**Spanning tree** of a connected graph $G$ is a spanning subgraph that is a tree.

**Theorem 4.10** Every connected graph has a spanning tree.

Assume a function $w$ assigning weight (cost) to edges of a graph $G$, that is $w: E(G) \rightarrow \mathbb{R}$.

**Minimum Spanning Tree Problem:** Find a spanning tree $T$ of $G$ minimizing $\sum_{e \in E(T)} w(e)$.

1: Find a minimum spanning tree of following graph.

![Graph](image)

**Kruskal’s greedy algorithm** [1956]: Order $E(G)$ increasingly according to $w$. Start with the empty spanning subgraph $T$ of $G$, take edges according to the ordering one by one and add if $T$ remains acyclic.

**Jarník’s** [1930] – **Prim’s** [1957] algorithm: Start with $T$ that is a single vertex of $G$. Find an edge $e$ of the smallest cost that has only one endpoint in $T$ and add $e$ to $T$.

**Borůvka’s** algorithm [1926]: Start with the empty spanning subgraph $T$ of $G$, note $T$ is a forest. For every connected component $C$ of $T$, add an edge $e$ of the smallest cost that has only one endpoint in $C$. Note the algorithm can run in parallel.

**Theorem 4.11** Kruskal’s algorithm produces minimum spanning tree.
2: Find minimum spanning tree of the following graph by running Kruskal’s, Jarník’s and Borůvka’s algorithms.

3: Let $C$ be a cycle in $G$ and let $T$ be a minimum spanning tree of $G$. Let $e$ be the edge of maximum weight in $C$. Show that $e \not\in E(T)$.

**Corollary 4.6** Every forest on $n$ vertices with $k$ components has $n - k$ edges.

**Theorem 4.7** Every connected graph on $n$ vertices has at least $n - 1$ edges.

**Theorem 4.8** If $G$ is connected graph on $n$ vertices with $n - 1$ edges, then $G$ is a tree.

**Theorem 4.8** If $G$ is acyclic graph on $n$ vertices with $n - 1$ edges, then $G$ is a tree.

**Theorem 4.9** Let $T$ be a tree on $k$ vertices. If $G$ is a graph with $\delta(G) \geq k - 1$ then $G$ contains a subgraph isomorphic to $T$.

**Hints:** 3 assume for contradiction $e \in E(T)$. 4.6 count edges in components; 4.7 Induction on $n$ and find a leaf; 4.8 use 4.7 and show no cycles; 4.8; 4.9 induction on $k$; 4.11 take MST with as many edges in common as output of the algorithm.

Reading for next time: All up to 4.3 (skip 2.5, 3.3, 3.4) - midterm on Feb 11 (Thursday).

text from G. Chartrand and P. Zhang. “A First Course in Graph Theory”