Complexity of locally injective homomorphism to the Theta graphs ⋆

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Abstract. A Theta graph is a multigraph which is a union of at least three internally disjoint paths that have the same two distinct end vertices. We show full computational complexity characterization of the problem of deciding the existence of a locally injective homomorphism from an input graph G to any fixed Theta graph.
Keywords: computational complexity; locally injective homomorphism; Theta graph

1 Introduction

Let G be a graph. We denote its set of vertices by $V(G)$ and its set of edges by $E(G)$. Graphs in this paper are generally simple. If they may have parallel edges or loops, we explicitly say so. We denote the degree of a vertex v by $\deg_G(v)$ and the set of all neighbors of v by $N_G(v)$. We omit G in the subscript if it is clear from the context. By $[n]$ we denote the set of integers $\{1,\dots,n\}$.

Let G and H be graphs. A *homomorphism* is an edge preserving mapping $f : G \to H$. A homomorphism is *locally injective (resp. surjective, bijective)* if $N(v)$ is mapped to $N(f(v))$ injectively (resp. surjectively, bijectively). A locally bijective homomorphism is also known as a *covering projection* or simply a *cover*. Similarly, locally injective homomorphism is known as a *partial covering projection* and a *partial cover*.

We consider the following decision problem. Let H be a fixed graph and G be an input graph. Determine the existence of a locally injective (surjective, bijective) homomorphism $f : G \to H$. We denote the problem by H-LIHom (resp. H-LSHom, H-LBHom). If there is no local restriction on the homomorphism, the problem is called H-Hom.

In this paper we consider the H-LIHom problem.

Problem: H-LIHom
Input: graph G
Question: Does there exist a locally injective homomorphism $f : G \to H$.

Locally injective homomorphisms are closely related to $H(2,1)$-labelings, which have applications in frequency assignment. Let H be a graph. An $H(2,1)$-*labeling* of a graph G is a mapping $f : V(G) \to V(H)$ such that every pair of

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adjacent vertices are mapped to distinct and nonadjacent vertices. Moreover, image of every pair of vertices in distance two is two distinct vertices. The mapping f corresponds to a locally injective homomorphism to the complement of H.

The computational complexity of H-HOM was fully determined by Hell and Nešetřil [9]. They show that the problem is solvable in polynomial time if H is bipartite and it is NP-complete otherwise.

The study of H-LSHOM was initiated by Kristiansen and Telle [13] and completed by Fiala and Paulusma [8] who gave a full characterization by showing that H-LSHOM is NP-complete for every connected graph on at least three vertices.

The complexity of locally bijective homomorphisms was first studied by Bodlaender [2] and Abello et al. [1]. Despite the effort [10–12] the complete characterization is not known.

Similarly for the locally injective homomorphism the dichotomy for the complexity is not known. Some partial results can be found in [4, 5, 7]. Fiala and Kratochvíl [6] also considered a list version of the problem and showed dichotomy.

Fiala and Kratochvíl [5] showed, that H-LBHOM is reducible in polynomial time to H-LIHOM. Hence it makes sense to study the complexity of H-LIHOM where H-LBHOM is solvable in polynomial time. This is the case for Theta graphs, which we consider in this paper. Note that no other direct consequences of complexity of H-HOM or H-SHOM to H-LIHOM are known.

Fiala and Kratochvíl [4] showed, that if Theta graph H contains only simple paths of length a, then H-LIHOM is always polynomial. They also showed that if H contains only simple paths of two different lengths a and b, then:

- if both a and b are odd, then H-LIHOM is polynomial,
- if a and b have different parity, then H-LIHOM is NP-complete,
- if both a and b are even, then H-LIHOM is as hard as H'-LIHOM, where H' is a Theta graph, that arise from H by replacing paths of length a, resp. b by paths of lengths $\frac{a}{2}$, resp. $\frac{b}{2}$.

The study of Theta graphs continues in the work of Fiala et al. [7], which proves NP-completeness for Theta graphs with exactly three odd different lengths of simple paths. We extend the last result to all Theta graphs, which finishes the complexity characterization of Theta graphs.

Theorem 1. *Let H be a Theta graph with simple paths of at least three distinct lengths. Then H-LIHOM problem is NP-complete.*

In the next section, we introduce several definitions and gadgets which we use in NP-hardness reductions. In Section 3 we state necessary Lemmas for the proof of Theorem 1. We postpone proofs of some Propositions and Lemmas to Appendix due to the page limit for the paper.

2 Definitions and Gadgets

A graph G is a *Theta graph* (or *Θ-graph*) if it is the union of at least three internally disjoint paths that have the same two distinct end vertices. We denote the two vertices of degree at least three by A and B. Note that if two paths of the union are of length one, the resulting graph have parallel edges.

A Θ-graph $\mathcal{T}$ is denoted by $\Theta(a_1^{t_1}, a_2^{t_2}, \ldots, a_n^{t_n})$, where $1 \leq a_1 < a_2 < \cdots < a_n$ and $t_i \geq 1$ for all $1 \leq i \leq n$, if $\mathcal{T}$ is the union of paths of lengths $a_1, a_2, \ldots, a_n$ and t_i are the corresponding multiplicities. We write a_i instead of a_i^1. We assume that $n \geq 3$ as the case $n \leq 2$ is already solved [4].

Throughout this section we assume that $\mathcal{T} = \Theta(a_1^{t_1}, \ldots, a_n^{t_n})$ is some Θ-graph.

Let G be a graph and $v_1, v_2, \ldots, v_n$ be a path in G. The path is *simple* if v_1 and v_n are vertices of degree at least three and all inner vertices of the path have degree two. We denote a simple path of length n by SP_n.

Let G be a graph and f be a locally injective homomorphism from G to $\mathcal{T}$. Note that f must map all vertices of degree at least three to A or B in $\mathcal{T}$. Hence every end vertex of every simple path of G must be mapped to A or B. We call a vertex *special* if it has degree at least three or if we insist that it is mapped to A or B. Note that A and B are also special vertices and if v is a special vertex of degree less than three, then adding extra pendant leaves forces, that v must be mapped to A or B. We need to control what are the possible mappings of simple paths. Let $v_1, v_2, \ldots, v_{l-1}, v_l$ be a simple path P. For a locally injective homomorphism f, define a function $g_f^P(v_1, v_l) = a_i$ if the edge v_1v_2 is mapped by f to an edge of SP_{a_i} in $\mathcal{T}$. We omit the superscript P if there is only one simple path containing v_1 and v_l.

We say that SP_n allows *decomposition* $a_i - a_j$ if there exists a graph H containing a simple path P of length n with end vertices u and v and a locally injective homomorphism $f : H \to \mathcal{T}$ such that $g_f^P(u, v) = a_i$ and $g_f^P(v, u) = a_j$. We denote the decomposition by $a_i -_k a_j$ (resp. $a_i -_c a_j$) if it forces that $f(u) = f(v)$ (resp. $f(u) \neq f(v)$).

In case of $x -_k y$ (resp. $x -_c y$) decomposition we say, that the decomposition keeps (resp. changes) the parity.

Proposition 1. *Every simple path SP_{a_i} always allows decomposition $a_i -_c a_i$ and does not allow decomposition $a_i -_k a_i$. Similarly, for $i \neq j$ holds that $SP_{a_i + a_j}$ always allows decomposition $a_i -_k a_j$ and never allows $a_i -_c a_j$.*

The proof of Proposition 1 as well as proofs of the other propositions is in Appendix.

Let M be a positive integer and $\mathcal{E} \subseteq \{a_1, a_2, \ldots a_n\}$. The following notation

$$\boldsymbol{M_{\mathcal{E}}^{\mathcal{T}}} \textbf{ :}\ x_1 - y_1, x_2 - y_2, \ldots, x_s - y_s, (z_1 - w_1), (z_2 - w_2), \ldots, (z_t - w_t)$$

describes the list of all decompositions $x - y$ of SP_M where $x, y \in \mathcal{E}$. Decompositions $x_i - y_i$ must be possible and decompositions $z_j - w_j$ are optional for all $i \in [t]$ and $j \in [s]$. Moreover, $-_k$ and $-_c$ can be used instead of just $-$.

Now we introduce gadget $B_z^{\mathcal{T}}$, which can be used for blocking a simple path of length z at some vertex. It has a central vertex y which is for every $i \in [n]$ connected by paths of length a_i to vertices v_j^i where $j \in [t_i]$. Moreover, every vertex v_j^i except v_1^z has two extra pendant leaves (so v_j^i is special). If X is a copy of $B_z^{\mathcal{T}}$, we refer to the vertex v_1^z by $X(w)$ or w if X is clear from the context. Moreover, we demand that w is special. See Figure 1.

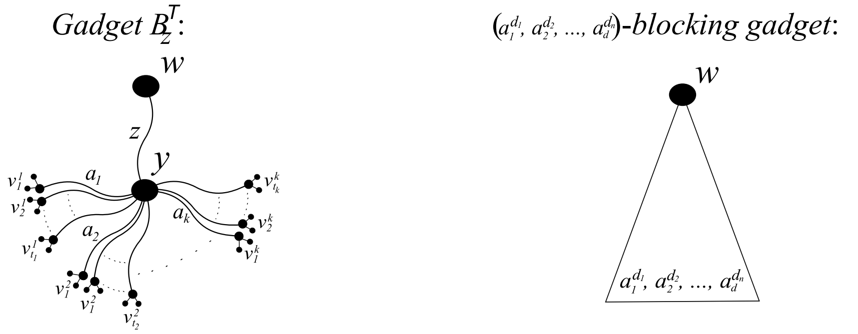

Fig. 1. $B_z^{\mathcal{T}}$ and a blocking gadget

Proposition 2. *Let G be a graph and let X be a copy of $B_z^{\mathcal{T}}$ in G. Moreover, w has degree at least three. Suppose, that there exists a locally injective homomorphism $f : G \to \mathcal{T}$. Then:*

$$g_f(w, y) = z = g_f(y, w)$$

The gadget $B_z^{\mathcal{T}}$ blocks usage of one z at w by forcing $\mathcal{T}$-LIHom to map the path wy to SP_z in $\mathcal{T}$.

We usually need to use several copies of the gadget $B_z^{\mathcal{T}}$ at once. Let d_1, $d_2, \ldots, d_n$ be nonnegative integers such that $d_i \le t_i$ for all $i \in [n]$. We define the $(a_1^{d_1}, \ldots, a_n^{d_n})$*-blocking gadget* to be the union of $t_i - d_i$ copies of $B_{a_i}^{\mathcal{T}}$ for every $i \in [n]$ where there is only one vertex w shared by all of them. If X is a copy of the blocking gadget, we refer to the vertex w by $X(w)$ or w if X is clear from the context. Note that we will consider only copies of the blocking gadget where vertex w is special.

In the notation we omit a^0 and the superscript d_i if $d_i = 1$. In further figures, we depict the $(a_1^{d_1}, \ldots, a_n^{d_n})$-blocking gadget by a triangle with one vertex corresponding to w and with inscribed text $a_1^{d_1}, \ldots, a_n^{d_n}$, see Figure 1.

Proposition 3. *Let G be a graph and X be a copy of $(a_1^{d_1}, a_2^{d_2}, \ldots, a_n^{d_n})$-blocking gadget in G where $\deg_G(w) \ge 3$. Let $P_1, P_2, \ldots, P_k$ be the all simple paths, starting at w with without any other intersection with the blocking gadget X and with end points $u_1, u_2, \ldots, u_k$. Suppose, that there exists a locally injective* *homomorphism* $f : G \to \mathcal{T}$*. Then* $k \leq \sum_{i=1}^{n} d_i$ *and*

$$\forall i \in [n] : \quad |\{u_j, g_f^{P_j}(w, u_j) = a_i\}| \leq d_i.$$

Note that the blocking gadget on its own is not sufficient for reducing $\Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ to $\Theta(a, b, c)$. The obstacle is that a simple path may have different possible inner decompositions and the blocking gadget cannot be used inside paths in general.

Apart from blocking some paths we also need to force that several special vertices are mapped to the same vertex (to A or B). Hence we introduce the following gadget.

Definition 1. *Let* $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ *be a* Θ*-graph. Let* $r \geq 2$ *be an integer and* N *be the smallest power of two such that* $N \geq 2r$*. Define a graph* $PC_a^{\mathcal{T}}(r)$ *(see Figure 2) with special vertices* $u_1, u_2, \ldots, u_{2N-1}, u'_1, u'_2, \ldots, u'_{N-1}$, $v_1, v_2, \ldots, v_{2N-1}, v'_1, v'_2, \ldots, v'_{N-1}$ *to be a graph constructed in the following way:*

- $\forall i \in \{1, 2, \ldots, N-1\}$*, connect vertex* u'_i *with vertices* u_i*,* u_{2i} *and* u_{2i+1} *by paths of lengths* c*,* a *and* b *(in this order),*
- $\forall i \in \{1, 2, \ldots, N-1\}$*, connect vertex* v'_i *with vertices* v_i*,* v_{2i} *and* v_{2i+1} *by paths of lengths* c*,* a *and* b *(in this order),*
- $\forall i \in \{2, 3, \ldots, N-1\}$*, take copies* U_i *and* V_i *of* (a, c)*-blocking gadget if* i *is even and* (b, c)*-blocking gadget if* i *is odd and identify vertex* u_i *with* $U_i(w)$ *and vertex* v_i *with* $V_i(w)$*,*
- $\forall i \in \{1, 2, \ldots, N-1\}$*, take copies* U'_i *and* V'_i *of* (a, b, c)*-blocking gadget and identify vertex* u'_i *with* $U'_i(w)$ *and vertex* v'_i *with* $V'_i(w)$*,*
- *identify vertex* u'_1 *with* v_1 *and vertex* v'_1 *with* u_1*.*

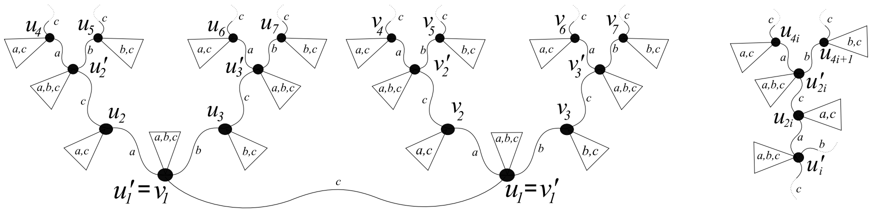

Fig. 2. Graph $PC_a^{\mathcal{T}}(N)$ and local neighborhood of vertices u_i and u'_i

Proposition 4. *Let* $r \geq 2$ *be an integer,* $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ *be a* Θ*-graph and let* Z *be a copy of graph* $PC_a^{\mathcal{T}}(r)$ *in a graph* G*. Let* N *be as in the definition of* $PC_a^{\mathcal{T}}(r)$*. Suppose, that there exists a locally injective homomorphism* $f : G \to \mathcal{T}$*, such that for all* $i \in [2N-1] : f(Z(u_i)), f(Z(v_i)) \in \{A, B\}$*. Then for all even* $i, j \in \{N, N+1, \ldots, 2N-1\}$ *the following hold:*

$$f(Z(u_i)) = f(Z(u_j)) \neq f(Z(v_j)) = f(Z(v_i)),$$

$$g_f(Z(u_i), Z(u'_{i/2})) = a = g_f(Z(v_j), Z(v'_{j/2})).$$

Let Z be a copy of $PC_a^{\mathcal{T}}(r)$. For $i \in [N]$ we define $Z(x_i)$ to be u_{N+2i-2} and $Z(y_i)$ to be v_{N+2i-2}. Similarly as the gadget $PC_a^{\mathcal{T}}(r)$, we define a gadget $PC_b^{\mathcal{T}}(r)$, with the only difference, that a and b are swapped in the construction. We call the graphs $PC_a^{\mathcal{T}}(r)$ and $PC_b^{\mathcal{T}}(r)$ *parity controllers*. With parity controllers we are able to create arbitrary many special vertices, which are mapped to the same vertex of $\mathcal{T}$ in every locally injective homomorphism to $\mathcal{T}$. Moreover, each of these special vertices is an end point of a path which must be mapped to a simple path of length a (resp. b) in $\mathcal{T}$.

For some $\mathcal{T}$, we reduce 3-SAT or NAE-3-SAT to $\mathcal{T}$-LIHOM. In the reduction we use copies the following gadget for representing clauses.

Let $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ be a Θ-graph. We define *$\mathcal{T}$-clause gadget* to be a graph with special vertices u_0, u_1, u_2, u_3 such that, for all $i \in \{1, 2, 3\}$, vertex u_i is connected to u_0 by a path of length $a + b + c$ and u_0 is identified with the vertex $X(w)$, where X is a copy of the (a, b, c)-blocking gadget. If Y is a copy of $\mathcal{T}$-clause gadget, we refer to the vertices u_j by $Y(u_j)$ or u_j if Y is clear from the context for all $j \in \{0, 1, 2, 3\}$. Note that we will consider only such copies of $\mathcal{T}$-clause gadget, that vertices u_1, u_2 and u_3 are special. See Figure 3.

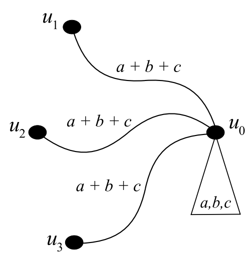

Fig. 3. $\mathcal{T}$-clause gadget

Let Y be a copy of the $\mathcal{T}$-clause gadget and $\gamma \in \{a, b\}$. We say, that $\mathcal{T}$ is *γ-positive* if and only if there exists a locally injective homomorphism $f : Y \to \mathcal{T}$ such that:

- $f(u_0) \neq f(u_1) = f(u_2) = f(u_3) \in \{A, B\}$,
- $g_f(u_1, u_0) = g_f(u_2, u_0) = g_f(u_3, u_0) = \gamma$.

Proposition 5. *Let $a < b < c$ be positive integers, such that $a + b \neq c$. Let $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ be a Θ-graph and Y be the $\mathcal{T}$-clause gadget. Let $\gamma \in \{a, b\}$ and $x, y, z \in \{\gamma, c\}$.*

Then there exists a locally injective homomorphism $f : Y \to \mathcal{T}$ satisfying:

- $f(u_0) \neq f(u_1) = f(u_2) = f(u_3) \in \{A, B\}$,
- $g_f(u_1, u_0) = x, \ g_f(u_2, u_0) = y, \ g_f(u_3, u_0) = z$

if and only if at least one of the following conditions hold:

- $\{x, y, z\} = \{\gamma, c\}$,
- $x = y = z = \gamma$ *and* $\mathcal{T}$ *is* γ*-positive.*

3 NP-Completeness reductions

In this section we give several lemmas, which each show NP-completeness for some Θ-graphs. Together, they cover all Θ-graphs and hence they imply Theorem 1. We present the proof only of Lemma 1. Proofs of the other lemmas are in Appendix.

Note that the lemmas show only NP-hardness as H-LIHOM is clearly in NP for any H.

In this section we assume that $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \dots, a_n^{t_n})$.

Lemmas are grouped into three blocks, which reflect what type of reduction is used. Reductions in each group are similar. The first group shows NP-hardness from 3-SAT and NAE-3-SAT.

Lemma 1. *Let* $\mathcal{T}$ *be a* Θ*-graph such that* $a + b \neq c$ *and*

$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}:}\ a-b, (a-a)$$
$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a-c, (a-a), (b-b)$$

then $\mathcal{T}$-LIHOM *is NP-complete.*

Proof. Let $\phi = \vee_{i=1}^{p}(c_i^1 \wedge c_i^2 \wedge c_i^3)$ be a boolean formula in conjunctive normal form with variables $s_1, s_2, \dots, s_r$ (where every clause has exactly 3 literals). Let var, neg and ord be functions from the set of all literals of the formula ϕ, such that $var(c_i^j)$ is the variable corresponding to the literal c_i^j, $neg(c_i^j)$ is 0 if the literal c_i^j is a positive occurrence of the variable $var(c_i^j)$ and $neg(c_i^j) = 1$ otherwise, and $ord(c_i^j)$ is the order of occurrence of the literal of the variable $var(c_i^j)$ in ϕ.

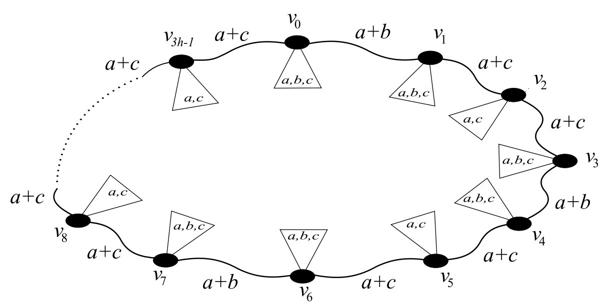

Fig. 4. Variable gadget α

Define *variable gadget* α *of order* h (see Figure 4) as a graph with special vertices $v_0, v_1, \dots, v_{3h-1}$ such that for all $i \in \{0, \dots, h-1\}$, vertices v_{3i} and v_{3i+1}

are connected by a path of length $a+b$ and vertices v_{3i+1} and v_{3i+2} as well as vertices v_{3i+2} and v_{3i+3} are connected by a path of length $a+c$ (all indices are counted by modulo $3h$). For every $i \in \{0,\dots,h-1\}$ we take two copies B_i^0 and B_i^1 of the (a,b,c)-blocking gadget and identify the vertex $B_i^0(w)$ with the vertex v_{3i} and the vertex $B_i^1(w)$ with the vertex v_{3i+1}, and for every $j = 0,\dots,h-1$ we take a copy B_i^2 of the (a,c)-blocking gadget and identify the vertex $B_i^2(w)$ with the vertex v_{3j+2}.

For every $i \in [r]$, let n_i be the number of occurrences of the variable s_i in the formula ϕ, let X_i be a copy of the variable gadget α of order n_i+1. For every $j \in [p]$ let Z_j be a copy of the the clause gadget and let Y be a copy of the parity controller $PC_b^{\mathcal{T}}(r)$. Now define a graph G_ϕ, which contains copies $X_1,\dots,X_r,Z_1,\dots,Z_p,Y$ and for every literal c_j^d of the formula ϕ, if $var(c_j^d) = s_i$ then we identify the vertices $X_i(v_{3ord(c_j^d)+neg(c_j^d)-3})$ and $Z_j(u_d)$. For every $i \in [r]$ we replace the copy of the (a,c)-blocking gadget on vertex $X_i(v_{3n_i+2})$ by a copy of the (a,b,c)-blocking gadget and identify vertices $X_i(v_{3n_i+2})$ and $Y(x_i)$ (clearly the combination of the (a,b,c)-blocking gadget and Y creates for the vertex $X_i(v_{3n_i+2})$ the same constraints as the (a,c)-blocking gadget), and to every vertex $X_i(v_j)$ and $Y(y_i)$ of degree less then three we add new pendant leaves (so all vertices $X_i(v_j)$ and $Y(y_i)$ are special).

We claim, that if $\mathcal{T}$ is b-positive then ϕ is satisfiable if and only if there exists a locally injective homomorphism from G_ϕ to $\mathcal{T}$. And if $\mathcal{T}$ is not b-positive then ϕ is NAE-satisfiable if and only if there exists a locally injective homomorphism from G_ϕ to $\mathcal{T}$. The fact that 3-SAT and NAE-3-SAT are NP-complete problems and $\mathcal{T}$-LIHom is in NP imply that $\mathcal{T}$-LIHom is NP-complete.

At first suppose that $\mathcal{T}$ is b-positive and there exists a locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Let X be one of the copies of the variable gadget α of order d in G_ϕ. Since $\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, (a-a)$, we know that $g_f(v_0,v_1) \in \{a,b\}$. If $g_f(v_0,v_1) = b$, then necessarily $g_f(v_1,v_0) = a$. But since there is a copy of the (a,b,c)-blocking gadget on vertex v_1 we know, that $g_f(v_1,v_2)$ is b or c. Since $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k c, (a-a), (b-b)$ if $g_f(v_1,v_2) = b$, then $g_f(v_2,v_1) = b$, which is not possible because of the copy of the (a,c)-blocking gadget on v_2 and so $g_f(v_1,v_2) = c$ and necessary $g_f(v_2,v_1) = a, g_f(v_2,v_3) = c, g_f(v_3,v_2) = a$ and then necessarily $g_f(v_3,v_4) = b$. And since the gadget X is symmetric, we can continue in the same way until we reach the vertex v_0 again. Then $\forall i \in \{0,\dots,d-2\}$ if there exists a simple path from v_{3i} to $Z(u_0)$ for some copy Z of the clause gadget, then $g_f(v_{3i}, Z(u_0)) = c$ (the corresponding literal is false) and analogically for the simple path from v_{3i+1} to $Z(u_0)$, for which holds $g_f(v_{3i+1}, Z(u_0)) = b$ (the corresponding literal is true). In this case we say that the variable corresponding to X is false.

If $g_f(v_0,v_1) = a$ then we use a similar idea as in the previous paragraph, but we argue in the counterclockwise order ($g_f(v_0,v_{3d-1})$ must be c, etc.) and analogically we get, that if appropriate simple paths exists then $g_f(v_{3i}, Z(u_0)) = b$ (the corresponding literal is true), resp. $g_f(v_{3i+1}, Z(u_0)) = c$ (the corresponding literal is false). In this case we say that the variable corresponding to X is true.

We claim that in this evaluation every clause of ϕ is satisfied. If not, then there exists a copy of the clause gadget Z corresponding to some clause and $g_f(u_1, u_0) = g_f(u_2, u_0) = g_f(u_3, u_0) = c$ in Z. Since there is a copy of the (a, b, c)-blocking gadget at vertex u_0, without loss of generality we suppose that $g_f(u_0, u_1) = c$. Thus the simple path u_0u_1 of length $a+b+c$ allows decomposition $c - c$. But this is not possible because $0 < a + b < c + a$ and $a + b \neq c$, a contradiction.

On the other side, if $\mathcal{T}$ is b-positive and formula ϕ is satisfiable, then there exists a locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Suppose that $e : \{s_1, \ldots, s_r\} \to \{true, false\}$ is a satisfying evaluation of the variables of ϕ and predefine a function $f : G_\phi \to \mathcal{T}$ in the following way. For every $i \in [r]$ let n_i be the number of occurrences of variable s_i in ϕ and let X_i be a copy of the variable gadget α corresponding to s_i, for every $j = 0, \ldots, 3n_i + 2$ define $f(v_j) = A$ and

- if $e(s_i) = true$, then for all $j \in \{0, .., n_i\} : g_f(v_{3j}, v_{3j+1}) = a$, $g_f(v_{3j+1}, v_{3j}) = b$, $g_f(v_{3j+1}, v_{3j+2}) = a$, $g_f(v_{3j+2}, v_{3j+1}) = c$, $g_f(v_{3j+2}, v_{3j+3}) = a$, $g_f(v_{3j+3}, v_{3j+2}) = c$,
- if $e(s_i) = false$, then for all $j \in \{0, .., n_i\} : g_f(v_{3j}, v_{3j+1}) = b$, $g_f(v_{3j+1}, v_{3j}) = a$, $g_f(v_{3j+1}, v_{3j+2}) = c$, $g_f(v_{3j+2}, v_{3j+1}) = a$, $g_f(v_{3j+2}, v_{3j+3}) = c$, $g_f(v_{3j+3}, v_{3j+2}) = a$.

It is now easy to extend the predefined function f to a locally injective homomorphism from the graph G_ϕ to $\mathcal{T}$.

If $\mathcal{T}$ is not b-positive, the proof is similar to the previous case with the only difference, that we must to prove that in any locally injective homomorphism $f : G_\phi \to \mathcal{T}$, for no copy Z of the clause gadget holds $g_f(u_1, u_0) = g_f(u_2, u_0) = g_f(_3, u_0) = b$. If such gadget Z exists, then necessarily $f(u_1) = f(u_2) = f(u_3)$ (because of parity controller Y and construction of variable gadgets). Because of Proposition 5 we have that $f(u_0) = f(u_1)$ and because of (a, b, c)-blocking gadget on vertex u_0 we have, that the simple path of length $a+b+c$ must allows decomposition $b -_k c$. But this is clearly not possible and so in every clause, there exists at least one positive and at least one negative literal. So NAE-3-SAT can be reduced to the $\mathcal{T}$-LIHom.

□

Lemma 2. *Let $\mathcal{T}$ be a Θ-graph such that $a + b \neq c$ and*

$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}} : \ a - b, (a - a)$$
$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}} : \ b - b, c - c, (a - a)$$

then $\mathcal{T}$-LIHom is NP-complete.

Lemma 3. *Let $\mathcal{T}$ be a Θ-graph such that $a + b \neq c$ and*

$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}} : \ a - b, (a - a)$$
$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}} : \ a - b, c - c, (a - a)$$

then $\mathcal{T}$-LIHom is NP-complete.

While in Lemmas 1, 2 and 3 we reduced 3-SAT, resp. NAE-3-SAT to the $\mathcal{T}$-LIHom, in the next Lemmas 4, 5, 6 and 7, the NP-complete problem of determining, if there exists a covering projection from a (simple) graph to the weight graph is reduced to the $\mathcal{T}$-LIHOM. The *weight graph* is a multigraph on vertices C and D joined by one edge and one loop at each of them. It is known, that covering projection (or simply *cover*) from a graph $G = (V, E)$ to the weight graph exists if and only if G is cubic and we can split the set of vertices V to two sets V_1 and V_2 such, that every vertex in V_1 has exactly two neighbors in V_1 and every vertex in V_2 has exactly two neighbors in V_2.

Lemma 4. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_k b, c - c, (b - b) \quad \text{or} \quad \boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_k b, c - c, (a - a)$$

then $\mathcal{T}$-LIHOM is NP-complete.

Lemma 5. *Let $\mathcal{T}$ be a Θ-graph for which $l \geq 2$. If there exists positive integer p such that*

$$\boldsymbol{(p)^{\mathcal{T}}_{a,b}:}\ a -_c a, b -_k b$$

then $\mathcal{T}$-LIHOM is NP-complete.

Lemma 6. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a - c, b -_c b, (a - a), (a - b)$$

then $\mathcal{T}$-LIHOM is NP-complete.

Lemma 7. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_k a, a -_k b, b -_k b, c - c$$

then $\mathcal{T}$-LIHOM is NP-complete.

It is well known, that we can color edges of every cubic bipartite graph with 3 colors in such a way, that all edges incident with one vertex have distinct colors, while determine, if such an edge 3-coloring exists for general cubic graphs is NP-complete problem. However, deciding if a given precoloring of a cubic bipartite graph can be extended to the proper edge 3-coloring of the whole graph is also NP-complete [3]. We prove Lemmas 8, 9 and 10 by reducing this problem to $\mathcal{T}$-LIHOM.

Lemma 8. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_c b, c - c, (b - b)$$

then $\mathcal{T}$-LIHOM is NP-complete.

Lemma 9. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a - c, b -_k b, (a - a), (a - b), (b -_c b)$$

then $\mathcal{T}$-LIHOM is NP-complete.

Lemma 10. *Let $\mathcal{T}$ be a Θ-graph where $k = 1$ and*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_c a, a -_c b, b -_c b, c - c$$

then $\mathcal{T}$-LIHOM is NP-complete.

The lemmas are main tools for proving the following two theorems. They clearly cover all Theta graphs and hence imply Theorem 1. Recall that k is the multiplicity of the shortest simple path in $\mathcal{T}$.

Theorem 2. *Let $\mathcal{T}$ be a Θ-graph where $k = 1$. Then $\mathcal{T}$-LIHOM is NP-complete.*

Theorem 3. *Let $\mathcal{T}$ be a Θ-graph where $k \geq 2$. Then $\mathcal{T}$-LIHOM is NP-complete.*

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Appendix

This Appendix contains all the missing proofs from Section 2 and Section 3.

Proposition 1. *Every simple path SP_{a_i} always allows decomposition $a_i -_c a_i$ and does not allow decomposition $a_i -_k a_i$. Similarly, for $i \neq j$ holds that $SP_{a_i+a_j}$ always allows decomposition $a_i -_k a_j$ and never allows $a_i -_c a_j$.*

Proof. Let $v_0, v_1, \ldots, v_{a_i}$ be vertices of a simple path SP_{a_i}. And let $A = u_0, u_1, \ldots, u_{a_i} = B$ be vertices of a simple path of length a_i in $\mathcal{T}$. Then we can define a locally injective homomorphism $f : SP_{a_i} \to \mathcal{T}$ as $f(v_k) = u_k$, for all $k = 0, 1, \ldots, a_i$. Such a homomorphism proves that SP_{a_i} allows the decomposition $a_i -_c a_i$.

On the other hand, assume that SP_{a_i} allows some decomposition $a_i -_k a_i$ and denote the corresponding locally injective homomorphism by $f : SP_{a_i} \to \mathcal{T}$. Without loss of generality we can suppose that $f(v_0) = u_0 = A$ and $f(v_1) = u_1$. Since for every $k \in \{1, 2, \ldots, a_i - 1\}$, vertex v_{k+1} must be mapped by f to some neighbor of $f(v_k)$ and f is locally injective, necessarily $f(v_{k+1}) = u_{k+1}$ and especially $f(v_{a_i}) = B$, a contradiction with the assumption that decomposition $a_i -_k a_i$ keeps the parity.

Next suppose that vertices of $SP_{a_i+a_j}$ are $v'_0, v'_1, \ldots, v'_{a_i+a_j}$ and let $B = w_0, w_1, \ldots, w_{a_j} = A$ be vertices of a simple path of length a_j in $\mathcal{T}$. Then we can define locally injective homomorphism $f' : SP_{a_i+a_j} \to \mathcal{T}$ as $f'(v'_k) = u_k$, for all $k = 0, 1, \ldots, a_i$ and $f'(v'_{a_i+k'}) = w_{k'}$, for all $k' = 1, 2, \ldots, a_j$. Such homomorphism proves that $SP_{a_i+a_j}$ allows the decomposition $a_i -_k a_j$.

For the rest of the proof assume that $SP_{a_i+a_j}$ allows decomposition $a_i -_c a_j$ and denote the corresponding locally injective homomorphism by $f' : SP_{a_i+a_j} \to \mathcal{T}$. Without loss of generality we can suppose that $f'(v'_0) = u_0 = A, f'(v'_1) = u_1, f'(v'_{a_i+a_j-1}) = w_1$ and $f'(v'_{a_i+a_j}) = w_0 = B$. Then necessarily $B = u_{a_i} = f'(v'_{a_i}) = w_{a_j} = A$, a contradiction with the assumption that decomposition $a_i -_c a_j$ changes the parity.

Proposition 2. *Let G be a graph and let X be a copy of $B_z^{\mathcal{T}}$ in G. Moreover, w has degree at least three. Suppose, that there exists a locally injective homomorphism $f : G \to \mathcal{T}$. Then:*

$$g_f(w, y) = z = g_f(y, w)$$

Proof. Since the only possible decomposition of SP_{a_1} is $a_1 - a_1$, necessarily for all $j \in [t_1] : g_f(y, v_j^1) = g_f(v_j^1, y) = a_i$. Since f is locally injective, it is clear that for all $i \geq 2, j \in [t_i] : g_f(v_j^i, y) \neq a_1 \neq g_f(y, v_j^i)$. Now we can continue by induction on i and in each step we show, that the only possible decomposition of SP_{a_i} is $a_i - a_i$ (because all shorter simple paths in $\mathcal{T}$ are already blocked). It means, that for very $i \in [n], j \in [t_i] : g_f(y, v_j^i) = g_f(v_j^i, y) = a_i$. $\square$

Proposition 3. *Let G be a graph and X be a copy of $(a_1^{d_1}, a_2^{d_2}, \ldots, a_n^{d_n})$-blocking gadget in G where $\deg_G(w) \geq 3$. Let $P_1, P_2, \ldots, P_k$ be the all simple paths, starting at w with without any other intersection with the blocking gadget X and with end points $u_1, u_2, \ldots, u_k$. Suppose, that there exists a locally injective homomorphism $f : G \to \mathcal{T}$. Then $k \leq \sum_{i=1}^{n} d_i$ and*

$$\forall i \in [n] : \quad |\{u_j, g_f^{P_j}(w, u_j) = a_i\}| \leq d_i.$$

Proof. In any locally injective homomorphism $f : X \to \mathcal{T}$, for every vertex $v \in V(X)$ holds that $deg_X(v) \leq deg_{\mathcal{T}}(f(v))$. Then we have $k + \sum_{i=1}^{n}(t_i - d_i) = deg_X(w) \leq deg_{\mathcal{T}}(f(w)) = \sum_{i=1}^{n} t_i$, which imply $k \leq \sum_{i=1}^{n} d_i$.

If for some $i \in [n] : |\{u_j, g_f^{P_j}(w, u_j) = a_i\}| > d_i$, then the number of simple paths beginning in w and ending in some $u \in V(G)$, such that $g_f(w, u) = a_i$ is more than $t_i - d_i + d_i = t_i$ (using Proposition 2 and the fact, that there are $t_i - d_i$ copies of $B_{a_i}^{\mathcal{T}}$ gadget on vertex w), which contradicts the locally injective constraints. □

Proposition 4. *Let $r \geq 2$ be an integer, $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ be a Θ-graph and let Z be a copy of graph $PC_a^{\mathcal{T}}(r)$ in a graph G. Let N be as in the definition of $PC_a^{\mathcal{T}}(r)$. Suppose, that there exists a locally injective homomorphism $f : G \to \mathcal{T}$, such that for all $i \in [2N-1] : f(Z(u_i)), f(Z(v_i)) \in \{A, B\}$. Then for all even $i, j \in \{N, N+1, \ldots, 2N-1\}$ hold the following:*

$$f(Z(u_i)) = f(Z(u_j)) \neq f(Z(v_j)) = f(Z(v_i)),$$

$$g_f(Z(u_i), Z(u'_{i/2})) = a = g_f(Z(v_j), Z(v'_{j/2})).$$

Proof. Because of the copies of the blocking gadgets on vertices u, u', v and v', it is clear that all simple paths of length a, resp. b and c must be mapped by f to the paths SP_a, resp. SP_b and SP_c in $\mathcal{T}$. That imply that for all even $i \in \{N, N+1, \ldots, 2N-1\} : g_f(Z(u_i), Z(u'_{i/2})) = a = g_f(Z(v_j), Z(v'_{j/2}))$.

By Proposition 1, we know that all these decompositions ($a - a$ of SP_a, $b - b$ of SP_b and $c - c$ of SP_c) change the parity and it trivially imply that $f(Z(u_i)) = f(Z(u_j)) \neq f(Z(v_j)) = f(Z(v_i))$ for all $i \in \{N, N+1, \ldots, 2N-1\}$. □

Proposition 5. *Let $a < b < c$ be positive integers, such that $a + b \neq c$. Let $\mathcal{T} = \Theta(a^k, b^l, c^m, a_4^{t_4}, \ldots, a_n^{t_n})$ be a Θ-graph and Y be the $\mathcal{T}$-clause gadget. Let $\gamma \in \{a, b\}$ and $x, y, z \in \{\gamma, c\}$.*

Then there exists a locally injective homomorphism $f : Y \to \mathcal{T}$ satisfying:

- $f(u_0) \neq f(u_1) = f(u_2) = f(u_3) \in \{A, B\}$,
- $g_f(u_1, u_0) = x,\ g_f(u_2, u_0) = y,\ g_f(u_3, u_0) = z$

if and only if at least one of the following conditions hold:

- $\{x, y, z\} = \{\gamma, c\}$

– *$x = y = z = \gamma$ and $\mathcal{T}$ is γ-positive.*

Proof. Without loss of generality assume that $\gamma = a$ (case $\gamma = b$ can be proved analogically). Suppose that $f : Y \to \mathcal{T}$ is a locally injective homomorphism, such that $f(u_0) \neq f(u_1) = f(u_2) = f(u_3) \in \{A, B\}$, $x = y = z$ and $g_f(u_1, u_0) = g_f(u_2, u_0) = g_f(u_3, u_0) = x$. If $x = a$ then $\mathcal{T}$ must a-positive by the definition of a-positive graph. Next suppose that $x = y = z = c$. Without loss of generality $g_f(u_0, u_1) = c$ (since there is an (a, b, c)-blocking gadget on vertex u_0). So we have $g_f(u_1, u_0) = g_f(u_0, u_1) = c$. But there exists no decomposition $c - c$ of a simple path of length $a + b + c$ (because $c \neq a + b$ and $c + a_i + c > a + b + c$ for every $i \in [n]$), a contradiction.

For the other implication we can assume that $\{x, y, z\} = \{a, c\}$ (if $x = y = z = a$ then $\mathcal{T}$ is a-positive by definition). It is clear that $(a + b + c)^{\mathcal{T}}_{a,b,c}$ always allows decompositions $a -_c b, a -_c c$ and $b -_c c$ and using these decompositions we can simply define a locally injective homomorphism $f : Y \to \mathcal{T}$, such that $f(u_0) \neq f(u_1) = f(u_2) = f(u_3) \in \{A, B\}$ and $g_f(u_1, u_0) = x,\ g_f(u_2, u_0) = y,\ g_f(u_3, u_0) = z$ □

Lemma 2. *Let $\mathcal{T}$ be a Θ-graph such that $a + b \neq c$ and*

$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}} : \ a - b, (a - a)$$
$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}} : \ b - b, c - c, (a - a)$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. At first suppose that $\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}}$ allows a decomposition $a - a$. Then necessarily $\boldsymbol{(b)^{\mathcal{T}}_{a,b,c}}$ also allows a decomposition $a - a$ (in fact $k \geq 2$ and b is divisible by a). Hence $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows $a - b$ (we can take a $b - b$ decomposition of SP_c and substitute one b to several a's) - contrary with the fact, that $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ does not allow a decomposition $a - b$. So necessarily $\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}} : \ a -_k b$ (a decomposition $a - b$ must keep the parity by Proposition 1).

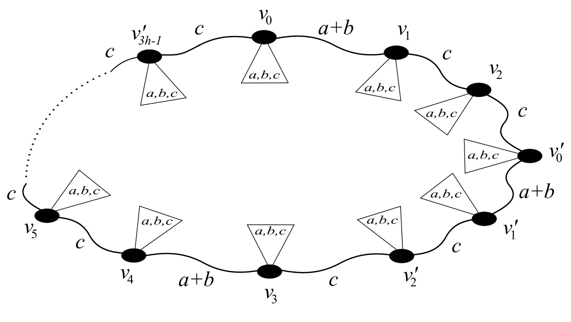

Fig. 5. Variable gadget β

Define a *variable gadget β of order h* (see Figure 5) as a graph with special vertices $v_0, v'_0, v_1, v'_1, \ldots, v'_{3h-1}$ such that, for all $i \in \{0, \ldots, h-1\}$, pairs of vertices (v_{3i}, v_{3i+1}) and (v'_{3i}, v'_{3i+1}) are connected by paths of length $a+b$ and pairs of vertices (v_{3i+1}, v_{3i+2}), (v_{3i+2}, v'_{3i}), (v'_{3i+1}, v'_{3i+2}) and (v'_{3i+2}, v_{3i+3}) are connected by paths of length c (all indices are counted modulo $3h$). For every $i \in \{0, \ldots, h-1\}$ we take six copies $X_i, Y_i, Z_i, X'_i, Y'_i, Z'_i$ of the (a,b,c)-blocking gadget and identify vertices $X_i(w), Y_i(w), Z_i(w), X'_i(w), Y'_i(w)$ and $Z'_i(w)$ with vertices $v_{3i}, v_{3i+1}, v_{3i+2}, v'_{3i}, v'_{3i+1}$ and v'_{3i+2} (in this order).

For a boolean formula $\phi = \vee_{i=1}^{p}(c_i^1 \wedge c_i^2 \wedge c_i^3)$ in conjunctive normal form with variables $s_1, s_2, \ldots, s_r$, where every clause has exactly 3 literals, define a graph G'_ϕ in the same way as in the proof of Lemma 1 with the only difference, that instead of copies of the variable gadget α, we use copies of the variable gadget β (with corresponding orders). In every copy X of the variable gadget β if $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows some decomposition $b -_c b$ then we identify all vertices v_{3i+2} and v'_{3i+2} with appropriate vertices of suitable a-parity controller (Proposition 4) in such a way, that all these vertices will be mapped to the same vertex in any locally injective homomorphism to the graph $\mathcal{T}$. If $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows only decomposition $b -_k b$, then we identify all vertices v_{3i+2} and v'_{3i+2} with appropriate vertices of suitable a-parity controller (Proposition 4) in such a way, that in any locally injective homomorphism f to the graph $\mathcal{T}$, all vertices v_{3i+2} will be mapped to the same vertex $f(v_0)$, all vertices v'_{3i+2} will be mapped to the same vertex $f(v'_0)$ and $f(v_0) \neq f(v'_0)$. We denote such graph by G_ϕ.

The reduction from the NP-complete problem (3-SAT or NAE-3-SAT) is very similar to the reduction used in the proof of Lemma 1. More precisely, if $\mathcal{T}$ is a-positive, then we reduce 3-SAT to the $\mathcal{T}$-LIHom problem, else we reduce NAE-3-SAT to $\mathcal{T}$-LIHom.

Suppose, that $\mathcal{T}$ is a-positive and there exists a locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Then we prove, that formula ϕ is satisfiable. Let us fix one copy X of the variable gadget β of order d. We know that $g_f(v_0, v_1) \in \{a, b\}$ (because of $\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b$). If $g_f(v_0, v_1) = a$ then $g_f(v_1, v_0) = b, g_f(v_1, v_2) = c = g_f(v_2, v_1)$ (because a's are blocked on the vertex v_2), $g_f(v_2, v'_0) = b = g_f(v'_0, v_2)$ and necessarily $g_f(v'_0, v'_1) = a$ and we can continue this way in the clockwise order. Then it is not hard to prove, that $\forall i \in \{0, \ldots, d-1\}$, if there exists a simple path from v_{3i} to $Z(u_0)$ for some copy Z of the clause gadget, then $g_f(v_{3i}, Z(u_0)) = c$ (the corresponding literal is false) and analogously for a simple path from v_{3i+1} to $Z(u_0)$, for which holds that $g_f(v_{3i+1}, Z(u_0)) = a$ (the corresponding literal is true). In this case we say, that the variable corresponding to X is false.

If $g_f(v_0, v_1) = b$, then we can continue in the same way in the counterclockwise order and we get $g_f(v_1, v_0) = a, g_f(v_1, v_2) = b = g_f(v_2, v_1), g_f(v_2, v'_0) = c = g_f(v'_0, v_2)$ and necessarily $g_f(v'_0, v'_1) = b$ and so on. Then it is not hard to prove, that $\forall i \in \{0 \ldots, d-1\}$, if there exists a simple path from v_{3i} to $Z(u_0)$ for some copy Z of the clause gadget, then $g_f(v_{3i}, Z(u_0)) = a$ (the corresponding literal is true) and analogically for a simple path from v_{3i+1} to $Z(u_0)$, for which holds

$g_f(v_{3i+1}, Z(u_0)) = c$ (the corresponding literal is false). In this case we say that the variable corresponding to X is true.

It is an easy exercise to show, that such an evaluation of the variables $s_1, s_2, \ldots, s_r$ is well defined and satisfy the formula ϕ (by Proposition 5).

On the other side, if $\mathcal{T}$ is a-positive and formula ϕ is satisfiable, then we show that there exists locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Suppose that $e : \{s_1, \ldots, s_r\} \to \{true, false\}$ is satisfying evaluation of variables of formula ϕ and we predefine a function $f : G_\phi \to \mathcal{T}$ in a following way. For every $i \in [r]$ let n_i be number of occurrences of the variable s_i in ϕ and let X_i be a copy of the variable gadget β of order n_i corresponding to the variable s_i, for every $j = 0, \ldots, n_i - 1$ define $f(X_i(v_{3j})) = f(X_i(v_{3j+1})) = A$ and

- if $e(s_i) = true$ then for all $j \in \{0, .., n_i - 1\}$; $g_f(v_{3j}, v_{3j+1}) = b$, $g_f(v_{3j+1}, v_{3j}) = a$, $g_f(v_{3j+1}, v_{3j+2}) = b = g_f(v_{3j+2}, v_{3j+1})$, $g_f(v_{3j+2}, v'_{3j}) = c = g_f(v'_{3j}, v_{3j+2})$, $g_f(v'_{3j}, v'_{3j+1}) = b$, $g_f(v'_{3j+1}, v'_{3j}) = a$, $g_f(v'_{3j+1}, v'_{3j+2}) = b = g_f(v'_{3j+2}, v'_{3j+1})$, $g_f(v'_{3j+2}, v_{3j+3}) = c = g_f(v_{3j+3}, v'_{3j+2})$, or
- if $e(s_i) = false$ then for all $j \in \{0, .., n_i - 1\}$; $g_f(v_{3j}, v_{3j+1}) = a$, $g_f(v_{3j+1}, v_{3j}) = b$, $g_f(v_{3j+1}, v_{3j+2}) = c = g_f(v_{3j+2}, v_{3j+1})$, $g_f(v_{3j+2}, v'_{3j}) = b = g_f(v'_{3j}, v_{3j+2})$, $g_f(v'_{3j}, v'_{3j+1}) = a$, $g_f(v'_{3j+1}, v'_{3j}) = b$, $g_f(v'_{3j+1}, v'_{3j+2}) = c = g_f(v'_{3j+2}, v'_{3j+1})$, $g_f(v'_{3j+2}, v_{3j+3}) = b = g_f(v_{3j+3}, v'_{3j+2})$

Now it is an easy exercise (with respect to the definition of the graph G_ϕ), that such predefined function f can be extended to a proper locally injective homomorphism from G_ϕ to $\mathcal{T}$.

If $\mathcal{T}$ is not a-positive, the proof is very similar to the previous case with the only difference, that in any locally injective homomorphism $f : G_\phi \to \mathcal{T}$ for every copy Z of the clause gadget at least one of the values $g_f(Z(u_1), Z(u_0))$, $g_f(Z(u_2), Z(u_0))$ and $g_f(Z(u_3), Z(u_0))$ must be c, what correspond to the situation, that in every clause there exists at least one literal evaluated by false. Hence we can reduce NAE-3-SAT to the $\mathcal{T}$-LIHom. □

Lemma 3. *Let $\mathcal{T}$ be a Θ-graph such that $a + b \neq c$ and*

$$\boldsymbol{(a+b)}^{\mathcal{T}}_{\boldsymbol{a,b,c}} : \ a - b, (a - a)$$
$$\boldsymbol{(c)}^{\mathcal{T}}_{\boldsymbol{a,b,c}} : \ a - b, c - c, (a - a)$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. The proof is very similar to the proof of Lemma 2 with too differences. Instead of asking if $\boldsymbol{(c)}^{\mathcal{T}}_{\boldsymbol{a,b,c}}$ allows a decomposition $b -_c b$, we ask if $\boldsymbol{(c)}^{\mathcal{T}}_{\boldsymbol{a,b,c}}$ allows a decomposition $a -_c b$ and instead of asking, if $\mathcal{T}$ is a-positive, we ask if $\mathcal{T}$ is b-positive. Then we define a graph G_ϕ in the same way as in the proof of Lemma 2, where $\phi = \vee_{i=1}^{p}(c_i^1 \wedge c_i^2 \wedge c_i^3)$ is a boolean formula in conjunctive normal form (CNF) with variables $s_1, s_2, \ldots, s_r$, where every clause has exactly 3 literals.

Now suppose that $\mathcal{T}$ is b-positive and let ϕ be a formula in CNF. We prove that ϕ is satisfiable if and only if there exists a locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Suppose that such mapping f exists. Take one copy X of the the variable gadget β of order d. We know, that $g_f(v_1, v_0) \in \{a, b\}$ (because $\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}}$ **:** $a-b, (a-a)$). If $g_f(v_1, v_0) = a$, then $g_f(v_1, v_2) = c = g_f(v_2, v_1)$ (because all a's are blocked on the vertex v_2), $g_f(v_2, v_0') = b$, $g_f(v_0', v_2) = a$ and necessary $g_f(v_0', v_1') = b$ and we can continue in this way in the clockwise order (and finally $g_f(v_0, v_1) = b$). Then it is not hard to prove, that $\forall i \in \{0, \ldots, d-1\}$, if there exists a simple path from v_{3i} to $Z(u_0)$ for some copy Z of the clause gadget, then $g_f(v_{3i}, Z(u_0)) = c$ (the corresponding literal is false) and analogically for a simple path from v_{3i+1} to $Z(u_0)$, for which holds that $g_f(v_{3i+1}, Z(u_0)) = b$ (the corresponding literal is true). In this case we say that the variable corresponding to X is false.

If $g_f(v_1, v_0) = b$ then $g_f(v_0, v_1) = a$ and we can continue in the same way as in the previous paragraph in the counterclockwise order and we get that $\forall i \in \{0, \ldots, d-1\}$, if there exists a simple path from v_{3i} to $Z(u_0)$ for some copy Z of the clause gadget, then $g_f(v_{3i}, Z(u_0)) = b$ (the corresponding literal is true) and analogically for a simple path from v_{3i+1} to $Z(u_0)$, for which holds that $g_f(v_{3i+1}, Z(u_0)) = c$ (the corresponding literal is false). In this case we say that the variable corresponding to X is true.

It is easy an exercise to show, that such evaluation of the variables $s_1, \ldots, s_r$ is a satisfying evaluation of formula ϕ.

On the other side, if $\mathcal{T}$ is b-positive and the formula ϕ is satisfiable, then we show, that there exists a locally injective homomorphism $f : G_\phi \to \mathcal{T}$. Suppose that $e : \{s_1, \ldots, s_r\} \to \{true, false\}$ is a satisfying evaluation of the variables of formula ϕ and we predefine a function $f : G_\phi \to \mathcal{T}$ in the following way. For every $i \in \{1, \ldots, r\}$, let n_i be the number of occurrences of the variable s_i in ϕ and let X_i be the copy of the variable gadget β of order n_i corresponding to the variable s_i, for every $j = 0, \ldots, n_i - 1$, define $f(X_i(v_{3j})) = f(X_i(v_{3j+1})) = A$ and

- if $e(s_i) = true$, then for all $j \in \{0, .., n_i - 1\}$; $g_f(v_{3j}, v_{3j+1}) = a$, $g_f(v_{3j+1}, v_{3j}) = b$, $g_f(v_{3j+1}, v_{3j+2}) = a$, $g_f(v_{3j+2}, v_{3j+1}) = b$, $g_f(v_{3j+2}, v_{3j}') = c$, $g_f(v_{3j}', v_{3j+2}) = c$, $g_f(v_{3j}', v_{3j+1}') = a$, $g_f(v_{3j+1}', v_{3j}') = b$, $g_f(v_{3j+1}', v_{3j+2}') = a$, $g_f(v_{3j+2}', v_{3j+1}') = b$, $g_f(v_{3j+2}', v_{3j+3}) = c$, $g_f(v_{3j+3}, v_{3j+2}') = c$,
- if $e(s_i) = false$, then for all $j \in \{0, .., n_i - 1\}$; $g_f(v_{3j}, v_{3j+1}) = b$, $g_f(v_{3j+1}, v_{3j}) = a$, $g_f(v_{3j+1}, v_{3j+2}) = c$, $g_f(v_{3j+2}, v_{3j+1}) = c$, $g_f(v_{3j+2}, v_{3j}') = b$, $g_f(v_{3j}', v_{3j+2}) = a$, $g_f(v_{3j}', v_{3j+1}') = b$, $g_f(v_{3j+1}', v_{3j}') = a$, $g_f(v_{3j+1}', v_{3j+2}') = c$, $g_f(v_{3j+2}', v_{3j+1}') = c$, $g_f(v_{3j+2}', v_{3j+3}) = b$, $g_f(v_{3j+3}, v_{3j+2}') = a$.

Now it is an easy exercise (with respect to the definition of the graph G_ϕ), that the predefined function f can be extended to a locally injective homomorphism of the graph G_ϕ to $\mathcal{T}$.

If $\mathcal{T}$ is not b-positive, then the only difference is that for every locally injective homomorphism $f : G_\phi \to \Theta$ and for every copy Z of the clause gadget

$g_f(Z(u_1), Z(u_0)) \neq b, g_f(Z(u_2), Z(u_0)) \neq b$ or $g_f(Z(u_3), Z(u_0)) \neq b$, what corresponds to the situation, that in every clause exists at least one literal evaluated by false, and so we can reduce NAE-3-SAT to the Θ-LIHom. □

Lemma 4. *Let $\mathcal{T}$ be a Θ-graph where*

$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c - c, (b - b)$ *or* $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c - c, (a - a)$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. We prove only the case, where $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c - c, (b - b)$. Since the only fact, we will use about a, b and c is that $a \neq b \neq c \neq a$, we can easily change this proof to a proof of the other the case where $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c - c, (a - a)$.

Let $H = (V_H, E_H)$ be a (simple) cubic graph with vertices $V_H = \{u_1, u_2, \dots, u_h\}$. For this graph we create a graph G_H in the following way: we take h copies $X_1, X_2, \dots, X_h$ of the (a, b, c)-blocking gadget and for every edge $u_i u_j \in E_H$ of H, we add a path of length c between vertices $X_i(w)$ and $X_j(w)$. Denote the vertices $X_1(w), X_2(w), \dots, X_h(w)$ simply by $v_1, v_2, \dots, v_h$.

To prove Lemma 4, it is sufficient to show, that the graph H covers the weight graph if and only if there exists a locally injective homomorphism from G_H to $\mathcal{T}$.

Suppose that $f : G_H \to \mathcal{T}$ is a locally injective homomorphism. For every $i \in [h]$ let u_{i1}, u_{i2} and u_{i3} be the distinct neighbors of the vertex u_i in the graph H. Since there is a copy of the (a, b, c)-blocking gadget on every vertex v_i, it is clear that $\{g_f(v_i, v_{i1}), g_f(v_i, v_{i2}), g_f(v_i, v_{i3})\} = \{a, b, c\}$. Since $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c - c, (b - b)$, an easy counting shows, that for every i, j such that $u_i u_j \in E_H$, $g_f(v_i, v_j) \neq b$ or $g_f(v_j, v_i) \neq b$ (decomposition $b - b$ can not occur on any SP_c).

Now we can define a partition of vertices of the graph H in the following way $V_1 = \{u \in V_H, f(u) = A\}$ and $V_2 = \{u \in V_H, f(u) = B\}$. Since every $c - c$ decomposition of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ changes the parity (Lemma 1), it is clear, that using such partition of V, we can easily construct a covering projection of H to the weight graph.

On the other side suppose, that there exists a covering projection from H to the weight graph. Fix one such cover and let V_1 and V_2 be the corresponding partitions of V_H and predefine a function $f : G_H \to \mathcal{T}$ in the following way:

- if $u \in V_1$ then $f(u) = A$
- if $u \in V_2$ then $f(u) = B$
- if $u_i \in V_1, u_j \in V_2; u_i u_j \in E_H$, then $g_f(v_i, v_j) = c = g_f(v_j, v_i)$

It is an easy exercise to show, that such predefined function f can be extended to a locally injective homomorphism from G_H to $\mathcal{T}$. □

Lemma 5. *Let $\mathcal{T}$ be a Θ-graph for which $l \geq 2$. If there exists a positive integer p such that*

$$\boldsymbol{(p)^{\mathcal{T}}_{a,b}}\ \textbf{:}\ a -_c a, b -_k b$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. We prove this Lemma similarly as Lemma 4 (reducing the cover to the weight graph to $\mathcal{T}$-LIHom). Let $H = (V_H, E_H)$ be a (simple) cubic graph with vertices $V_H = \{u_1, u_2, \ldots, u_h\}$. We take h copies $X_1, X_2, \ldots, X_h$ of the (a, b, b)-blocking gadget and for every edge $u_iu_j \in E_H$ of H, we add a path of length p between the vertices $X_i(w)$ and $X_j(w)$ and denote such graph by G_H.

Now it is an easy exercise (similar to the one in the proof of Lemma 4) to show that H covers the weight graph if and only if there exists a locally injective homomorphism from G_H to $\mathcal{T}$. □

Lemma 6. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}\textbf{:}\ \ a - c, b -_c b, (a - a), (a - b)$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. We prove this Lemma similarly as Lemma 4 (reducing the cover to the weight graph to $\mathcal{T}$-LIHom). Let $H = (V_H, E_H)$ be a (simple) cubic graph with vertices $V_H = \{u_1, u_2, \ldots, u_h\}$. Then we take h copies $X_1, X_2, \ldots, X_h$ of the (a, b, c)-blocking gadget and for every edge $u_iu_j \in E_H$ of H, we add a path of length $a + c$ between the vertices $X_i(w)$ and $X_j(w)$ and denote such graph by G_H.

Similarly as in Lemma 4, but using the fact, that neither decomposition $a - a$ nor $a - b$ of $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ can occur in any locally injective homomorphism from G_H to $\mathcal{T}$, it is not hard to prove, that H covers the weight graph if and only if there exists a locally injective homomorphism from G_H to $\mathcal{T}$. □

Lemma 7. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}\textbf{:}\ \ a -_k a, a -_k b, b -_k b, c - c$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. We prove this Lemma similarly as Lemma 4 (reducing the cover to the weight graph to $\mathcal{T}$-LIHom). Let $H = (V_H, E_H)$ be a (simple) cubic graph with vertices $V_H = \{u_1, u_2, \ldots, u_h\}$. Take h copies $X_1, X_2, \ldots, X_h$ of the (a, b, c)-blocking gadget and for every edge $u_iu_j \in E_H$ of H, we add a path of length c between the vertices $X_i(w)$ and $X_j(w)$. Denote such graph by G_H.

Similarly as in Lemma 4, it is not hard to prove, that in every locally injective homomorphism $f : G_H \to \mathcal{T}$, every vertex $u \in V_H$ has exactly two neighbors mapped to the vertex $f(u) \in \mathcal{T}$ (by mapping f). So if there exists a locally injective homomorphism from G_H to $\mathcal{T}$, then H covers the weight graph. On the other side if H covers the weight graph, then we can easily find a locally injective homomorphism from G_H to the $\mathcal{T}$ (similarly as in the proof of Lemma 4). □

Lemma 8. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}\textbf{:}\ \ a -_c b, c - c, (b - b)$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. Let $H = (V_H, E_H)$ be a (simple) cubic bipartite graph with parts $V_1 = \{u_1, u_2, \ldots, u_h\}$ and $V_2 = \{u'_1, u'_2, \ldots, u'_h\}$. Then we take $2h$ copies $X_1, X'_1, X_2, X'_2, \ldots, X_h, X'_h$ of the (a, b, c)-blocking gadget and denote the vertices $X_1(w), X'_1(w), X_2(w), X'_2(w), \ldots, X'_h(w)$ simply by $v_1, v'_1, v_2, v'_2, \ldots, v'_h$. Let $E_1, E_2, E_3 \subset E_H$ be disjoint sets of precolored edges (without loss of generality, we can assume, that we use colors $1, 2$ and 3 and that E_i is the set of edges precolored by color i), then:

- $\forall u_i u'_j \in E_1$: take a copy X of the gadget $B_a^{\mathcal{T}}$ and a copy Y of the gadget $B_b^{\mathcal{T}}$ and identify v_i with $X(w)$ and v'_j with $Y(w)$
- $\forall u_i u'_j \in E_2$: take a copy X of the gadget $B_b^{\mathcal{T}}$ and a copy Y of the gadget $B_a^{\mathcal{T}}$ and identify v_i with $X(w)$ and v'_j with $Y(w)$
- $\forall u_i u'_j \in E_3$: take two copies X, Y of the gadget $B_c^{\mathcal{T}}$ and identify v_i with $X(w)$ and v'_j with $Y(w)$
- $\forall u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3)$: add a path of length c between vertices v_i and v'_j

Denote this graph by G^H. To prove Lemma 8, it is enough to show, that the edge precoloring of the graph H with 3 colors can be extended to a proper 3-edge coloring of the graph H if and only if there exists a locally injective homomorphism from G^H to $\mathcal{T}$.

Suppose that $f : G^H \to \mathcal{T}$ is a locally injective homomorphism. Then we extend the precoloring in the following way:

- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3), g_f(v_i, v'_j) = a$ and $g_f(v'_j, v_i) = b$, then color the edge $u_i u'_j$ by color 1
- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3), g_f(v_i, v'_j) = b$ and $g_f(v'_j, v_i) = a$, then color the edge $u_i u'_j$ by color 2
- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3), g_f(v_i, v'_j) = c$ and $g_f(v'_j, v_i) = c$, then color the edge $u_i u'_j$ by color 3

From the construction of graph G^H and allowed decompositions of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ (recall that since there is a copy of the (a, b, c)-blocking gadget on every vertex v_i and v'_i, the only occurred decompositions of SP_c can be $a - b$ and $c - c$) it is an easy exercise to show, that every edge in the graph H has exactly one color and all edges incident to one vertex in the graph H have different colors.

Now suppose that a 3-edge precoloring of the cubic bipartite graph H can be extended to a proper 3-edge coloring with colors $\{1, 2, 3\}$ and fix such extension function $col : E_H \setminus (E_1 \cup E_2 \cup E_3) \to \{1, 2, 3\}$. We fix one SP_a, one SP_b and one SP_c path in $\mathcal{T}$ and predefine a function $f : G^H \to \mathcal{T}$ in the following way:

- $\forall i \in \{1, 2, \ldots, h\}$: $f(v_i) = A$ and $f(v'_i) = B$
- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3)$ and $col(u_i u'_j) = 1$, then $g_f(v_i, v'_j) = a$ and $g_f(v'_j, v_i) = b$
- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3)$ and $col(u_i u'_j) = 2$, then $g_f(v_i, v'_j) = b$ and $g_f(v'_j, v_i) = a$
- if $u_i u'_j \in E_H \setminus (E_1 \cup E_2 \cup E_3)$ and $col(u_i u'_j) = 3$, then $g_f(v_i, v'_j) = c$ and $g_f(v'_j, v_i) = c$

It is again an easy exercise to show, that f can be extended to a locally injective homomorphism from G^H to $\mathcal{T}$. $\square$

Lemma 9. *Let $\mathcal{T}$ be a Θ-graph where*

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a-c, b-_k b, (a-a), (a-b), (b-_c b)$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a-_k c, b-_k b$ then we can prove Lemma 9 in a similar way as Lemma 8, with the difference that instead of simple paths SP_a, SP_b, SP_c, we will use simple paths SP_a, SP_c and SP_b (recall that in the proof of Lemma 8 the only fact about a, b and c we used, was that $a \neq b \neq c \neq a$) and we have to be a little bit more careful, when we predefine a function $f : G^H \to \mathcal{T}$, because now we need for all $i \in [h]$: $f(v_i) = A = f(v_i')$ (all other steps are exactly the same).

If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ allows decompositions $b-_c b, a-a$ or $a-b$, then necessarily $k \geq 2$ or $l \geq 2$ (since these decompositions can only use SP_a and SP_b paths and there exists a decomposition $b-_k b$). Take the graph G^H from the previous case (when $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a-_k c, b-_k b$), substitute all copies of (a,b,c)-blocking gadget by copies of the (a^2,b,c)-blocking gadget (if $k \geq 2$) or by copies of the (a,b^2,c)-blocking gadget (if $k=1$) and use the appropriate a-parity controller (if $k \geq 2$) or b-parity controller (if $k=1$) to create a graph $\overline{G^H}$ such, that in every locally injective homomorphism from $f : \overline{G^H} \to \mathcal{T}$, $\forall i,j : f(v_i) = f(v_j')$.

Now it is an easy exercise to show, that if $f : \overline{G^H} \to \mathcal{T}$ is a locally injective homomorphism, then decompositions of all Θ-paths between the vertices v_i and v_j' must be $a-_k c$ or $b-_k b$, and these decompositions define a proper 3-edge coloring of the graph H (similarly as in the proof of Lemma 8). On the other side, if a partial 3-edge precoloring of the graph H can be extended to a proper 3-edge coloring, then it is not hard to find a locally injective homomorphism from $\overline{G^H}$ to $\mathcal{T}$ (again analogically as in the proof of Lemma 8). $\square$

Lemma 10. *Let $\mathcal{T}$ be a Θ-graph where $k=1$ and*

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a-_c a, a-_c b, b-_c b, c-c$$

then $\mathcal{T}$-LIHom is NP-complete.

Proof. Since $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a-_c a, a-_c b, b-_c b, c-c$, it is easy to show that $l \geq 2$ (because if $l=1$, then there cannot be two consecutive paths neither of length a nor b in any decomposition of SP_c). Then it is clear that $\boldsymbol{(c+b)^{\mathcal{T}}_{a,b,c}}$ allows $a-_k a$ (just take $a-_c a$ decomposition of SP_c and insert one path of length b to this decomposition) and $c-_k b$. On the other side $\boldsymbol{(c+b)^{\mathcal{T}}_{a,b,c}}$ does not allow $c-c$ and $c-a$ (because $c-c$ is too short, resp. too long and if it allows decomposition $c-a$, then $\boldsymbol{(b)^{\mathcal{T}}_{a,b,c}}$ must allow a decomposition $a-a$, which is not possible since $k=1$) and we have

$$\boldsymbol{(b+c)^{\mathcal{T}}_{a,b,c}:}\ a-_k a, b-_k c, (a-_c a), (a-b), (b-b)$$

And now we can prove this Lemma in the same way as Lemma 9. The only difference is that instead of simple paths SP_a, SP_b and SP_c, we use simple paths SP_b, SP_a, and SP_c (recall, that in the proof of Lemma 9 we used only fact, that $a \neq b \neq c \neq a$). □

Theorem 2. *Let $\mathcal{T}$ be a Θ-graph where $k = 1$. Then $\mathcal{T}$-LIHom is NP-complete.*

Proof. If $c = a + b$ then clearly $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k b, c -_c c, (a - a)$ (Proposition 1) and using Lemma 4 we have, that $\mathcal{T}$-LIHom is NP-complete. For the rest of the proof, we assume, that $c \neq a + b$.

By analysis of possible decompositions it is not hard to prove, that:

$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}} \textbf{ :} \ a -_k b, (a - a)$$

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}} \textbf{ :} \ a -_k c, (a - a), (a - b), (b - b)$$

If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ does not allow any decomposition $a - b$, then $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k c, (a - a), (b - b)$ and $\mathcal{T}$-LIHom is NP-complete by Lemma 1. So we can assume, that $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ always allows some decomposition $a - b$.

If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ does not allow any decomposition $a-a$, then $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ **:** $a-_k c, a - b, (b-b)$. It can be shown, that in this case $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows decompositions $c-c$ and $b-b$ (because in every decomposition $a-b$ of SP_{a+c}, the only convenient simple path of $\mathcal{T}$ following the former SP_a is SP_b). On the other side, $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ neither allows decompositions $a - c, b - c$ (because of its total length) nor $a - b$ (because $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ does not allow $a-a$), so we have $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $b-b, c-c, (a-a)$ and $\mathcal{T}$-LIHom is NP-complete by Lemma 2.

Now we can assume, that $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ **:** $a -_k c, a - a, a - b, (b - b)$ and by a similar argument, as in the previous paragraph, we have:

$$\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}} \textbf{ :} \ a - b, b - b, c - c, (a - a)$$

We continue by distinguishing two cases depending on the existence of decomposition $a - a$ of $(c)^{\mathcal{T}}_{a,b,c}$:

1. If $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a - b, b - b, c - c$, then we know that there exists $a - b$ decomposition of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$, but there is no $a - a$ decomposition. It means (using $c \neq a + b$), that there is only one decomposition $a - b$ of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ (and necessarily $c = a + b + b + \cdots + b$) and this decomposition either changes or keeps the parity. Using the fact, that every $c - c$ decomposition of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ changes the parity (Proposition 1), we have two possibilities:
   (a) If decomposition $a - b$ of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ changes the parity, then $\mathcal{T}$-LIHom is NP-complete by Lemma 8.
   (b) If decomposition $a - b$ of $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ keeps the parity, then $\mathcal{T}$-LIHom is NP-complete by Lemma 4.
2. If $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ **:** $a - a, a - b, b - b, c - c$, then we have the following three possibilities:
   (a) If all $a - a, a - b, b - b$ decompositions keep the parity, then $\mathcal{T}$-LIHom is NP-complete by Lemma 7.

(b) If all $a-a, a-b, b-b$ decompositions change the parity, then $\mathcal{T}$-LIHom is NP-complete by Lemma 10.

(c) If there exists (at least one) parity changing $a-a, a-b$ or $b-b$ decomposition and (at least one) parity keeping $a-a, a-b$ or $b-b$ decomposition, then there must exist numbers $i, j, i', j' \in \mathbb{N}_0$ such that $c = ia + jb = i'a + j'b; i > i', i \leq j+1, i' \leq j'+1$ and $i+j$ have different parity than $i'+j'$ (these couples i, j, resp. i', j' correspond to the appropriate decompositions, because these decompositions can use only simple paths SP_a and SP_b) and necessarily $l \geq 2$ (because $k = 1$). From all such quadruples i, j, i', j', we choose the one with minimal $i - i'$ and define $p = (i-i')a + (i-i'-1)b = (j'-j+i-i'-1)b$. It is clear, that $(i-i') + (i-i'-1)$ is odd and $j'-j+i-i'-1$ is even, so $\boldsymbol{(p)^{\mathcal{T}}_{a,b}}$ allows decompositions $a -_c a$ and $b -_k b$. If $\boldsymbol{(p)^{\mathcal{T}}_{a,b}}$ allows one of the decompositions $a -_k a$, $b -_c b$ or $a - b$, it means, that there exists $i'', j'' \in \mathbb{N}_0$ such that $p = i''a + j''b, i'' \leq j''+1, 0 < i'' < i - i'$. But in this case, we can find a quadruple i_0, j_0, i'_0, j'_0, with smaller $i_0 - i'_0$ - which is a contradiction to our choice of the quadruple i, j, i', j'. So we have:
$$\boldsymbol{(p)^{\mathcal{T}}_{a,b}:}\ a -_c a, b -_k b$$
and $\mathcal{T}$-LIHom is NP-complete by Lemma 5.

$\square$

Theorem 3. *Let $\mathcal{T}$ be a Θ-graph where $k \geq 2$. Then $\mathcal{T}$-LIHom is NP-complete.*

Proof. If $c = a + b$, then similarly as in the previous proof of Theorem 2, $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}\ a -_k b, c -_c c, (a-a)$ and by Lemma 4 we get, that $\mathcal{T}$-LIHom is NP-complete. For the rest of the proof, we assume, that $c \neq a + b$.

By a case analysis, it is not hard to prove, that:
$$\boldsymbol{(a+b)^{\mathcal{T}}_{a,b,c}:}\ a -_k b, (a-a)$$
$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a -_k c, (a-a), (a-b), (b-b)$$

If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ does not allow any decomposition $a-b$, then $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}$ $a -_k c, (a-a), (b-b)$ and $\mathcal{T}$-LIHom is NP-complete by Lemma 1. So we can assume, that $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ always allows some decomposition $a-b$.

If $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ does not allow any decomposition $b-b$, then $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ a -_k c, a-b, (a-a)$. It can be shown, that in this case $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ neither allows decompositions $a-c, b-c$ nor $b-b$ (because if $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows $b-b$, then $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ allows $b-b$ as well - contrary) and $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows decompositions $c -_c c$ and $a-b$ (because in decomposition $a-b$ of $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ the only convenient simple path of $\mathcal{T}$ following the former SP_a is SP_a, because in the other case $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}}$ allows decomposition $b-b$). So we have $\boldsymbol{(c)^{\mathcal{T}}_{a,b,c}:}$ $a-b, c-c, (a-a)$ and $\mathcal{T}$-LIHom is NP-complete by Lemma 3.

Now we can assume, that $\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}}$ allows some decomposition $b-b$. If there exists such $b-b$ decomposition which keeps the parity, then $\mathcal{T}$-LIHom is NP-complete by Lemma 9, else we have

$$\boldsymbol{(a+c)^{\mathcal{T}}_{a,b,c}:}\ \ a-b, a-_c, b-_c b, (a-a)$$

and then $\mathcal{T}$-LIHOM is NP-complete by Lemma 6. $\square$