INTERIOR-POINT METHODS (11.1)

(BOYD, VANDENBERGHE - CONVEX OPTIMIZATION)

$$(P) \begin{cases} \text{MIN } f(x) \\ \text{S.T. } g_i(x) \leq 0 \quad i=1\dots m \end{cases}$$

f, g CONVEX TWICE CONT. DIFFERENTIABLE

x^* OPT. SOLUTION EXISTS

(P) SUPERCONSISTENT (STRICTLY FEASIBLE)

$\exists x \quad g_i(x) < 0 \quad i = 1 \dots m$

KKT THEOREM $\Leftrightarrow$ $\lambda^* \in \mathbb{R}^m$ ST

1) $\lambda_i^* \geq 0 \quad \forall i$

2) $\lambda_i^* g_i(x^*) = 0 \quad \forall i$

3) $\nabla f(x^*) + \sum \lambda_i^* \nabla g_i(x^*) = 0$

COVERS

LP, QP, QCQP, GP

(SDP IF ONE TRIES)

(11.2) LOGARITHMIC BARRIER

IDEA:

CHANGE (P) TO

$$\min \; f(x) + \sum_{i=1}^{m} I(g_i(x))$$

WHERE

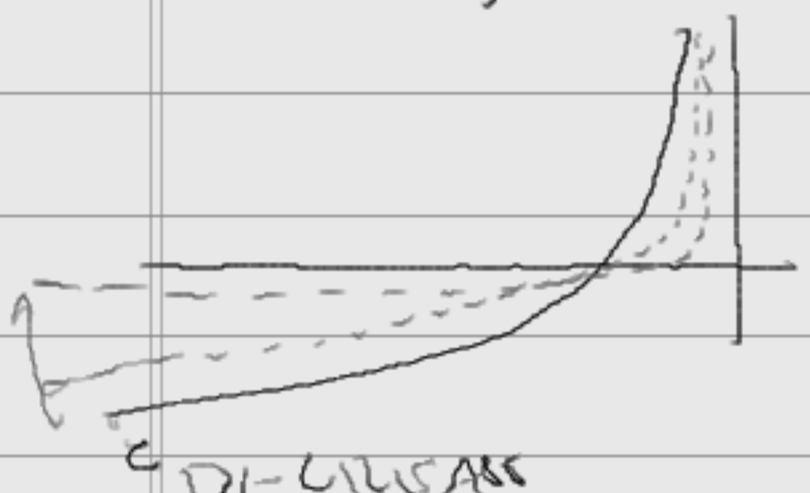
$$I(u) = \begin{cases} 0 & \text{IF } u \le 0 \\ +\infty & \text{IF } u > 0 \end{cases}$$

INDICATOR FUNCTION FOR $\mathbb{R}_-$

(LIKE "TOTAL" PENALTY FUNCTION

APPROXIMATION:

$I(u) \simeq -c \cdot \log(-u) \qquad c > 0$



APPROXIMATION IMPROVES AS $c \to 0$ DIFFICULT TO SOLVE

$t \to +\infty$ IS BETTER APPROX (WILD HESS.)

$$\Rightarrow \min \; f(x) - \frac{1}{t} \sum \log(-g_i(x))$$

→ UNCONSTRAINED, SMOOTH APPROXIMATION OF

DEF: LOGARITHMIC BARRIER FUNCTION

$$\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$$

$\forall x$ S.T. $g_i(x) < 0$ ← INTERIOR OF FEASIBLE SOLUTIONS

$\therefore$ $\phi(x)$ IS CONVEX SINCE $-\log$ IS CONVEX STRICTLY DECREASING & g_i CONVEX

NOTE:

$$\nabla\phi(x) = \sum \frac{1}{-g_i(x)} \nabla g_i(x)$$

$$H\phi(x) = \sum \frac{1}{g_i(x)^2} \nabla g_i(x) \nabla g_i(x)^T +$$

$$\overset{\shortparallel}{\nabla^2\phi(x)} \quad + \sum \frac{1}{-g_i(x)} H g_i(x)$$

FOR $t > 0$ DEFINE $x^*(t)$ AS SOLUTION TO

$$\min \; t f(x) + \phi(x)$$

{SUPPOSE ALWAYS UNIQUE}

CENTRAL PATH: $\{x^*(t) \mid t > 0\}$

SO FOR $x^*(t)$:

$$0 = t \nabla f(x^*(t)) + \nabla \phi(x^*(t))$$

$$0 = t \nabla f(x^*(t)) + \sum_{i=1}^{m} \frac{1}{-g_i(x^*(t))} \nabla g_i(x^*(t))$$

EX FOR LINEAR PROGRAMMING

$$(LP) \begin{cases} \min & c^T x \\ \text{s.t.} & Ax \leq b \end{cases} \qquad \min \; t c^T x + \phi(x)$$

$A \in \mathbb{R}^{m \times n}$

$$\phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

WHERE a_i^T IS i-ROW OF A

$$\nabla \phi(x) = \sum_{i}^{m} \frac{1}{b_i - a_i^T x} a_i$$

$$\left[H\phi(x) = \sum_{i=1}^{m} \frac{1}{(b_i - a_i^T x)^2} a_i a_i^T \right.$$

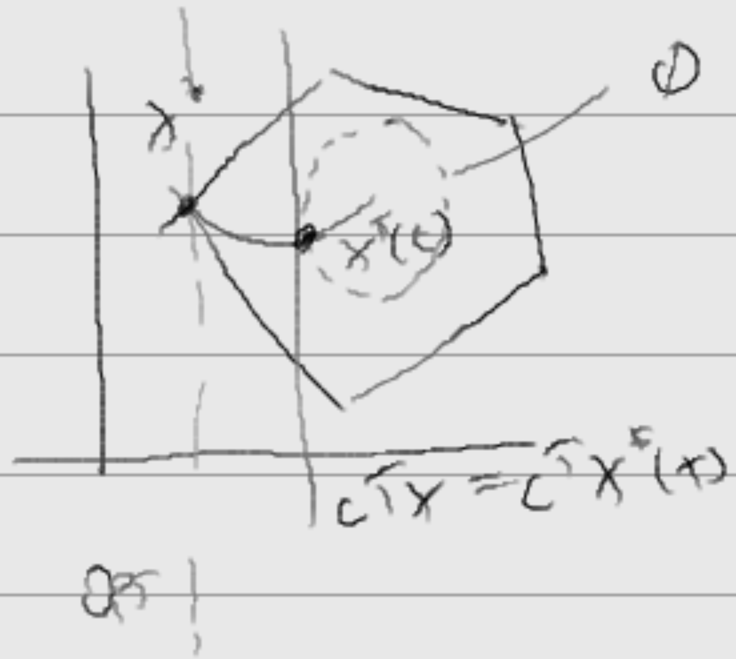
JUSTIFIES THAT JUST $\nabla = 0$ IS FINE $\Big]$

$$0 = t c + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x} a_i$$

$$t c = -\nabla \phi(x)$$

PARALLEL TO C

BONUS: $\nabla \phi(x^*(t))$ IS $\perp$ TO LEVEL CURVE OF $\phi(x) = \phi(x^*) \Rightarrow$ $c^T x = c^T x^*(t)$ IS A TANGENT OF LEVEL CURVES FOR ϕ



OPT!

DUAL POINTS FROM CENTRAL PATH

$x^*(t)$ GIVES POIN IN DUAL:

RECALL:

$$(P) \begin{cases} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \end{cases}$$

$$(DP) \begin{cases} \max & h(\lambda) = \inf_x L(x, \lambda) = f(x) + \sum \lambda_i g_i \\ & \lambda \geq 0 \end{cases}$$

λ FEASIBLE IF $h(\lambda) \neq -\infty$

GRADIENT CONDITION

$$\lambda_i = \frac{-1}{t\, g_i(x^*(t))}$$

POSITIVITY

1, $\lambda > 0$ SINCE ≤ 0

2, $0 = t \nabla f(x^*(t)) + \sum_{i=1}^{m} \frac{1}{-g_i(x^*(t))} \nabla g_i(x^*(t))$

$$0 = \nabla f(x^*(t)) + \sum \lambda_i \nabla g_i(x^*(t))$$

HENCE $x^*(t)$ MINIMIZES

$$L(x, \lambda) = f(x) + \sum \lambda_i g_i(x)$$

$\Rightarrow \lambda$ IS FEASIBLE. — CALL IT $\lambda^*(t)$

$$h(\lambda^*(t)) = f(x^*(t)) + \sum_{i=1}^{m} \lambda^*(t)_i\, g_i(x^*(t))$$

$$= f(x^*(t)) - m/t$$

$\Rightarrow$ DUALITY GAP m/t AS

$$f(x^*(t)) - h(\lambda^*(t)) \leq m/t$$

AS $t \to \infty$ BY $x^*(t) \to x^*$!

USING KKT CONDITIONS

X IS OPT SOLUTION $X^*(t)$ IFF $\exists \lambda$

1) $g_i(x) \leq 0 \quad \forall i$

2) $\lambda \geq 0$

3) $\nabla f(x) + \sum \lambda_i \nabla g_i(x) = 0$

4) $-\lambda_i f_i(x) = 1/t$

RECALL IN KKT $\lambda_i f_i(x) = 0$

t MAKES DEFORMATION OF KKT CONDITIONS