Graphs whose positive semi-definite matrices have nullity at most three

Hein van der Holst

Eindhoven University of Technology

Given a graph $G = (V, E)$, let $\mathcal{M}_G$ be the set of all real-valued symmetric $V \times V$ matrices with

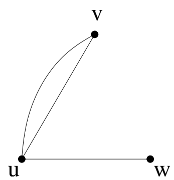
- $m_{i,j} = 0$ if $i \neq j$ and i and j are not connected by any edge,
- $m_{i,j} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge,
- $m_{i,j} \in \mathbb{R}$ if $i \neq j$ and i and j are connected by multiple edges, and
- $m_{i,j} \in \mathbb{R}$, for all $i \in V$.

$\mathrm{mnull}(G)$ is the largest nullity attained by any positive semi-definite matrix $M \in \mathcal{M}_G$.

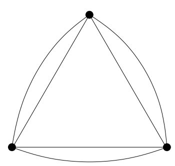
Problem: characterize the graphs $G = (V, E)$ with $\mathrm{mnull}(G) \leq k$ for some fixed integer k.

Why allow multiple edges?

Examples:

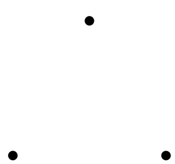
$G_1 =$ 

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{G_1}$$

$G_2 =$ 

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{G_2}$$

$\mathrm{mnull}(G_2) = 3.$

$G_3 =$ 

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_{G_3}$$

$\mathrm{mnull}(G_3) = 3.$

In general it is not true that

$$\operatorname{mnull}(H) \leq \operatorname{mnull}(G)$$

if H is a minor of G. Makes it difficult to find the structure of graphs with $\operatorname{mnull}(G) \leq k$.

A minor of a graph G is a graph obtained from a subgraph of G by contracting some edges.

Contracting edge e.

Solution: add Strong Arnol'd Hypothesis as condition to $\operatorname{mnull}(G)$.

A matrix M fulfills the Strong Arnol'd Hypothesis if $X = 0$ is the only symmetric $V \times V$ matrix $X = (x_{i,j})$ with

- $x_{i,j} = 0$ if i and j are connected or $i = j$, and
- $MX = 0$.

This is the same Strong Arnol'd Hypothesis as used in the definition of $\mu(G)$.

Definition 1 (Colin de Verdière). $\nu(G)$ *is the largest the nullity attained by any positive semi-definite matrix* $M \in \mathcal{M}_G$ *satisfying the Strong Arnol'd Hypothesis.*

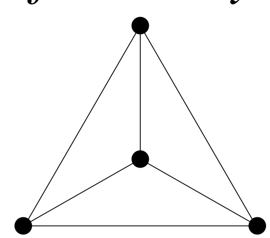
Theorem 1. $\nu(H) \leq \nu(G)$ *if* H *is a minor of* G.

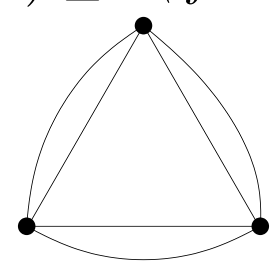
The graphs satisfying $\nu(G) \leq k$ have a forbidden minor characterization.

Characterizations:

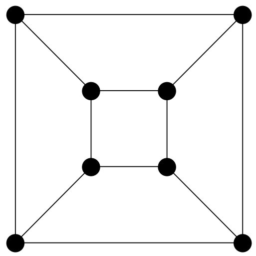
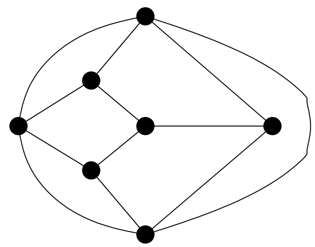
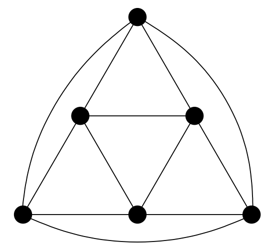
Theorem 2 (Colin de Verdière). $\nu(G) \leq 1$ *if and only if* G *is a forest.*

Theorem 3 (Kotlov, vdH). $\nu(G) \leq 2$ *if and only if* $\mathrm{la}(G) \leq 2$ *(if and only if* G *has no minor isomorphic to:*

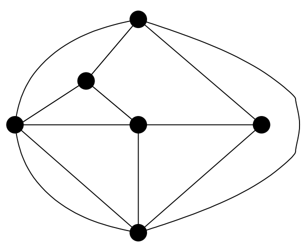
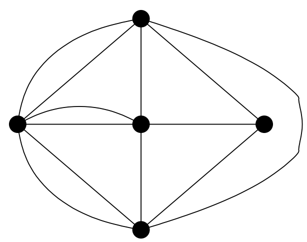
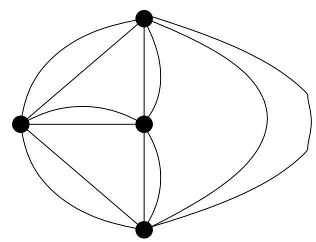




Theorem 4 (vdH). $\nu(G) \leq 3$ *if and only if* G *has no minor isomorphic to*

K_5 *or to:*

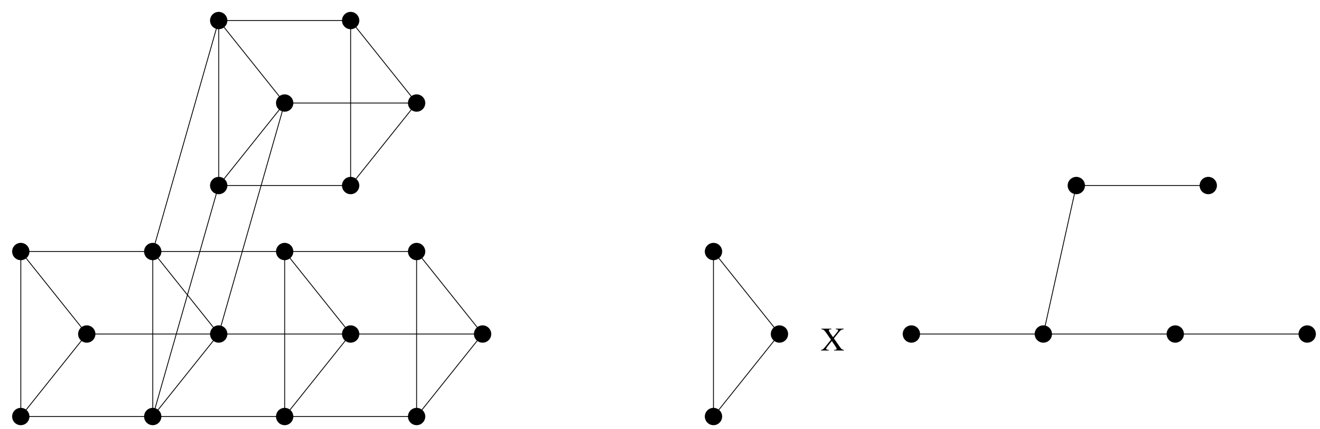
The asymptotic behaviour of ν is the same as the tree-width of a graph:

$\nu(G)$ is large $\leftrightarrow$ tree-width is large (Colin de Verdière)

(The tree-width plays a fundamental role in the Graph Minors project of Robertson and Seymour.)

Definition 2. $\mathrm{la}(G) = \min\{k \mid G$ *is a minor of* $K_k \times T$ *for some tree* $T\}$.

Given graphs $G = (V, E)$ and $H = (W, F)$, $G \times H$ is the graph with vertex set $V \times W$ and edge-set $\{(v \times w_1, v \times w_2) \mid v \in V, (w_1, w_2) \in F\} \cup \{(v_1 \times w, v_2 \times w) \mid w \in W, (v_1, v_2) \in E\}$.



Theorem 5. $\nu(G) \leq \mathrm{la}(G)$.

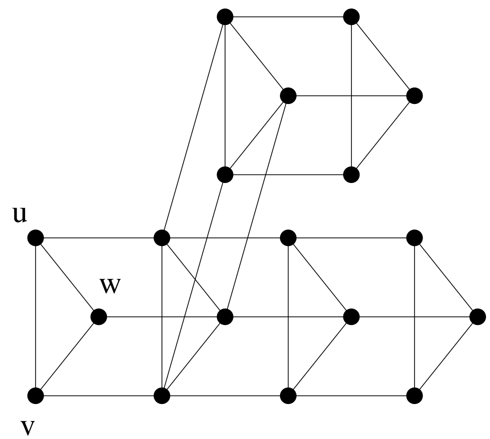
How do we prove this?

For $x \in \mathbb{R}^V$, define

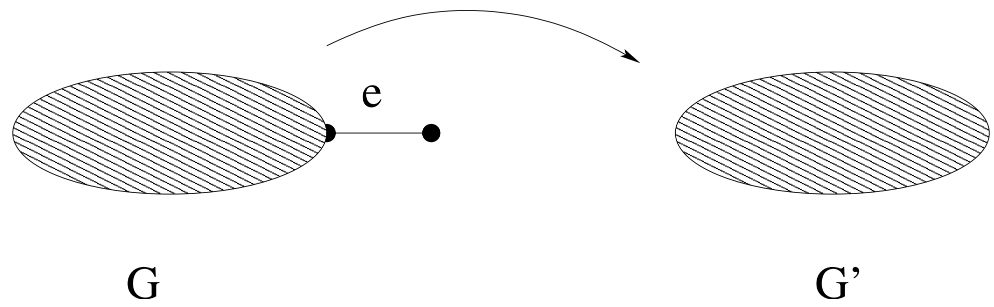
$$\operatorname{supp}(x) = \{v \in V \mid x_v \neq 0\}$$

Lemma 6 (A Courant Nodal Theorem). *Let $M \in \mathcal{M}_G$ be positive semi-definite and satisfying the Strong Arnol'd Hypothesis. If $x \in \ker(M)$ is nonzero, then $G[\operatorname{supp}(x)]$ is connected.*

To show that $\nu(G) \leq \mathrm{la}(G)$ it suffices to show that $\nu(K_k \times T) \leq k$.

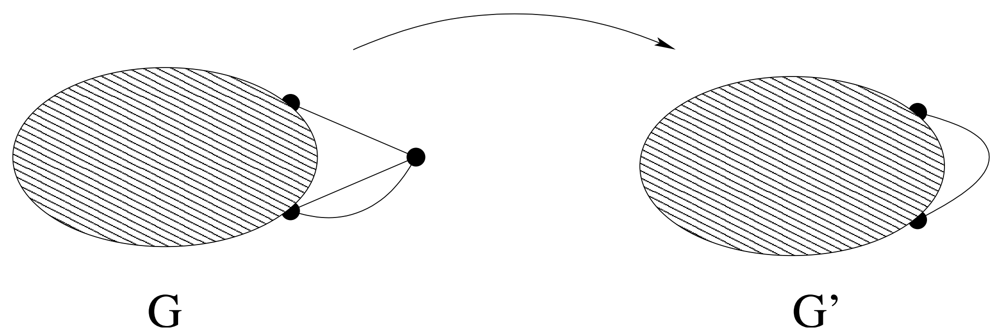


A similar Courant Nodal Theorem holds for the general case where M need not satisfy the Strong Arnol'd Hypothesis. In this case we need to consider $x \in \ker(M)$ with minimal support.



$$\mathrm{mnull}(G) = \mathrm{mnull}(G')$$

This show that trees have $\mathrm{mnull}(G) \leq 1$.

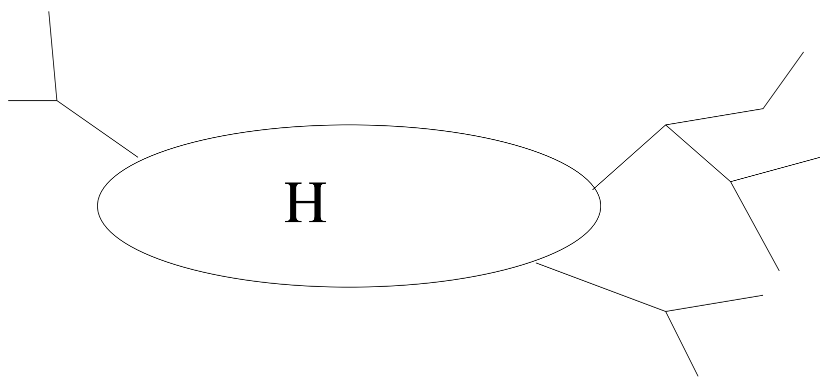


$$\mathrm{mnull}(G) = \mathrm{mnull}(G')$$

Characterizations:

Theorem 7. $\mathrm{mnull}(G) \leq 1 \Leftrightarrow$ *G is a tree.*

Theorem 8. $\mathrm{mnull}(G) \leq 2 \Leftrightarrow$ *G is either the disjoint union of two trees, or G is connected and G can be obtained from a 2-connected graph H with* $\mathrm{la}(H) \leq 2$ *by attaching trees to it.*



What about graphs satisfying $\operatorname{mnull}(G) \leq 3$?

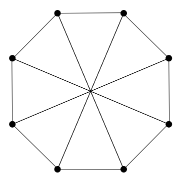
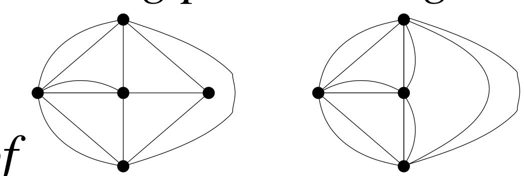
Disjoint unions and 1-separations are easy to deal with.

Lemma 9. *Let G be the disjoint union of G_1 and G_2. Then $\operatorname{mnull}(G) \leq k$ if and only if $\operatorname{mnull}(G_1) + \operatorname{mnull}(G_2) \leq k$.*

Lemma 10. *Let (G_1, G_2) be a 1-separation of a connected graph G. Then $\operatorname{mnull}(G) \leq 3$ if and only if $\operatorname{mnull}(G_1) + \operatorname{mnull}(G_2) \leq 4$.*

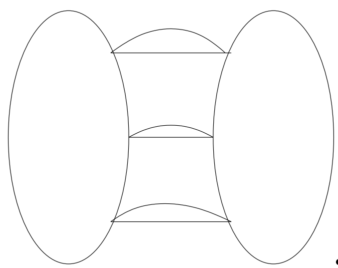
2-separations are not so easy to deal with.

Theorem 11. *Let* G *be a* 3*-connected graph. Then* $\mathrm{mnull}(G) \le 3$ *if and only if one of the following is true:*

1. *G is*  *after removing parallel edges and G contains no minor isomorphic to one of* 
2. $\mathrm{la}(G) \le 3$*, or*
3. *five classes more.*

One can verify in polynomial time that a 3-connected graph G has $\mathrm{mnull}(G) \le 3$.

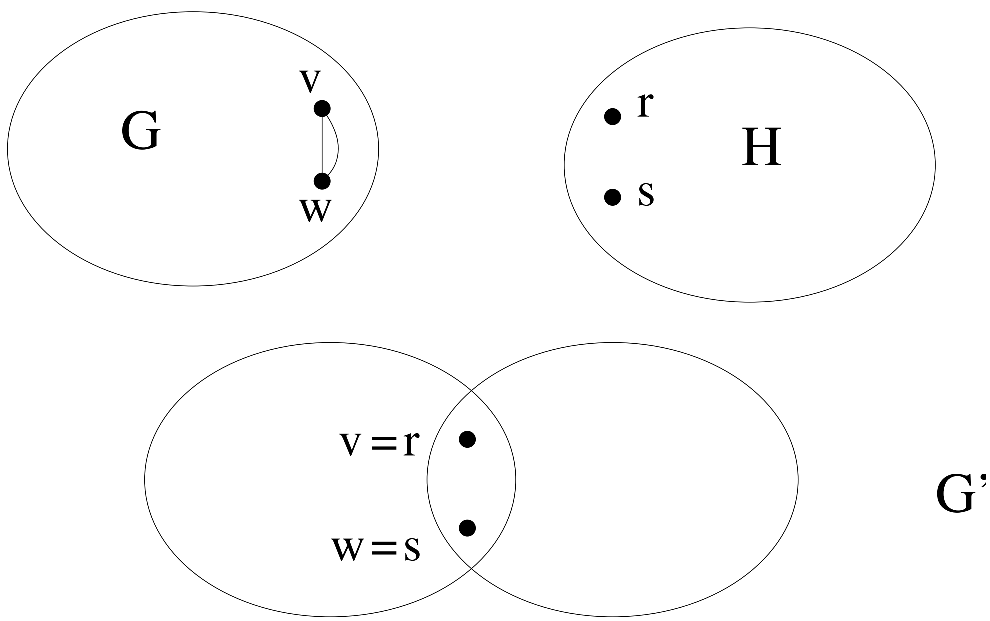
Theorem 12. *If G is 3-connected, then $\operatorname{mnull}(G) \geq 4$ if and only if one of the following is true:*

1. $\nu(G) \geq 4$*, or*
2. *G has the form*  .

Let G be as in 2, with $\nu(G) \leq 3$. If $M \in \mathcal{M}_G$ is positive semi-definite and has $\operatorname{mnull}(M) \geq 4$, then M does not satisfy the Strong Arnol'd Hypothesis.

Lemma 13. *Let G be a graph. If we replace in G an edge by a path, then the new graph G' has* $\mathrm{mnull}(G') = \mathrm{mnull}(G)$.

Lemma 14. *Let G be a graph. If we replace in G two parallel edges by a graph H containing a circuit and which is such that the graph H' obtained from H by adding parallel edges between r and s has* $\mathrm{la}(H') \leq 2$*, then the new graph G' has* $\mathrm{mnull}(G') = \mathrm{mnull}(G)$.



This does not deal with all 2-separations of a graph G.