Chapter 5.4 Menger’s Theorem

Paths $P_1$ and $P_2$ are **internally disjoint** if their intersection contains only endpoints.

**Theorem 5.16 (Menger’s Theorem)** Let $u$ and $v$ be nonadjacent vertices in a graph $G$. The minimum number of vertices in a $u-v$ separating set equals the maximum number of internally disjoint $u-v$ paths in $G$.

**Proof.** Let the minimum separating set be $U$. Use induction on $|U| = k$. Then use induction on the number of vertices and edges.

1: Case 1: vertices $u$ and $v$ have a common neighbor $x \in U$.

2: Case 2: There exists vertex in $U$ not adjacent to $u$ and a vertex not adjacent to $v$.

3: Case 3: Every $U$ has all vertices adjacent to $u$ and none to $v$ or vice versa.

**Theorem 5.17** A nontrivial graph $G$ is $k$-connected for some integer $k \geq 2$ if and only if for each pair $u, v$ of distinct vertices of $G$ there are at least $k$ internally disjoint $u-v$ paths in $G$.

4: Prove theorem 5.17 for complete graphs.

5: Show $\Leftarrow$ direction.

6: Show $\Rightarrow$ direction if $u$ and $v$ are not adjacent.

7: Show $\Rightarrow$ direction if $u$ and $v$ are adjacent.

8: Let $G$ be a $k$-connected graph and let $S$ be any set of $k$ vertices. Show that if a graph $H$ is obtained from $G$ by adding a new vertex $w$ and joining $w$ to the vertices of $S$, then $H$ is also $k$-connected.
Show that if \( G \) is a \( k \)-connected graph and \( u, v_1, v_2, \ldots, v_k \) are \( k + 1 \) distinct vertices of \( G \), then there exist internally disjoint \( u - v_i \) paths \( (1 \leq i \leq k) \) in \( G \).

**Theorem 5.20** If \( G \) is a \( k \)-connected graph, \( k \geq 2 \), then every \( k \) vertices of \( G \) lie on a common cycle of \( G \).

We prove Theorem 5.20 by growing a cycle. Let \( S = \{v_1, \ldots, v_k\} \). Since \( k \geq 2 \), there is a cycle containing \( v_1 \) and \( v_2 \). Let \( C \) be a cycle containing vertices \( \{v_1, \ldots, v_l\} \). We will use the previous question to extend the cycle.

**10:** Show that if \( C \) is a cycle of length \( l \) formed by vertices \( \{v_1, \ldots, v_l\} \), then there exists a cycle containing vertices \( \{v_1, \ldots, v_l, v_{l+1}\} \).

**Solution:** Consider internally disjoint paths from \( C \) to \( v_{l+1} \). Any two consecutive will make the cycle longer.

**11:** Show that if \( C \) is a cycle of length \( > l \) containing vertices \( \{v_1, \ldots, v_l\} \), then there exists a cycle containing vertices \( \{v_1, \ldots, v_l, v_{l+1}\} \).

**Solution:** Consider internally disjoint paths from \( l + 1 \) vertices of \( C \) to \( v_{l+1} \). Take the initial parts of the cycles until they first time hit \( C \). Since there are \( l + 1 \), one can use pigeonhole principle to show that two are in the same segment that has no internal vertices of \( v_1, \ldots, v_l \) and it can be used to add \( v_{l+1} \).

**12:** 5.33 Let \( G \) be a 5-connected graph and let \( u, v \) and \( w \) be three distinct vertices of \( G \). Prove that \( G \) contains two cycles \( C \) and \( C' \) that have only \( u \) and \( v \) in common but neither of which contains \( w \).

**Solution:** Use that there are 5 internally disjoint paths between \( u \) and \( v \) and \( w \) is in at most one of them.

**Harary graph** \( H_{r,n} \) is a graph on \( n \) vertices \( v_1, \ldots, v_n \) that form a cycle \( C \) defined as follows. If \( r = 2k \) is event, then \( H_{r,n} = C^k \) (recall that we take power of cycle). If \( r = 2k + 1 \) is odd and \( n = 2l \) is even, then \( H_{r,n} \) is obtained from \( C^k \) by adding edges \( v_i v_{i+l} \), where \( 1 \leq i \leq l \). If \( r = 2k + 1 \) is odd and \( n = 2l \) is odd, then \( H_{r,n} \) is obtained from \( C^k \) by adding edges \( v_i v_{i+l+1} \), where \( 1 \leq i \leq l \) and edge \( v_1 v_{l+1} \).

**13:** Draw \( H_{2,6} \), \( H_{3,8} \), \( H_{3,9} \).

**14:** Show that for any two integers \( r, n \) with \( 2 \leq r < n \) holds \( \kappa(H_{r,n}) = r \).

**15:** Open Prove or disprove that if \( G \) is a 3-connected graph, then no longest cycle in \( G \) is induced.

Reading for next time - Chapter 5.4.