Let $G$ be a graph. A cut is $\{uv : u \in X, v \notin X, uv \in E(G)\}$ for some $X \subseteq V(G)$.

1: Let $X \subseteq V(G)$ create a cut $C$. Let $u \in X$ and $v \notin X$. Show that every $u - v$ path contains at least one edge of $C$.

Red-Blue meta algorithm for MST. Let $G$ be a graph and $w$ be a weight assignment to $E(G)$. Assume that all weights are distinct. Start with all edges being uncolored. Apply the following rules as long as possible.

- if $e \in E$ is in a cycle $C$ where $e$ is the heaviest edge, color $e$ red
- if there is a cut where $e \in E$ is the lightest edge, color $e$ blue.

Blue edges for a minimum spanning tree.

2: Show that red edge cannot be in MST.

3: Show that blue edge must be in MST.

4: Show that blue edges form a tree

5: Show that every edge gets colored.

6: Show that no edge satisfies both red and blue criteria. (i.e. every edge has one color).

7: What is the number of spanning trees of $C_5$?

8: Count the number of spanning trees of $K_4$. 

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**Theorem 4.15** (Cayley 1889) The number of distinct trees of order $n$ with a specified vertex set is $n^{n-2}$.

Proof by double counting (there are many others).

Let $F_{n,k}$ be the set of all rooted forests with $k$ components on $n$ (distinguishable) vertices. Assume edges directed away from the roots.

9: What is $|F_{n,1}|$?

Solution: 

10: What is $|F_{n,n}|$?

Solution: $|F_{n,n}| = 1$. All vertices are isolated.

Forest $F$ contains forest $F'$ if $F'$ can be obtained from $F$ by deleting edges.

Call a sequence $F_1, \ldots, F_k$ of forests refining sequence if $F_i \in F_{n,i}$ and $F_i$ contains $F_{i+1}$ for all $i$.

For a fixed forest $F_k$ in $F_{n,k}$ denote by

- $N(F_k)$ the number of rooted trees containing $F_k$.
- $N^*(F_k)$ the number of refining sequences ending in $F_k$.

11: Suppose $F_1 \in F_{n,1}$ contains $F_k$. How many refining sequences are there that start at $F_1$ and end at $F_k$?

12: Express $N^*(F_k)$ using $N(F_k)$ and the previous question.

Now we try to count $N^*(F_k)$ the other way by adding edges to $F_k$ and building a tree.

13: How many ways can you build $F_{k-1}$ that contains $F_k$?

14: Express $N^*(F_k)$ by iterating the previous question.

15: Combine results of $N^*(F_k)$ to count $N(F_k)$.

16: Use $N(F_k)$ to count the number of rooted trees on $n$ vertices (pick the correct $k$).

17: Count the number of spanning trees of $K_7$. 

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