Recall \textit{k-factor} of a graph \( G \) is a spanning \( k \)-regular subgraph

A graph \( G \) is \textit{k-factorable} if its edges can be decomposed into \( k \)-factors.

**Theorem 8.15** Every \( r \)-regular bipartite graph, \( r \geq 1 \), is 1-factorable.

1: Prove Theorem 8.15 by induction on \( r \).

**Theorem 8.14** \( K_{2k} \) is 1-factorable.

2: Find 1-factorization of \( K_6 \). Use the following 1-factor \( F_1 \) to find the 1-factorization.

![Diagram of \( K_6 \) and \( F_1 \)]

3: Generalize the 1-factorization of \( K_6 \) to factorization of any \( K_{2k} \) for \( k \geq 3 \).

**Theorem 8.16** A graph \( G \) is 2-factorable if and only if \( G \) is \( r \)-regular for some positive even integer \( r \).

4: Show that if \( G \) is 2-factorable, then \( G \) is \( r \)-regular, where \( r \) is some even integer.

Now we plan to prove the other direction of Theorem 8.16. Let \( G \) be a \( 2k \)-regular graph on \( n \) vertices for some integer \( k \). We want to show \( G \) has a 2-factor and then use induction like in 8.15.

Notice that \( G \) is Eulerian so it has an Eulerian trail \( T \). Remember, edges do not repeat but vertices may repeat.

Let \( G \) have vertices \( v_1, \ldots, v_n \). Create a bipartite graph \( H \), whose vertex set \( U = \{u_1, \ldots, u_n\} \) and \( W = \{w_1, \ldots, w_n\} \) and \( u_i, w_j \) is an edge if \( v_i v_j \) appear in this order next to each other on the trail \( T \).

from G. Chartrand and P. Zhang. “A First Course in Graph Theory”
5: Suppose \( G = K_5 \). Find some Eulerian trail \( T \) in \( G \) and construct the corresponding graph \( H \).

6: Let \( F_H \) be a 1-factor in \( H \). What is correspondence of edges in \( F_H \) and edges in \( G \) and why? (Use \( K_5 \).)

This way we managed to find one 2-factor \( F \) in \( G \). Since \( G - F \) is \((2k - 2)\)-regular, we can use induction. Spanning subgraph of \( G \) is called a factor. A graph is factorable into factors \( F_1, F_2, \ldots, F_k \), if edges of the factors form a partition of edges of \( G \). If all factors are isomorphic to \( F \), then \( G \) is \( F \)-factorable.

7: Let \( 3K_3 \) be a disjoint union of three triangles. Show that \( K_9 \) is \( 3K_3 \)-factorable.

8: Show that \( K_{2k} \) can be factorized into \( k - 1 \) 2-factors and one 1-factor. (Hint: use that \( K_{2k} \) is 1-factorable.)

9: Let \( G \) be a connected graph on at least 4 vertices such that every edge of \( G \) belongs to a 1-factor in \( G \). Show that \( G \) is 2-connected.

10: Prove that if the bridges of a 3-regular graph \( G \) lie on a single path, then \( G \) has a 1-factor.

11: Is there a 2-factorization of \( K_7 \) in which no 2-factor is a Hamiltonian cycle?

12: Show that \( K_{2k} \) can be factorized into \( k - 1 \) Hamilton cycles and one 1-factor. (Hint: construction like where \( K_{2k} \) is 1-factorable.)

13: Find a multiset of cycles in Petersen’s graph such that every edge is in exactly two of the cycles.

14: Open Let \( G \) be a bridgeless cubic graph. Is there a multiset of cycles in \( G \) such that every edge is in exactly two of the cycles. (cycle double cover conjecture)