree-splitting?

**TOURNAMENT $(k_1, k_2)$-OUTDEGREE-SPLITTING**

**Input:** A tournament $T$.

**Question:** Does $T$ admit a $(k_1, k_2)$-outdegree-splitting?

**TOURNAMENT $(1, 1)$-OUTDEGREE-SPLITTING** is a particular case of $(1, 1)$-OUTDEGREE-SPLITTING, and thus is polynomial-time solvable. In Theorem 3, we show that, more generally, for any $k_1, k_2$, TOURNAMENT $(k_1, k_2)$-OUTDEGREE-SPLITTING can be solved in $O(n^{k_1}k_2^{3k_2/2+3})$ time. We then describe a faster algorithm solving TOURNAMENT $(1, k_2)$-OUTDEGREE-SPLITTING. It runs in $O(n^3)$ time for $k_2 \geq 2$ and in $O(n^2)$ time for $k_2 = 1$. In view of these results, it is natural to ask the following.
Problem 12. Is Tournament Outdegree-Splitting fixed-parameter tractable with \((k_1, k_2)\) as a parameter? In other words, can we solve Tournament Outdegree Splitting in \(F(k_1, k_2)P(n)\) time, where \(F\) is an arbitrary computable function and \(P\) is a polynomial in the order \(n\) of the input tournament?

Finally, in Section 5 we present some possible directions for further research.

2 Definitions and folklore on tournaments

The score sequence of a tournament \(T\), denoted by \(s(T)\), is the non-decreasing sequence of outdegrees of its vertices. Landau [6] characterized the non-decreasing sequences of integers that are score sequences.

Theorem 13 (Landau [6]). A non-decreasing sequence of non-negative integers \((s_1, s_2, \ldots, s_n)\) is a score sequence if and only if:

(i) \(s_1 + s_2 + \cdots + s_i \geq \binom{i}{2}\), for \(i = 1, 2, \ldots, n - 1\), and

(ii) \(s_1 + s_2 + \cdots + s_n = \binom{n}{2}\).

Condition (ii) in the above theorem implies directly the following proposition.

Proposition 14. Every tournament of order \(2k\) has minimum degree less than \(k\).

Corollary 15. \(f_T(k_1, k_2) \geq k_1 + k_2 + 1\).

Proof. Let \(T\) be a \((k_1 + k_2)\)-regular tournament of order \(2k_1 + 2k_2 + 1\). In every bipartition \((V_1, V_2)\) of \(V(T)\), either \(|V_1| \leq 2k_1\) or \(|V_2| \leq 2k_2\). Thus, by Proposition 14 either \(\delta^+(T[V_1]) < k_1\) or \(\delta^+(T[V_2]) < k_2\).

An \(\ell\)-cycle is a cycle of length \(\ell\). A tournament \(T\) is transitive if it contains no cycles. The score sequence of a transitive tournament of order \(n\) is \((0, 1, \ldots, n - 1)\).

We denote by \(tt_3(T)\) the number of transitive subtournaments of order 3 in \(T\) and by \(c_3(T)\) number of 3-cycles in \(T\). Since a tournament of order 3 is either a transitive tournament or a 3-cycle, we have

\[ tt_3(T) + c_3(T) = \binom{|T|}{3}. \]

Now if \(v\) is a vertex, the number of transitive subtournaments of order 3 with source \(v\) is \(\binom{\delta^+(v)}{2}\). Hence

\[ tt_3(T) = \sum_{v \in V(T)} \binom{\delta^+(v)}{2}. \]

A digraph \(D\) is strongly connected or strong if there is a path from \(u\) to \(v\) for every \(u, v \in V(D)\). A digraph \(D\) is \(k\)-strong if \(D - X\) is strong for every \(X \subseteq V(D)\) where \(|X| \leq k - 1\). A (strong) component of \(D\) is a strong subdigraph of \(D\) which is maximal by inclusion.

Let \(T\) be a tournament. Let \(T_1, T_2, \ldots, T_m\) be the components of \(T\). Then \((V(T_1), V(T_2), \ldots, V(T_m))\) is a partition of \(V(T)\) and without loss of generality, we may suppose that \(T_i \rightarrow T_j\) whenever \(i < j\). In this case we say that \(T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_m\) is the decomposition of \(T\). Component \(T_1\) is said to be the initial component of \(T\) and \(T_m\) its terminal component.

A vertex is pancyclic in a digraph \(D\) if, for every \(3 \leq \ell \leq |D|\), it is contained in an \(\ell\)-cycle. To contain a pancyclic vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to Camion’s theorem [4]. Moon [8] showed that this condition is sufficient.
Theorem 16 (Moon [8]). Every vertex of a strong tournament is pancyclic.

We sometimes use the results of this section without referring to them.

3 Small sub tournament of minimum outdegree k

We now prove Theorem 7. In fact, we prove a more general theorem whose particular case with \(U = \emptyset\) is Theorem 7.

Theorem 17. Let \(T\) be a tournament with minimum outdegree at least \(k\) and \(U \subseteq V(T)\) be a subset of vertices. There is a sub tournament \(T'\) of \(T\) with minimum outdegree \(k\) such that \(U \subseteq V(T')\) and \(|V(T')| \leq |U| + k^2/2 + 3k/2 + 1\).

Proof. For every \(p\), we prove the result for all sets \(U\) of size \(p\) by induction on \(|V(T)|\), the result holding trivially if \(|V(T)| \leq p + k^2/2 + 3k/2 + 1\).

Let \(T\) be a tournament of order at least \(p + k^2/2 + 3k/2 + 2\) with minimum outdegree at least \(k\) and \(U\) a set of \(p\) vertices of \(T\). Let \(S\) be the set of vertices of degree \(k\) in \(T\). There are \(k|S|\) arcs with their tail in \(S\). Among them \(|S|(|S|-1)/2\) are in \(S\) and the remaining ones have their heads out of \(S\). Hence \(|N^+(S)| \leq |S| + k|S| - |S|(|S|-1)/2\). Now the polynomial \(P(x) = (k + 3/2)x - x^2/2\) increases on \([0, k + 3/2]\) and decreases on \([k + 3/2, +\infty]\). Moreover \(P(k + 1) = P(k + 2) = k^2/2 + 3k/2 + 1\).

Consequently, \(|N^+(S)| \leq k^2/2 + 3k/2 + 1\).

Since \(|V(T)| \geq p + k^2/2 + 3k/2 + 2\), there is a vertex \(v\) which is not in \(N^+(S) \cup U\). Thus \(T - v\) has minimum outdegree at least \(k\) and by induction \(T - v\) (and thus also \(T\)) has a sub tournament \(T'\) with minimum outdegree \(k\) such that \(U \subseteq V(T')\) and \(|V(T')| \leq |U| + k^2/2 + 3k/2 + 1\).

The bound \(k^2/2 + 3k/2 + 1\) is Theorem 7 is tight in the following sense.

Proposition 18. For every non-negative integer \(k\), and for every \(n \geq k^2/2 + 3k/2 + 1\), there is a tournament \(T(n,k)\) of order \(n\) and a set \(W \subset V(T)\) of order \(n - k^2/2 + 3k/2 + 1\) such that for every \(U \subset W\), every sub tournament \(T'\) of order \(n\) and with minimum outdegree \(k\) such that \(U \subseteq V(T')\) has order at least \(|U| + k^2/2 + 3k/2 + 1\).

Proof. Consider the disjoint union of a strong tournament \(S\) of order \(k + 1\) and a transitive tournament \(TT\) of order \(k(k + 1)/2\). Set \(V(S) = \{s_1, \ldots, s_{k+1}\}\). Partition \(V(TT)\) into \(k+1\) sets \(A_1, \ldots, A_{k+1}\) such that \(|A_i| = k - d^+_S(s_i)\). This is possible since \(\sum_{i=1}^{k+1} d^+_S(s_i) = k(k+1)/2\), so \(\sum_{i=1}^{k+1} (k - d^+_S(s_i)) = |V(TT)|\).

Now for each \(i\), add the arc \(s_ia\) for all \(a \in A_i\) and all the arcs \(bs_i\) for all \(b \in V(TT) \setminus A_i\). The resulting tournament \(R\) has order \(k^2/2 + 3k/2 + 1\) and minimum outdegree \(k\).

Let \(R'\) be a sub tournament of \(R\) with outdegree at least \(k\).

It must contains a vertex of \(S\), because all sub tournaments of \(TT\) are transitive. But each element \(s\) of \(S\) has outdegree exactly \(k\) in \(R\), so if \(s \in V(R')\), then \(N^+_R(s) \subset V(R')\). Since \(S\) is strong, it has a hamiltonian cycle by Camion’s Theorem, and so \(V(S) \subset V(R')\). But by construction, every vertex in \(R\) is dominated by a vertex in \(S\), and thus must be in \(R'\). Hence \(R = R'\).

Set \(p = n - k^2/2 + 3k/2 + 1\). Let \(T(n,k)\) be a tournament obtained from the disjoint union of \(R\) and the transitive tournament \(TT_p\) of order \(p\) by adding all arcs from \(TT_p\) towards \(R\). Then, for any set \(U \subset V(TT_p)\), every sub tournament of \(T(n,k)\) with minimum outdegree \(k\) containing \(U\) must also contain \(V(R)\) and thus has order at least \(|U| + k^2/2 + 3k/2 + 1\).

We can then easily derive Theorem 8 from Theorem 17.
Proof of Theorem 8. Let $T$ be a tournament of minimum outdegree at least $m$. The tournament $T - U_2$ has minimum outdegree at least $k_1$ because $|U_2| = u_2$. Thus, by Theorem 17, there exists a subtournament $T_1$ of $T - U_2$ with minimum degree at least $k_1$ and order at most $k_1^2/2 + 3k_1/2 + u_1 + 1$ such that $U_1 \subseteq V(T_1)$. Set $V_1 = V(T_1)$, $T_2 = T - T_1$, and $V_2 = V(T_2)$. By definition, $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. Now $\delta^+(T_2) \geq \delta^+(T) - |V_1| \geq k_2$. Hence $(V(T_1), V(T_2))$ is a $(k_1, k_2)$-outdegree-splitting. □

3.1 Outdegree-critical tournaments

Theorem 7 can be rephrased in terms of $k$-outdegree-critical tournament. A tournament $T$ is said to be $k$-outdegree-critical if it has minimum outdegree $k$ and all its proper subtournaments have outdegree less than $k$. Theorem 7 implies that all $k$-outdegree-critical tournaments have bounded size. Hence a natural problem is the following.

Problem 19. Describe the $k$-outdegree-critical tournaments.

The unique 1-outdegree-critical tournament is the 3-cycle.

We now show that the 2-outdegree-critical tournaments are those depicted in Figure 1.

Theorem 20. Every tournament with minimum outdegree 2 has a subtournament isomorphic to one of those depicted in Figure 1.

Proof. By induction on $|V(T)|$, the result holding trivially when $|V(T)| < 5$. 

Figure 1: The 2-outdegree-critical tournaments.
Let $T$ be a tournament of order at least 5 with minimum outdegree 2. Every vertex $v$ has an inneighbour $u$ such that $d^+(u) = 2$, for otherwise $T-v$ has minimum outdegree at least 2 and by induction $T-u$ (and thus also $T$) has a subtournament with minimum outdegree 2 and with order 5 or 6.

Let $S$ be the set of vertices of outdegree 2 in $T$. By the previous remark, $S$ is not empty and $T[S]$ has minimum indegree at least 1. Hence $T[S]$ contains a 3-cycle $C = (x_1, x_2, x_3)$. For $i = 1, 2, 3$, let $y_i$ be the outneighbour of $x_i$ in $V(T) \setminus \{x_1, x_2, x_3\}$. If all $y_i$ are distinct, then each $y_i$ dominates $\{x_1, x_2, x_3\} \setminus \{x_i\}$, and so $T[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is one of the tournaments $A_6$, $B_6$, $C_6$ and $D_6$. If $y_1 = y_2 = y_3$, let $z_1$ and $z_2$ be the two outneighbours of $y_1$. These two vertices dominate $\{x_1, x_2, x_3\}$, so $T[\{x_1, x_2, x_3, y_1, z_1, z_2\}]$ is isomorphic to $E_6$. If $y_1 = y_2 \neq y_3$, then $y_3$ dominates $x_1$ and $x_2$, and $y_1$ dominates $x_3$. If $y_1$ dominates $y_3$, then $T[\{x_1, x_2, x_3, y_1, y_3\}]$ is isomorphic to $R_5$. If $y_1$ is dominated by $y_3$, let $z$ be an outneighbour of $y_1$ distinct from $x_3$. The vertex $z$ dominates $\{x_1, x_2\}$, so $T[\{x_1, x_2, x_3, y_1, y_3, z\}]$ is isomorphic to $C_6$ or $D_6$.

\[\square\]

4 (1, $k$)-outdegree-splitting of tournaments

4.1 Improved upper bound for $f_T(1, k)$

A 3-cycle $C$ in a tournament $T$ is said to be $k$-good if $\delta^+(T-C) \geq k$. Clearly, if $C$ is a $k$-good 3-cycle, then $(V(C), V(T-C))$ is a (1, $k$)-splitting of $T$.

Lemma 21. Let $k$ be an integer and let $T$ be a strong tournament with minimum outdegree at least $k + 2$. Let $S$ be the set of vertices with outdegree $k + 2$ in $T$. If $T$ has no $k$-good 3-cycle, then the following hold.

(i) Every arc is dominated by a vertex in $S$.

(ii) For every vertex $v$, the subtournament $T[N^-(v) \cap S]$ has minimum indegree 1 and at least five vertices.

(iii) $|V(T)| \leq \frac{1}{10}(k + 7)(k + 8)$.

Proof. Suppose that $T$ contains no $k$-good 3-cycle. A 3-cycle $C$ in $T$ is $S$-dominated if there is a vertex $x \in S$ dominating $C$. Clearly, a 3-cycle in $T$ is $k$-good if and only if it is not $S$-dominated. Hence all 3-cycles are $S$-dominated.

(i) Let $uv$ be an arc. Since $T$ is strong, there is a 3-cycle $C$ containing $u$ by Theorem 16. This cycle is dominated by a vertex $s \in S$. If $s$ dominates $v$, then $s$ dominates the arc $uv$. If not, then $uv$ is a 3-cycle. This cycle is dominated by a vertex in $s'$ in $S$, which thus dominates $uv$.

(ii) Let $v$ be a vertex of $T$. By (i), $v$ is dominated by a vertex in $S$, so $N^-(v) \cap S$ is not empty. For any vertex $s \in N^-(v) \cap S$, the arc $sv$ is dominated by a vertex $s' \in S$, which is distinct from $s$. Hence $T[N^-(v) \cap S]$ has indegree at least 1 and thus contains a 3-cycle $s_1s_2s_3$. This 3-cycle is dominated by a vertex $s \in S$.

Assume first $s \rightarrow v$. By (i) the arc $sv$ is dominated by a vertex $s'$ of $S$. Clearly $s' \notin \{s_1, s_2, s_3\}$, because dominates $s_1s_2s_3$. Hence $s_1, s_2, s_3, s', s''$ are five vertices in $N^-(v) \cap S$.

Assume now that $v \rightarrow s$. Then $ss_1v$ is a 3-cycle which is dominated by a vertex $s'$. This vertex in $N^-(v) \cap S$ and is distinct from $s_2, s_3$ because it dominates $s$. Furthermore, by (i) there is a vertex $t$ of $S$ dominating $s'v$. If $t \notin \{s_1, s_2, s_3\}$, then $s_1, s_2, s_3, s', t$ are five vertices in $N^-(v) \cap S$. So we may assume that $t \in \{s_1, s_2, s_3\}$ and, without loss of generality, $t = s_2$. Now, there is a vertex $s''$ dominating the 3-cycle $ss_2v$. This vertex is distinct from $s_1, s_3$ because it dominates $s$, and is distinct from $s'$ because it dominates $s_2$. Hence, $s_1, s_2, s_3, s', s''$ are five vertices in $N^-(v) \cap S$.
Conjecture 25. Let \( k \) be a positive integer. If \( T \) is a \((k + 2)\)-regular tournament, then \( T \) contains a \( k \)-good 3-cycle.

A first step to prove this conjecture is the following.

Conjecture 24. Let \( k \) be a positive integer. If \( T \) is a tournament with minimum outdegree at least \( k + 2 \), then \( T \) contains a \( k \)-good 3-cycle.

Proof. It is sufficient to prove the result for strong tournaments. Indeed if \( T \) is not strong, then its terminal component \( T' \) has also outdegree at least \( k + 2 \). Moreover, every 3-cycle that is \( k \)-good in \( T' \) is also \( k \)-good in \( T \).

Henceforth, we may assume that \( T \) is strong. Let \( S \) be the set of vertices with outdegree \( k + 2 \) in \( T \).

• Assume \( k \in \{1, 2\} \). Then every vertex of \( S \) has outdegree at most 4 in \( T[S] \), so \( T[S] \) has a vertex with indegree at most 4. Thus, by Lemma 21(ii), \( T \) has a 1-good 3-cycle.

• Assume \( k = 3 \). Since \( \delta^+(T) \geq 5 \), then \( |V(T)| \geq 11 \). By Lemma 21(iii), we have the result if \( |V(T)| > 11 \). Henceforth we may assume \( |V(T)| = 11 \), so \( T \) is 5-regular. Hence \( tt_3(T) = \sum_{v \in V(T)} \binom{5}{2} \) = 110. Thus \( c_3(T) = \binom{11}{3} = 165 \). Now a tournament of order 5 contains at most five 3-cycles, and it contains exactly five if and only if it is \( R_5 \) the 2-regular tournament on 5-vertices. If all the 3-cycles are dominated, the outneighbourhood of every vertex induces an \( R_5 \). But then a vertex \( u \) dominates at most two inneighbours of any other vertex \( v \). Now if \( T \) had no \( k \)-good 3-cycles, then by Lemma 21(ii), for every vertex \( v \) the subtournament \( T[N^{-}(v)] \) would have a 3-cycle, which cannot be dominated and thus is \( k \)-good, a contradiction.

• Assume \( k = 4 \). Since \( \delta^+(T) \geq 6 \), then \( |V(T)| \geq 13 \). By Lemma 21(iii), we have the result if \( |V(T)| > 13 \). Henceforth we may assume \( |V(T)| = 13 \), so \( T \) is 6-regular. It is possible to test all 6-regular graphs on 13 vertices using a simple computer program and verify that each of them has at least one good 3-cycle.

Corollary 23. For \( k \in \{1, 2, 3, 4\} \), \( f_T(1, 2) = k + 2 \).

Proof. Let \( k \in \{1, 2, 3, 4\} \). Theorem 22 implies \( f_T(1, k) \leq k + 2 \) and Corollary 15 yields \( f_T(1, k) \geq k + 2 \).

We believe that Theorem 22 can be extended to all values of \( k \).

(iii) By (ii), every vertex has at least four inneighbours in \( S \). Thus

\[
|V(T)| = |N^+(S)| \leq |S| + \frac{1}{5} \left( (k + 2)|S| - \binom{|S|}{2} \right).
\]

But the polynomial \( Q(x) = x + \frac{1}{5} \left( (k + 2)x - \binom{k + 2}{2} \right) = \frac{1}{10}x(2k + 15 - x) \) increases on \([0, k + 15/2] \) and decreases on \([k + 15/2, +\infty[ \) and \( Q(k + 7) = Q(k + 8) = \frac{1}{10}(k + 7)(k + 8) \). Consequently, \( |V(T)| \leq \frac{1}{10}(k + 7)(k + 8) \). □
If true Conjecture 24 would be best possible.

**Proposition 26.** Let $k$ be a positive integer. For any $n \geq 3k + 3$, there is a tournament of order $n$ with minimum outdegree $k + 1$ that does not admit any $(1,k)$-outdegree-splitting.

**Proof.** Let $n \geq 3k + 3$. Let $T$ be a tournament of order $n$ whose vertex set can be partitioned into $(X_1, X_2, \{x\})$ such that $X_1 \rightarrow X_2$, $X_2 \rightarrow x$, $x \rightarrow X_1$, $T[X_1]$ is a transitive tournament of order $n - 2k - 2$, and $T[X_2]$ is a $k$-regular tournament.

Clearly, $\delta^+(T) = k + 1$. Let us now prove that $T$ has no $(1,k)$-outdegree-splitting.

Suppose for a contradiction that $T$ admits a $(1,k)$-outdegree-splitting $(V_1, V_2)$. The set $V_2$ must contain a vertex in $X_2$ because $T[X_1 \cup \{x\}]$ is transitive. The subtournament $T[V_1]$ contains a 3-cycle $C$. This cycle either contains $x$ or is contained in $C_1$.

- If $C$ contains $x$, then $C = xx_1x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. But $T[X_2]$ is $k$-regular, so it is strong. Thus there is a vertex $u$ of $V_2 \cap X_2$ dominating a vertex in $V_1 \cap X_2$. Thus $u$ has outdegree at most $k - 1$ in $T[V_2]$, a contradiction.

- If $C$ is contained in $C_1$, then $|V_2 \cap X_2| \leq 2k - 2$. Therefore $T[V_2 \cap X_2]$ has a vertex $u$ with outdegree less than $k - 1$. This vertex $u$ has outdegree less than $k$ in $T[V_2]$, a contradiction.

\[ \square \]

### 4.2 Existence of $k$-good 3-cycles

A result of Song [11] states that every 2-strong tournament of order at least 6 can be split into a 3-cycle and a strong subtournament unless it is $P_7$, the Paley tournament of order 7. Since $P_7$ is 3-regular, it has a 1-good 3-cycle by Theorem 22 Therefore we obtain the following.

**Theorem 27.** Every 2-strong tournament of order at least 6 has a 1-good 3-cycle and thus admits a $(1,1)$-outdegree-splitting.

In fact, having a 1-good 3-cycle is equivalent to having a $(1,1)$-outdegree-splitting.

**Proposition 28.** Let $T$ be a tournament. Then $T$ has a $(1,1)$-outdegree-splitting if and only if it has a 1-good 3-cycle $C$.

**Proof.** As we already observed, if $C$ is a 1-good 3-cycle, then $T$ has a $(1,1)$-outdegree-splitting.

Conversely, suppose that $T$ admits a $(1,1)$-outdegree-splitting $(V_1, V_2)$. Then for $i = 1, 2$, $T[V_i]$ contains a 3-cycle $C_i$. Let $S_2$ be the largest set such that $V_2 \subseteq S_2 \subseteq V(T - C_1)$ and $\delta^+(T[S_2]) \geq 1$. If $S_2 = V(T - C_1)$, then $C_1$ is $k$-good. If not, then let $R = V(T - C_1) \setminus S_2$. By definition, $S_2 \rightarrow R$. Thus $\delta^+(T - C_2) \geq 1$, and $C_2$ is $k$-good.

Unfortunately, Proposition 28 cannot be generalized for larger value of $k$ in the sense that there are tournaments with a $(1,k)$-splitting and no $k$-good 3-cycles. Furthermore, there are such tournaments with minimum outdegree $k + 1$; this shows that the condition of having minimum outdegree $k + 2$ in Conjecture 24 is best possible.

**Proposition 29.** Let $k$ be an integer greater than 1. There exists a tournament of order at $3k + 3$ with minimum outdegree $k + 1$ such that $T$ has a $(1,k)$-splitting but no $k$-good 3-cycles.
Proposition 32. Let $D$ be a digraph of order $n$. If $D$ contains two disjoint digraphs $D_1$, $D_2$ such that $\delta^+(D_i) = k_i$ for $i = 1, 2$ and $d_D^+(v) \geq k_1 + k_2 - 1$ for all $v \in V(D - (D_1 \cup D_2))$, then $D$ admits a $(k_1, k_2)$-outdegree-splitting. Moreover such a $(k_1, k_2)$-outdegree-splitting can be found in $O(n^2)$ time.
The second one is an algorithmic version of Theorem \[17\]

**Proposition 33.** Let \( T \) be a tournament with minimum outdegree at least \( k \). One can find in \( O(n^3) \) time a subtournament \( T' \) of \( T \) with minimum outdegree \( k \) such that \(|V(T')| \leq k^2/2 + 3k/2 + 1\).

**Proof.** By the proof of Theorem \[17\] if \(|V(T)| > k^2/2 + 3k/2 + 1\), then it contains a vertex \( x \) such that \( T - x \) has minimum outdegree at least \( k \). Such a vertex can be found in \( O(n^2) \) time, by finding the set \( S \) of vertices with outdegree \( k \), and taking \( x \) not in \( S \cup N^+(S) \). We then recursively apply the procedure to \( T - x \). As we reduce the order of the tournament at most \( n \) times, we find the desired subtournament \( T' \) in \( O(n^3) \) time.

**Lemma 34.** Let \( T \) be a tournament and \( v \) a vertex of \( T \). If \( T \) has a \((1, k)\)-outdegree-splitting \((V_1, V_2)\) with \( v_1 \in V_1 \), then there is a 3-cycle \( C_1 \) in \( T[V_1] \) such that \( v \in V(C_1) \) or \( V(C_1) \subseteq N^+(v) \).

**Proof.** Let \( N_1 = N^+(v) \cap V_1 \). If \( T[N_1] \) has a cycle, then it is the desired 3-cycle. Otherwise, \( T[N_1] \) is a transitive tournament. Now the sink \( w \) of \( T[N_1] \) has an outneighbour \( u \) in \( T[V_1] \), which is necessarily an inneighbour of \( v \), by definition of \( N_1 \). Therefore \( uwv \) is the desired 3-cycle.

**Theorem 35.**

(i) **TOURNAMENT \((1, 1)\)-OUTDEGREE-SPLITTING** can be solved in \( O(n^2) \) time;

(ii) for all \( k \geq 2 \), **TOURNAMENT \((1, k)\)-OUTDEGREE-SPLITTING** can be solved in \( O(n^3) \) time.

**Proof.** (i) Let us describe a procedure \((1, 1)\)-split \((T)\) that given a tournament \( T \) returns ‘yes’ if it admits a \((1, 1)\)-outdegree-splitting, and returns ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine \( \delta^+(T) \). This can be done in \( O(n^2) \) time.

1. If \( \delta^+(T) = 0 \), then the tournament \( T \) has no \((1, 1)\)-outdegree-splitting, and we return ‘no’.

2. If \( \delta^+(T) \geq 3 \), the answer is ‘yes’, by Corollary \[23\]

3. If \( \delta^+(T) \in \{1, 2\} \), let \( v \) be a vertex of degree 1 or 2 in \( T \). Without loss of generality, one may look for a \((1, 1)\)-outdegree-splitting \((V_1, V_2)\) of \( T \) such that \( v \in V_1 \). For every \( w \in N^+(v) \) and \( u \in N^+(w) \setminus N^+(v) \), we check whether \( T - \{u, v, w\} \) contains a 3-cycle. If yes for at least one choice of \( \{u, v, w\} \), the answer is ‘yes’ by Proposition \[3\] since \( \delta^+(T) \geq k \). If not, then return ‘no’.

This is valid by Lemma \[14\]

Given its score sequence, checking if a tournament of order \( n \) contains a 3-cycle can be done in \( O(n) \) by checking whether the score sequence is distinct from \((0, 1, 2, \ldots, n - 1)\), the score sequence of the transitive tournament. Since the score sequence of \( T - \{u, v, w\} \) can be obtained in linear time from the list of outdegrees of \( T \), checking if \( T - \{u, v, w\} \) contains a cycle can be done in \( O(n) \) time.

Now since \( v \) has degree at most 2, the procedure considers at most \( 2(n - 1) \) subtournaments \( T - \{u, v, w\} \). Therefore \((1, 1)\)-split runs in \( O(n^2) \) time.

(ii) Let us describe a procedure \((1, k)\)-split \((T)\) that given a tournament \( T \) returns ‘yes’ if \( T \) if it admits a \((1, k)\)-outdegree-splitting, and return ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine \( \delta^+(T) \). This can be done in \( O(n^2) \) time.

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1. If \( \delta^+(T) = 0 \), then the tournament \( T \) has no \((1, k)\)-outdegree-splitting, and we return ‘no’.

2. If \( 1 \leq \delta^+(T) \leq k - 1 \), let \( U_1 \) be the set of vertices of degree less than \( k \) in \( T \). Clearly, for any \((1, 2)\)-outdegree-splitting \((V_1, V_2)\) of \( T \), \( U_1 \subseteq V_1 \). Let \( v \) be a vertex of \( U_1 \). For every \( w \in N^+(v) \) and \( u \in N^+(v) \setminus N^+(v) \), we check whether \( T - (U_1 \cup \{u, v, w\}) \) contains a subtournament of minimum outdegree \( k \) using the procedure \text{Outdegree-}k\text{-Subtournament} described below. If yes for at least one choice of \( \{u, v, w\} \), the answer is ‘yes’ by Proposition \( \ref{prop:32} \) since all vertices of \( V(T) \setminus U_1 \) have outdegree at least \( k \) in \( T \). If not, then return ‘no’. This is valid by Lemma \( \ref{lem:34} \).

3. If \( \delta^+(T) \geq k \), then we first find a subtournament \( T' \) of \( T \) with \( \delta(T') \geq k \) and \( |V(T')| \leq k^2/2 + 3k/2 + 1 \). If \( T - T' \) contains a 3-cycle, then \( T \) admits a \((1, k)\)-outdegree-splitting by Proposition \( \ref{prop:3} \) and so we return ‘yes’. If not then \( T - T' \) is a transitive tournament and all 3-cycles of \( T \) intersect \( T' \) and therefore there are at most \((k^2/2 + 3k/2 + 1)n^2 \) of them. For each 3-cycle \( C \), we check with \text{Outdegree-}k\text{-Subtournament} whether \( T - C \) contains a subtournament of minimum outdegree \( k \). If yes, for one of them, then we return ‘yes’ because there is a \((1, k)\)-outdegree-splitting by Proposition \( \ref{prop:3} \) if not, then we return ‘no’.

Remark 36. In the above procedure, one can shorten Step 3 if \( \delta^+(T) \geq k + 2 \). In this case, by Corollary \( \ref{cor:23} \) we can directly return ‘yes’.

The procedure \text{Outdegree-}k\text{-Subtournament}(T)\) takes as an input the tournament \( T \) as well as its list of outdegrees and a list \( L \) of vertices in the transitive tournament \( T - T' \) ordered in increasing order of their outdegrees. Observe that the list of outdegrees is already computed when \text{degree-}1\text{-split} call this procedure and the order of \( T - T' \) can be computed just once after computing \( T' \). First, we alter the list of outdegrees by keeping the outdegrees for vertices in \( T' \) but for vertices in \( T - T' \) we count only outneighbours in \( T' \). At each step, \text{Outdegree-}k\text{-Subtournament} first checks \( V(T') \) and returns ‘no’ if \( V(T') = \emptyset \), otherwise it tries to find a vertex \( v \) with \( d^+(v) < k \). Notice that possible candidates for \( v \) are only vertices in \( T' \) and the first \( k \) vertices in \( L \). If there is no such vertex \( v \), it returns ‘yes’. Otherwise it removes \( v \) and tries again. If \( v \in V(T') \), then it decreases the outdegree of all inneighbours of \( v \) and if \( v \notin V(T') \), then it decreases outdegrees only for inneighbours from \( V(T') \). The total time spent on a vertex \( v \in V(T') \) is \( O(n) \), which gives \( O(V(T')n) = O(n) \) in total. The total time spent on a vertex \( v \notin V(T') \) is \( O(1) \), which gives \( O(n) \) in total. Therefore, \text{Outdegree-}k\text{-Subtournament} runs in \( O(n) \) time.

Now Step 1 runs in constant time. In Step 2, there are at most \( k + 1 \) candidates for \( w \), and thus \text{Outdegree-}k\text{-Subtournament} is called less than \((k + 1)n \) times. Therefore Step 2 runs in \( O(n^3) \) time. Step 3 first finds a small subtournament \( T' \) with outdegree \( k \), which can be done in \( O(n^3) \) time by Proposition \( \ref{prop:33} \). Then it runs \( O(n^2) \) times \text{Outdegree-}k\text{-Subtournament}. Therefore Step 3 runs in \( O(n^3) \) time.

Overall \((1, k)\)-split runs in \( O(n^3) \) time.

\begin{proof}

The procedure \((1, k)\)-split \((T)\) can be modified to find a \((1, k)\)-outdegree-splitting if it exists, using Proposition \( \ref{prop:32} \) instead of Proposition \( \ref{prop:3} \). In contrast, the procedure \((1, 1)\)-split \((T)\) cannot be instantly modified into a procedure that finds a \((1, 1)\)-outdegree-splitting if it exists. However, using a similar approach, we now describe such a procedure.

Theorem 37. One can find a \((1, 1)\)-outdegree-splitting of a tournament in \( O(n^2) \) time.
Proof. Let us describe a procedure \((1, 1)\)-findsplit\((T)\) that returns a \((1, 1)\)-outdegree-splitting of the tournament \(T\) if it admits one, and return ‘no’, otherwise.

We first compute the outdegree of every vertex and we determine \(\delta^+(T)\).

If \(T\) contains a vertex of outdegree 0, then we return ‘no’. If \(\delta^+(T) \geq 4\), then we pick a vertex \(x\) and find a 3-cycle \(C\) containing \(x\). Such a cycle can be found in \(O(n^2)\) by testing if there is an arc from \(N^+(x)\) to \(N^-(x)\). We return \((V(C), V(T - C))\). This is valid since \(\delta^+(T - C) \geq \delta^+(T) - |V(C)| \geq 1\).

If \(\delta^+(T) \leq 3\), we choose a vertex \(v\) such that \(d^+(v) \in \{1, 2, 3\}\). If \(T[N^+(v)]\) induces a 3-cycle, then we check whether \(T - N^+(v)\) contains a cycle \(C\). If yes, we extend \((T[N^+(v)], C)\) into a \((1, 1)\)-outdegree-splitting by Proposition 32. If not, for every \(w \in N^+(v)\) and \(u \in N^+(w)\ \setminus N^+(v)\), we check if \(T - \{u, v, w\}\) contains a cycle \(C(uvw)\). If yes for at least one choice of \(\{u, w\}\), then we extend \((uvw, C(uvw))\) into a \((1, 1)\)-outdegree-splitting by Proposition 32 and we return ‘no’ otherwise. This is valid by Lemma 34.

Since there are at most three candidates for \(w\), there are \(O(n)\) cases to check. Therefore \((1, 1)\)-findsplit runs in \(O(n^2)\) time.

Remark 38. The proof of Proposition 28 yields a \(O(n^2)\)-time procedure to find a 1-good 3-cycle given a \((1, 1)\)-outdegree-splitting. Combining this procedure with \((1, 1)\)-findsplit, we obtain a \(O(n^2)\)-time algorithm that finds a 1-good 3-cycle in a tournament if it exists, and returns ‘no’ otherwise.

6 Further research
6.1 Splittable score sequences

Being \((1, 1)\)-outdegree-splittable is not determined by the score sequence. For example, the two tournaments depicted Figure 2 have score sequences \((2, 2, 2, 3, 4)\) but the one to the left has no \((1, 1)\)-outdegree-splitting (See Proposition 26) while the one to the right admits the \((1, 1)\)-outdegree-splitting \((\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})\).

Figure 2: Non-(1, 1)-outdegree-splittable and (1, 1)-outdegree-splittable tournaments with the same score sequence

However there are score sequences \(s\) such that all tournaments with score sequence \(s\) are \((1, 1)\)-outdegree-splittable. Such score sequences are said to be \((1, 1)\)-outdegree-splittable. For example, Theorem 22 implies that \((s_1, \ldots, s_n)\) is \((1, 1)\)-outdegree-splittable.

Problem 39. Which score sequences are \((1, 1)\)-outdegree-splittable?
6.2 Erdős-Posa property for digraphs with minimum outdegree \( k \)

McCuaig’s algorithm [7] relies on the theorem stating that a digraph \( D \) has either two disjoint cycles or a set \( S \) of at most three vertices such that \( D - S \) is acyclic. More generally, Reed et al. [9] showed that cycles in digraphs have the Erdős-Posa property.

**Theorem 40** (Reed et al. [9]). For every positive integer \( n \), there exists an integer \( t(n) \) such that for every digraph \( D \), either \( D \) has \( n \) pairwise-disjoint cycles, or there exists a set \( T \) of at most \( t(n) \) vertices such that \( D - T \) is acyclic.

It is then natural to ask whether digraphs with maximum outdegree \( k \) have the the Erdős-Posa property.

**Problem 41.** Let \( k \) be a fixed integer. For every positive integer \( n \), does there exist an integer \( t_k(n) \) such that for every digraph \( D \), either \( D \) has \( n \) pairwise-disjoint subdigraphs with minimum outdegree \( k \), or there exists a set \( T \) of at most \( t_k(n) \) vertices such that \( \delta^+(D - T) < k \) ?

6.3 Strong connectivity and outdegree-splitting with prescribed vertices

Any \( f_T(k_1, k_2) \)-strong tournament has minimum outdegree at least \( f_T(k_1, k_2) \) and thus admits a \( (k_1, k_2) \)-outdegree-splitting. Therefore, it is natural to ask the following.

**Problem 42.** What is the minimum integer \( h_T(k_1, k_2) \) such that every \( h_T(k_1, k_2) \)-strong tournament \( T \) of order at least \( 2k_1 + 2k_2 + 2 \) contains a \( (k_1, k_2) \)-outdegree-splitting?

The condition \( |V(T)| \geq 2k_1 + 2k_2 + 2 \) is the above problem is just to avoid the small tournaments that cannot have any \( (k_1, k_2) \)-outdegree-splitting for cardinality reasons. Clearly, \( h_T(k_1, k_2) \leq f_T(k_1, k_2) \). But it is very likely that \( h_T(k_1, k_2) \) is smaller than \( f_T(k_1, k_2) \). As mentioned in the beginning of Subsection 4.2, a result of Song [11] implies that \( h_T(1, 1) \leq 2 \) (in fact \( h_T(1, 1) = 2 \) because a 1-strong tournament \( T \) with a vertex \( v \) such that \( T - v \) is a transitive tournament has clearly no \( (1, 1) \)-outdegree-splitting.) whereas \( f_T(1, 1) = 3 \).

One might also ask similar questions for outdegree-splitting with prescribed vertices (as in Theorem 8). Bang-Jensen et al. [2] proved that if \( T \) is a tournament of order 8 and \( xy \) an arc in \( T \) such that \( T \setminus xy \) is 2-strong, then \( T \) contains an outdegree-1-splitting \( (V_x, V_y) \) with \( x \in V_x \) and \( y \in V_y \).

References


