MINIMUM RANK OF GRAPHS WITH LOOPS*

CHASSIDY BOZEMAN†, ANNAVICTORIA ELLSWORTH*, LESLIE HOGBEN*‡,
JEPHIAN CHIN-HUNG LIN*, GABI MAURER*, KATHLEEN NOWAK*,
AARON RODRIGUEZ*, AND JAMES STRICKLAND*

June 2, 2014

Abstract. A loop graph $\mathfrak{G}$ is a finite undirected graph that allows loops but does not allow multiple edges. The set $\mathcal{S}(\mathfrak{G})$ of real symmetric matrices associated with a loop graph $\mathfrak{G}$ of order n is the set of symmetric matrices $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $a_{ij} \neq 0$ if and only if $ij \in E(\mathfrak{G})$. The minimum (maximum) rank of a loop graph is the minimum (maximum) of the ranks of the matrices in $\mathcal{S}(\mathfrak{G})$. We characterize loop graphs having minimum rank at most two (by forbidden induced subgraphs and graph complements) and loop graphs having minimum rank equal to the order of the graph. A Schur complement reduction technique is used to determine the minimum ranks of cycles with various loop configurations; we also determine the minimum ranks of complete graphs and paths with various configurations of loops. Unlike simple graphs, loop graphs can have maximum rank less than the order of the graph. We establish a method to determine the maximum rank of a loop graph, and also present some results on which ranks between minimum and maximum can be realized. Interesting open questions remain.

Keywords. loop graph, minimum rank, maximum nullity, zero forcing number, spanning generalized cycle, Schur complement, matrix, graph

AMS subject classifications. 05C50, 15A03, 15B57

1. Introduction.

A simple graph does not allow loops or multiple edges, whereas a loop graph allows (but does not require) loops. The set of edges of a simple graph describes the nonzero pattern of the off-diagonal entries of the graph's family of real symmetric matrices (with no constraints on diagonal entries), whereas the edges of a loop graph completely describe the nonzero pattern of the graph's family of real symmetric matrices. Note that a loop graph that does not contain any loops requires matrices to have a zero diagonal, whereas a simple graph (which by definition has no loops) does not require a zero diagonal. To avoid confusion, we use G etc. to denote a simple graph and $\mathfrak{G}$ etc. to denote a loop graph. Formal definitions and other terminology can be found in Section 1.1.

*Received by the editors on Month x, 200x. Accepted for publication on Month y, 200y Handling Editor: .

†Department of Mathematics, Iowa State University, Ames, IA 50011, USA (cbozeman, avellsw, LHogben, chlin, gjmaurer, knowak, rodr45, jamess)@iastate.edu.

‡American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306, USA (hogben@aimath.org).

The minimum rank problem for a simple or loop graph is to determine the minimum of the ranks among the matrices in the family described by the simple or loop graph. The minimum rank problem for simple graphs has been studied extensively (see [14] and the references therein). Some work has been done on loop graphs, including complete determination of minimum rank for loop trees [12], cut-vertex reduction [23], and characterizations of extreme minimum rank for loop graphs that do not have loops [16]. However, far fewer results have been obtained about the minimum rank problem for loop graphs than for simple graphs.

Characterizations of loop graphs having minimum rank at most two (by forbidden induced subgraphs and graph complements) are established in Section 2; the proof makes use of the analogous characterizations for simple graphs [5], but the forbidden subgraphs are substantially different. In Section 3 the characterization of minimum rank equal to order if and only if there is a unique spanning generalized cycle is extended from loopless loop graphs (zero diagonal minimum rank) [16] to every loop graph. Section 4 presents a technique for reducing the minimum rank problem for a specific graph to a smaller one through the use of the Schur complement. This method is applied to cycles in Section 5, which contains characterizations of the minimum ranks of complete graphs, paths, and cycles with various configurations of loops. Unlike simple graphs, loop graphs can have maximum rank less than the order of the graph [16]. Thus it is of interest to determine the maximum rank and which ranks between minimum and maximum can be realized; these topics are discussed in Section 6. Section 7 presents examples showing that certain results, including the Graph Complement Conjecture and Colin de Verdière type parameters do not extend to loop graphs, and asks questions for future research.

1.1. Notation and terminology. For a *loop graph* $\mathfrak{G} = (V(\mathfrak{G}), E(\mathfrak{G}))$, the finite nonempty set of vertices is denoted by $V(\mathfrak{G})$ and the set of edges $E(\mathfrak{G})$ is a set of two element multisets of vertices (i.e., the two vertices in an edge need not be distinct). A *loop* is an edge with two copies of one vertex. The edge $\{u, v\}$ is often denoted by uv (or in the case of a loop $\{u, u\}$ by uu). A *simple graph* $G = (V(G), E(G))$ is defined analogously, except an edge is a two element set of vertices (i.e., the two vertices in an edge must be distinct). The subgraph $\mathfrak{G}[W]$ of $\mathfrak{G} = (V, E)$ *induced* by $W \subseteq V$ is the subgraph with vertex set W and edge set $\{wu \mid wu \in E \text{ and } w, u \in W\}$. If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are disjoint loop graphs, the *join* $\mathfrak{G}_1 \vee \mathfrak{G}_2$ of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is the graph with vertex set $V(\mathfrak{G}_1 \vee \mathfrak{G}_2) = V(\mathfrak{G}_1) \dot\cup V(\mathfrak{G}_2)$ and edge set $E(\mathfrak{G}_1 \vee \mathfrak{G}_2) = E(\mathfrak{G}_1) \cup E(\mathfrak{G}_2) \cup E$, where E consists of all the edges uv with $u \in V(\mathfrak{G}_1)$ and $v \in V(\mathfrak{G}_2)$. A *trail* $(v_1, v_2, \ldots, v_t)$ in a simple or loop graph is a subgraph with vertex set $\{v_1, v_2, \ldots, v_t\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{t-1}v_t\}$ with no repeated edges (vertices may be repeated). A *path* is a trail with no repeated vertices and a cycle is a trail in which $t \geq 3$, $v_1 = v_t$, and all other vertices are distinct; a *k-cycle* is a cycle with k vertices.

For a symmetric $n \times n$ real matrix A, the *loop graph* of A is $\mathfrak{G}(A) = (V, E)$ where $V = \{1, \dots, n\}$ and $E = \{uv \,|\, a_{uv} \neq 0\}$ (the definition of the simple graph $\mathcal{G}(A)$ is analogous except the stipulation $u \neq v$ is made in defining E). Let $\mathfrak{G} = (V, E)$ be a loop graph of order n (normally $V = \{1, \dots, n\}$; otherwise we associate V with $\{1, \dots, n\}$). The set of real symmetric matrices described by $\mathfrak{G}$ is

$$\mathcal{S}(\mathfrak{G}) = \{A \in \mathbb{R}^{n\times n} : A^\top = A \text{ and } \mathfrak{G}(A) = \mathfrak{G}\}$$

(the definition for a simple graph G is analogous using $\mathcal{G}(A)$). The adjacency matrix $A_\mathfrak{G}$ is in $\mathcal{S}(\mathfrak{G})$ and analogously for a simple graph. The *minimum rank* and the *maximum nullity* of a loop graph $\mathfrak{G}$ are

$$\operatorname{mr}(\mathfrak{G}) = \min\{\operatorname{rank} A \mid A \in \mathcal{S}(\mathfrak{G})\} \quad \text{and} \quad \operatorname{M}(\mathfrak{G}) = \max\{\operatorname{null} A \mid A \in \mathcal{S}(\mathfrak{G})\}$$

(the definitions for a simple graph G are analogous). Clearly $\operatorname{mr}(\mathfrak{G}) + \operatorname{M}(\mathfrak{G}) = |\mathfrak{G}|$.

By definition, the complete graph on n vertices, $\mathfrak{K}_n$, has all loops, and the complete bipartite graph on s and t vertices, $\mathfrak{K}_{s,t}$, has no loops. For any loop graph $\mathfrak{G}$, $\mathfrak{G}^0$ denotes the loop graph having the same underlying simple graph as $\mathfrak{G}$ but no loops, and $\mathfrak{G}^\ell$ has the same underlying simple graph as $\mathfrak{G}$ but all loops.

The *union* of loop graphs $\mathfrak{G}_i = (V_i, E_i)$ is $\bigcup_{i=1}^h \mathfrak{G}_i = (\cup_{i=1}^h V_i, \cup_{i=1}^h E_i)$; if the V_i are pairwise disjoint, then the union can be denoted by $\dot{\bigcup}_{i=1}^h \mathfrak{G}_i$. Let $\mathfrak{G} = (V, E)$ be a loop graph of order n. The *complement* of $\mathfrak{G}$ is the loop graph $\overline{\mathfrak{G}} = (V, \overline{E})$ where $\overline{E} = E(\mathfrak{K}_n) \setminus E$. Vertex u is a *neighbor* of vertex v in $\mathfrak{G}$ if and only if $uv \in E$ (this applies to loops also, so u is a neighbor of itself if and only if it has a loop). The set of neighbors of v is denoted by $N_\mathfrak{G}(v)$ (or $N(v)$ if $\mathfrak{G}$ is clear). The *degree of vertex v in $\mathfrak{G}$ is* $\deg_\mathfrak{G} v = |N_\mathfrak{G}(v)|$ (and $\deg_\mathfrak{G} v$ can be denoted by $\deg v$ if $\mathfrak{G}$ is clear). The *minimum degree* of $\mathfrak{G} = (V, E)$ is $\delta(\mathfrak{G}) := \min\{\deg v \,|\, v \in V\}$.

Let A be an $n \times n$ matrix. For $\alpha, \beta \subseteq \{1, 2, \dots, n\}$, the submatrix of A lying in rows indexed by α and columns indexed by β is denoted by $A[\alpha, \beta]$; $A[\alpha, \alpha]$ is also denoted by $A[\alpha]$ and is a *principal* submatrix. We also define $A[\alpha, \alpha) := A[\alpha, \overline{\alpha}]$, $A(\alpha, \alpha] := A[\overline{\alpha}, \alpha]$, and $A(\alpha) := A[\overline{\alpha}]$ where $\overline{\alpha} = \{1, 2, ..., n\} \setminus \alpha$.

1.2. Generalized cycles and the characteristic polynomial. A *generalized cycle* $\mathcal{C}$ of a loop graph $\mathfrak{G}$ is a subgraph of $\mathfrak{G}$ where each connected component is one of the following: a cycle, an edge (meaning an edge and its two distinct endpoints), or a loop (meaning a vertex v and its edge vv). Generalized cycles are also called $[1, 2]$-*factors*. The *order* of $\mathcal{C}$ is the number of vertices in $\mathcal{C}$. A generalized cycle of order $|\mathfrak{G}|$ is said to be a *spanning* generalized cycle (or a *perfect* $[1, 2]$-factor). The following notation is adapted from [16]. Given a generalized cycle $\mathcal{C}$, define $\operatorname{nc}(\mathcal{C})$ to be the number of distinct cycles in $\mathcal{C}$, and $\operatorname{ne}(\mathcal{C})$ to be the number of even components of $\mathcal{C}$, that is, the number of edges plus the number of cycles of even order. The set of all generalized cycles of order k of a loop graph $\mathfrak{G}$ is denoted by $\operatorname{cyc}_k(\mathfrak{G})$. With a generalized cycle $\mathcal{C}$, we associate a permutation of the vertices of $\mathcal{C}$ as follows. For each cycle in $\mathcal{C}$, fix an orientation and then associate a directed graph cycle $(v_1, v_2, \ldots, v_k)$ with the cyclic permutation $(v_1 v_2 \cdots v_k)$. Each edge component v_1v_2 of $\mathcal{C}$ is associated with the 2-cycle (v_1v_2) and each loop v_1v_1 is associated with the permutation (v_1), which fixes v_1. The permutation $\pi_{\mathcal{C}}$ is defined to be the product of these associated permutation cycles. Note that there are $2^{\operatorname{nc}(\mathcal{C})}$ different choices for the orientation of the cycles of $\mathcal{C}$, and each choice yields a permutation that has the same sign as $\pi_{\mathcal{C}}$, namely $(-1)^{\operatorname{ne}(\mathcal{C})}$. The sum of all $k \times k$ principal minors of an $n \times n$ matrix $A = [a_{ij}] \in \mathcal{S}(\mathfrak{G})$ is denoted $S_k(A)$, and can be expressed using generalized cycles [19] as

$$S_k(A) = \sum_{\mathcal{C} \in \operatorname{cyc}_k(\mathfrak{G}(A))} (-1)^{\operatorname{ne}(\mathcal{C})} 2^{\operatorname{nc}(\mathcal{C})} a_{i_1 \pi_{\mathcal{C}}(i_1)} \cdots a_{i_k \pi_{\mathcal{C}}(i_k)}$$

where $\{i_1, i_2, \ldots, i_k\}$ is the vertex set of $\mathcal{C}$ and the sum over the empty set is defined to be zero. The characteristic polynomial $p_A(x)$ of A is

$$x^n - S_1(A)x^{n-1} + S_2(A)x^{n-2} + \cdots + (-1)^{n-1} S_{n-1}(A)x + (-1)^n S_n(A).$$

Note that for $A \in \mathcal{S}(\mathfrak{G})$, $\det A = S_n(A)$ can be computed using spanning generalized cycles, and if $\mathfrak{G}$ has a unique generalized cycle then $\det A \neq 0$. The next remark extends and generalizes Remark 1.4 in [16].

Remark 1.1. For a real symmetric matrix A with $p_A(x) = x^n + c_1x^{n-1} + \cdots + c_kx^{n-k} + \cdots + c_n$, $\operatorname{rank} A = \max\{k \mid c_k \neq 0\}$. Let $\mathfrak{G}$ be a loop graph of order n. If $\mathfrak{G}$ has no generalized cycle of order k, then $\operatorname{rank} A \neq k$ for all $A \in \mathcal{S}(\mathfrak{G})$. If $\mathfrak{G}$ has no generalized cycle of order k for all $k > m$, then $\operatorname{rank} A \leq m$ for all $A \in \mathcal{S}(\mathfrak{G})$, and hence $\operatorname{mr}(\mathfrak{G}) \leq m$.

1.3. Additional results. This section contains some obvious extensions to loop graphs of well-known results for minimum rank of simple graphs, and summarizes additional known results for loop graphs that we will use.

Observation 1.2. *Let $\mathfrak{G}$ be a loop graph.*

1. *If $\mathfrak{G}$ is obtained from the simple graph G by adding some configuration of loops to G, then $\operatorname{mr}(G) \leq \operatorname{mr}(\mathfrak{G})$.*
2. *If $\mathfrak{H}$ is an induced subgraph of $\mathfrak{G}$, then $\operatorname{mr}(\mathfrak{H}) \leq \operatorname{mr}(\mathfrak{G})$.*
3. *$\mathfrak{G}$ has no edges if and only if $\operatorname{mr}(\mathfrak{G}) = 0$.*
4. *If the connected components of $\mathfrak{G}$ are $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_t$, then*

$$\operatorname{mr}(\mathfrak{G}) = \sum_{i=1}^{t} \operatorname{mr}(\mathfrak{G}_i).$$

A loop graph $\mathfrak{T}$ is a forest if $\mathfrak{T}$ does not have any cycles, and a tree is a connected forest. Note that a forest is permitted to have loops. The technique in the next remark is known for matrices described by simple graphs (see, for example, [10]), and the same inductive reasoning applies to loop graphs.

REMARK 1.3. Suppose $A \in \mathcal{S}(\mathfrak{G})$ and $\mathfrak{T}$ is a loopless forest that is a subgraph of $\mathfrak{G}$. If $\mathfrak{G}$ has a loop at vertex v, then the v, v-entry of $B := \frac{1}{a_{vv}} A$ is one. There exists a nonsingular diagonal matrix D such that $(DBD)_{uv} = 1$ for every $uv \in E(\mathfrak{T})$ and $(DBD)_{vv} = 1$ (v is chosen as the root). Observe that for D nonsingular diagonal, $B \in \mathcal{S}(\mathfrak{G})$ implies $DBD \in \mathcal{S}(\mathfrak{G})$ and $\operatorname{rank}(DBD) = \operatorname{rank} A$, so when showing that a matrix in $\mathcal{S}(\mathfrak{G})$ realizing a specific rank does not exist, without loss of generality we can assume the entries associated with the edges of a loopless forest are all one, and one nonzero diagonal entry can be assumed to be one (if such exists).

The zero forcing number was introduced in [2] for simple graphs and extended to loop graphs in [20]. Let $\mathfrak{G} = (V, E)$ be a loop graph, with each vertex colored either white or blue. If exactly one neighbor v of u is white, then change the color of v to blue (the possibility that $u = v$ is permitted); this is the *color-change rule* for loop graphs. When the color-change rule is applied to u changing the color of v, we say u *forces* v, and write $u \to v$. Given an initial coloring of $\mathfrak{G}$, the *final coloring* is the result of applying the color-change rule until no more changes are possible. A *zero forcing set* for $\mathfrak{G}$ is a subset of vertices B such that if initially the vertices in B are colored blue and the remaining vertices are colored white, then all the vertices of $\mathfrak{G}$ are blue in the final coloring. The *zero forcing number* $\mathrm{Z}(\mathfrak{G})$ is the minimum cardinality of a zero forcing set $B \subseteq V$.

THEOREM 1.4. [20] *For every loop graph* $\mathfrak{G}$*,* $\mathrm{M}(\mathfrak{G}) \leq \mathrm{Z}(\mathfrak{G})$*. If* $\mathfrak{T}$ *is a forest, then* $\mathrm{M}(\mathfrak{T}) = \mathrm{Z}(\mathfrak{T})$.

2. Low minimum rank. In this section we characterize loop graphs having minimum rank at most two. Minimum rank at most three was characterized for loop graphs that have no loops in [16], where it was shown that:

- $\operatorname{mr}(\mathfrak{G}^0) = 0$ if and only if $\mathfrak{G}^0$ has no edges.
- $\operatorname{mr}(\mathfrak{G}^0) \neq 1$.
- For $\mathfrak{G}^0$ connected, $\operatorname{mr}(\mathfrak{G}^0) = 2$ if and only if $\mathfrak{G}^0 = \mathfrak{K}_{n_1,n_2}$ with $n_1, n_2 \geq 1$.
- For $\mathfrak{G}^0$ connected, $\operatorname{mr}(\mathfrak{G}^0) = 3$ if and only if $\mathfrak{G}^0 = \mathfrak{K}_{n_1,n_2,\dots,n_t}$ with $t \geq 3$ and $n_i \geq 1$ for $i = 1, \dots, t$.

OBSERVATION 2.1. *A loop graph* $\mathfrak{G}$ *has* $\operatorname{mr}(\mathfrak{G}) = 0$ *if and only if* $\mathfrak{G} = \overline{\mathfrak{K}_n}$*, and* $\operatorname{mr}(\mathfrak{G}) = 1$ *if and only if* $\mathfrak{G} = \mathfrak{K}_s \,\dot{\cup}\, \overline{\mathfrak{K}_r}$ *with* $s \geq 1$ *and* $r \geq 0$.

We extend Barrett, van der Holst, and Loewy's characterizations of simple graphs having minimum rank at most two to loop graphs, but with a different set of forbidden induced subgraphs, the set $\mathcal{F}_{\mathrm{mr2}}$ shown in Figure 2.1 (see Theorem 2.3 below).

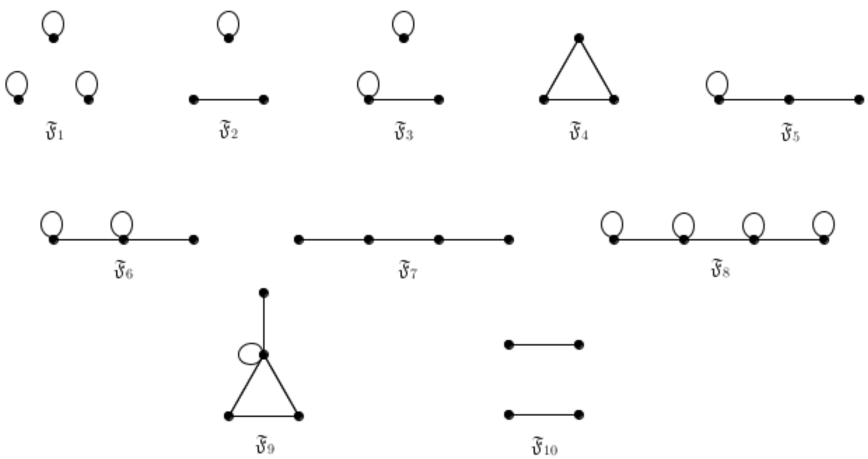

FIG. 2.1. *The set $\mathcal{F}_{\mathrm{mr2}} = \{\mathfrak{F}_1, \ldots, \mathfrak{F}_{10}\}$ of forbidden induced subgraphs for minimum rank at most two.*

Following the definitions of F-free and $\mathcal{F}$-free for simple graphs in [5], we say a loop graph $\mathfrak{G}$ is $\mathfrak{F}$-free if $\mathfrak{G}$ does not contain $\mathfrak{F}$ as an induced subgraph, and for a set $\mathcal{F}$ of loop graphs, $\mathfrak{G}$ is $\mathcal{F}$-free if $\mathfrak{G}$ is $\mathfrak{F}$-free for all $\mathfrak{F} \in \mathcal{F}$. The next theorem, due to Barrett, van der Holst, and Loewy, will be used:

THEOREM 2.2. [5, Theorem 6] *Let G be a simple graph. The following are equivalent:*

1. $\mathrm{mr}(G) \le 2$.
2. *G is $\{P_4,$ dart, $\ltimes$, $K_{3,3,3}, P_3 \dot\cup K_2, 3K_2\}$-free.*
3. *$\overline{G} = \left(K_{s_1} \dot\cup K_{s_2} \dot\cup K_{p_1,q_1} \dot\cup \cdots \dot\cup K_{p_k,q_k}\right) \vee K_r$ for some nonnegative $s_1, s_2, k,$ p_i, q_i, r with $p_i + q_i \ge 1$ for $i = 1, \ldots, k$.*

The next theorem characterizes loop graphs having minimum rank at most two.

THEOREM 2.3. *Let $\mathfrak{G}$ be a loop graph. The following are equivalent:*

1. $\mathrm{mr}(\mathfrak{G}) \le 2$.
2. *$\mathfrak{G}$ is $\mathcal{F}_{\mathrm{mr2}}$-free for the set $\mathcal{F}_{\mathrm{mr2}}$ of loop graphs shown in Figure 2.1.*
3. *$\overline{\mathfrak{G}} = \left(\mathfrak{K}_{s_1} \dot\cup \mathfrak{K}_{s_2} \dot\cup \mathfrak{K}_{p_1,q_1} \dot\cup \cdots \dot\cup \mathfrak{K}_{p_k,q_k}\right) \vee \mathfrak{K}_r$ for some nonnegative $s_1, s_2, k,$ p_i, q_i, r with $p_i + q_i \ge 1$ for $i = 1, \ldots, k$.*
4. *$\mathfrak{G} = \left(\mathfrak{K}_{s_1,s_2} \vee (\mathfrak{K}_{p_1} \dot\cup \mathfrak{K}_{q_1}) \vee \cdots \vee (\mathfrak{K}_{p_k} \dot\cup \mathfrak{K}_{q_k})\right) \dot\cup \overline{\mathfrak{K}_r}$ for some nonnegative s_1, s_2, k, p_i, q_i, r with $p_i + q_i \ge 1$ for $i = 1, \ldots, k$.*

Proof. We modify conditions (3) and (4) by removing the isolated vertices from the latter:

$$\overline{\mathfrak{G}} = \mathfrak{K}_{s_1} \dot\cup \mathfrak{K}_{s_2} \dot\cup \mathfrak{K}_{p_1,q_1} \dot\cup \cdots \dot\cup \mathfrak{K}_{p_k,q_k} \tag{2.1}$$

for some nonnegative s_1, s_2, k, p_i, q_i with $p_i + q_i \geq 1$ for $i = 1, \ldots, k$ and $k \geq 1$ or $s_1, s_2 \geq 1$, and

$$\mathfrak{G} = \mathfrak{K}_{s_1,s_2} \vee (\mathfrak{K}_{p_1} \dot\cup \mathfrak{K}_{q_1}) \vee \cdots \vee (\mathfrak{K}_{p_k} \dot\cup \mathfrak{K}_{q_k}) \tag{2.2}$$

for some nonnegative s_1, s_2, k, p_i, q_i with $p_i + q_i \geq 1$ for $i = 1, \ldots, k$ and $k \geq 1$ or $s_1, s_2 \geq 1$. We prove that conditions (1), (2), (2.1), and (2.2) are equivalent for loop graphs with positive minimum degree. The result then follows, since taking a disjoint union of $\mathfrak{G}$ and $\overline{\mathfrak{K}_r}$ is equivalent to bordering a matrix $M \in \mathcal{S}(\mathfrak{G})$ with blocks of zeros. So henceforth we assume $\delta(\mathfrak{G}) \geq 1$.

(1)⇒(2) Every graph in $\mathcal{F}_{\text{mr2}}$ has minimum rank greater than two. So, if $\mathfrak{G}$ contains some $\mathfrak{F}_i \in \mathcal{F}_{\text{mr2}}$ as an induced subgraph, then $\text{mr}(\mathfrak{G}) \geq 3$.

(2)⇒(2.1) Assume $\mathfrak{G}$ is $\mathcal{F}_{\text{mr2}}$-free. It is easy to check that any loop configuration of any of the six graphs P_4, dart, $\ltimes$, $K_{3,3,3}$, $P_3 \dot\cup K_2$, and $3K_2$ contains at least one induced subgraph in $\mathcal{F}_{\text{mr2}}$ (see Appendix [7] for details). Thus the associated simple graph G of $\mathfrak{G}$ is $\{P_4$, dart, $\ltimes$, $K_{3,3,3}$, $P_3 \dot\cup K_2$, $3K_2\}$-free, and so by Theorem 2.2, $\overline{G} = \left(K_{s_1} \dot\cup K_{s_2} \dot\cup K_{p_1,q_1} \dot\cup \cdots \dot\cup K_{p_k,q_k}\right) \vee K_r$ for some nonnegative s_1, s_2, k, p_i, q_i, r with $p_i + q_i \geq 1$ for $i = 1, \ldots, k$.

Hence G is of the form $\left(K_{s_1,s_2} \vee (K_{p_1} \dot\cup K_{q_1}) \vee \cdots \vee (K_{p_k} \dot\cup K_{q_k})\right) \dot\cup \overline{K_r}$ and $\mathfrak{G}$ is G with a certain loop configuration. We show that without loss of generality we may assume $r = 0$. Since $\delta(\mathfrak{G}) \geq 1$, every vertex in $\overline{K_r}$ must have a loop. Suppose first that the simple graph of $\mathfrak{G}$ is $G = \overline{K_r}$. Since $\mathfrak{G}$ is $\mathfrak{F}_1$-free, $r \leq 2$, and so $\mathfrak{G} = \mathfrak{K}_1 \dot\cup \mathfrak{K}_1$ or $\mathfrak{G} = \mathfrak{K}_1$, both of which have the required form. Now suppose that $\mathfrak{G} = \mathfrak{H} \dot\cup (\overline{K_r})^\ell$ with $|\mathfrak{H}| \geq 1$ and $r \geq 1$. Since $\mathfrak{G}$ is $\{\mathfrak{F}_2, \mathfrak{F}_3\}$-free, every non-loop edge of $\mathfrak{H}$ must have loops on both of its endpoints. Since $\delta(\mathfrak{G}) \geq 1$ and $\mathfrak{G}$ is $\mathfrak{F}_1$-free, $\mathfrak{H}$ can have at most one connected component and $r = 1$. If $\mathfrak{H} \neq \mathfrak{K}_s$, then there would be some pair of vertices v and u such that $\mathfrak{H}$ does not contain the edge uv, in which case $\mathfrak{G}$ would contain $\mathfrak{F}_1$. So $\mathfrak{H} = \mathfrak{K}_s$, and $\mathfrak{G} = \mathfrak{K}_s \dot\cup \mathfrak{K}_1$, which has the required form. Since the cases with $r \geq 1$ all have the required form, we now assume $r = 0$.

Thus we assume $\mathfrak{G}$ has the form $K_{s_1,s_2} \vee (K_{p_1} \dot\cup K_{q_1}) \vee \cdots \vee (K_{p_k} \dot\cup K_{q_k})$ with some loop configuration, so the complement of $\mathfrak{G}$ is $K_{s_1} \dot\cup K_{s_2} \dot\cup K_{p_1,q_1} \dot\cup \cdots \dot\cup K_{p_k,q_k}$ with the complementary loop configuration. A loop graph is $\mathcal{F}_{\text{mr2}}$-free if and only if its complement is $\mathcal{H}_{\text{mr2}}$-free for the set $\mathcal{H}_{\text{mr2}}$ shown in Figure 2.2.

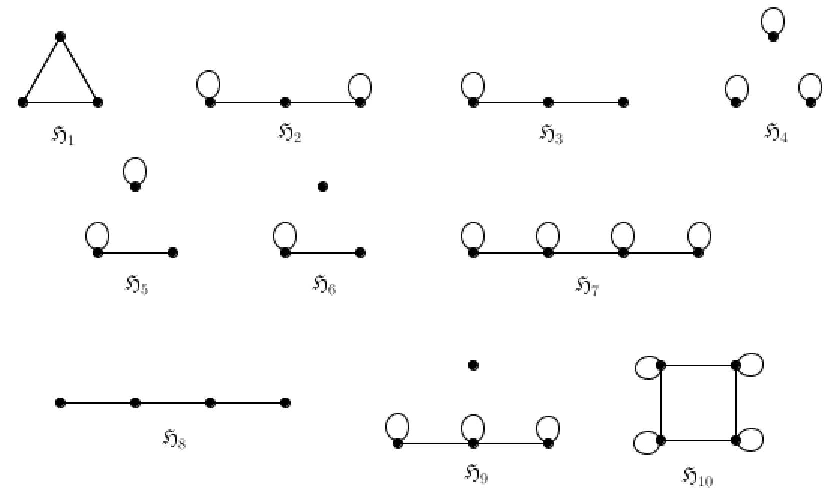

FIG. 2.2. *The set $\mathcal{H}_{\mathrm{mr2}} = \{\mathfrak{H}_1, \ldots, \mathfrak{H}_{10}\}$ of complements of forbidden induced subgraphs for minimum rank ≤ 2, where $\mathfrak{H}_i$ is the complement of $\mathfrak{F}_i$ in Figure 2.1.*

Consider a matrix $M \in \mathcal{S}(\overline{\mathfrak{G}})$, which has the form

$$M = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_k \end{bmatrix}$$

where $A_i \in \mathcal{S}(K_{s_i})$, $B_i \in \mathcal{S}(K_{p_i,q_i})$, and all other entries are zero. We now want to consider the diagonals of these block matrices of type A, representing a complete simple graph, and type B, representing a complete bipartite simple graph, given that $\overline{\mathfrak{G}}$ does not contain any of the subgraphs $\mathfrak{H}_1, \ldots, \mathfrak{H}_{10}$ of Figure 2.2. With the allowed forms of block matrices, we show $M \in \mathcal{S}(\mathfrak{K}_{s_1} \dot\cup \mathfrak{K}_{s_2} \dot\cup \mathfrak{K}_{p_1,q_1} \dot\cup \cdots \dot\cup \mathfrak{K}_{p_k,q_k})$ for appropriate s_i, p_i, q_i, k.

A matrix of type A represents a complete simple graph, so all off-diagonal entries are nonzero. If there are three loopless vertices in the complete graph, the graph contains $\mathfrak{H}_1$. So the zero-nonzero pattern of a matrix of type A have one of the three following forms A_α, $\alpha \in \{a, b, c\}$:

$$A_a = \begin{bmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix}, \; A_b = \begin{bmatrix} 0 & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}, \; A_c = \begin{bmatrix} 0 & * & * & \cdots & * \\ * & 0 & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

Let s_α denote the dimension of the matrix A_α for $\alpha \in \{a, b, c\}$. Notice that the graphs corresponding to A_b and A_c already contain a K_2 with one loop if $s_b \geq 2$ and $s_c \geq 3$. If we take the disjoint union one of these graphs with any other graph, the union will contain $\mathfrak{H}_5$ or $\mathfrak{H}_6$. So if A_b appears, then $M = A_b$, and similarly for A_c. Observe that if $M = A_b$ with $s_b \geq 2$ or $M = A_c$ with $s_c \geq 3$, then $\delta(\mathfrak{G}) = 0$. So, we need consider only A_a of any size, A_b with $s_b = 1$ (this matrix represents an isolated vertex without a loop), and A_c with $s_c = 2$ (this matrix represents two isolated vertices without loops).

A matrix of type B represents a $K_{p,q}$ with a certain number of loops. We cannot have more than two vertices with loops in either one of the partition sets, because the vertices of a partition are not connected and with at least 3 loops in one partition set, $\mathfrak{G}(B)$ would contain $\mathfrak{H}_4$. Therefore B must have one of the following six forms:

$$B_a = \begin{bmatrix} & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \\ * & \cdots & * & & & \\ \vdots & \ddots & \vdots & & & \\ * & \cdots & * & & & \end{bmatrix}, \; B_b = \begin{bmatrix} * & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \\ * & \cdots & * & & & \\ \vdots & \ddots & \vdots & & & \\ * & \cdots & * & & & \end{bmatrix},$$

$$B_c = \begin{bmatrix} * & & & & * & \cdots & * \\ & * & & & & & \\ & & & & \vdots & \ddots & \vdots \\ & & & & * & \cdots & * \\ * & & \cdots & * & & & \\ \vdots & & \ddots & \vdots & & & \\ * & & \cdots & * & & & \end{bmatrix}, \; B_d = \begin{bmatrix} * & & & * & \cdots & * \\ & & & \vdots & \ddots & \vdots \\ & & & * & \cdots & * \\ * & \cdots & * & * & & \\ \vdots & \ddots & \vdots & & & \\ * & \cdots & * & & & \end{bmatrix},$$

$$B_e = \begin{bmatrix} * & & & & * & \cdots & * \\ & * & & & & & \\ & & & & \vdots & \ddots & \vdots \\ & & & & * & \cdots & * \\ * & & \cdots & * & * & & \\ \vdots & & \ddots & \vdots & & & \\ * & & \cdots & * & & & \end{bmatrix}, \; B_f = \begin{bmatrix} * & & & & * & & \cdots & * \\ & * & & & & & & \\ & & & & \vdots & & \ddots & \vdots \\ & & & & * & & \cdots & * \\ * & & \cdots & * & * & & & \\ & & & & & * & & \\ \vdots & & \ddots & \vdots & & & & \\ * & & \cdots & * & & & & \end{bmatrix}.$$

Let p_β and q_β be the number of vertices in the two partitions for $\beta \in \{a, b, c, d, e, f\}$. Notice that if we allow either p_β or q_β to be zero, the corresponding matrix represents a union with isolated vertices (with or without loops). Since we handle this case separately, here we assume $p_\beta \geq 1$ and $q_\beta \geq 1$. If the matrices other than B_a are too big, we show that the corresponding bipartite graphs have an induced $\mathfrak{H}_2$ or $\mathfrak{H}_3$, and so are prohibited:

If $p_b \geq 2$, then $\mathfrak{G}(B_b)$ contains $\mathfrak{H}_3$ (since we assume $q_b \geq 1$). For $p_b = 1, q_b \geq 1$, $\mathfrak{G}(B_b)$ contains a K_2 with one loop. Thus, in this case we cannot have a union with other graphs because the union would contain $\mathfrak{H}_5$ or $\mathfrak{H}_6$, so $M = B_b$. But $\mathfrak{G}(B_b) = \mathfrak{K}_1 \vee \overline{\mathfrak{K}_{q_b}}$, so $\delta(\mathfrak{G}) = 0$ and this case is excluded.

By construction of B_c, $p_c \geq 2$, so $\mathfrak{G}(B_c)$ contains $\mathfrak{H}_2$ (since we assume $q_c \geq 1$), and this case is excluded.

If $p_d \geq 2$ and $q_d \geq 2$, $\mathfrak{G}(B_d)$ contains $\mathfrak{H}_3$, so without loss of generality $p_d = 1$. If $q_d \geq 2$ the corresponding graph contains a K_2 with one loop and can therefore not be in a union with another graph, and as for B_b this case is excluded. So B_d can only appear with $p_d = 1, q_d = 1$, and $\mathfrak{G}(B_d) = \mathfrak{K}_2$.

By construction of B_e, $p_e \geq 2$. For $q_e \geq 2$, $\mathfrak{G}(B_e)$ contains $\mathfrak{H}_2$. So B_e can only appear with $p_e \geq 2$ and $q_e = 1$. But then $\mathfrak{G}(B_e)$ contains an induced $\mathfrak{P}_3^\ell$. So if we have a union of $\mathfrak{G}(B_e)$ with another graph, the union contains $\mathfrak{H}_4$ or $\mathfrak{H}_9$. Thus $M = B_e$ and $\mathfrak{G}(B_e) = (\mathfrak{K}_1 \dot\cup \mathfrak{K}_1 \dot\cup \overline{\mathfrak{K}_1} \dot\cup \cdots \dot\cup \overline{\mathfrak{K}_1}) \vee \mathfrak{K}_1$, so $\delta(\mathfrak{G}) = 0$ and this case is excluded.

Also notice that $\mathfrak{G}(B_f)$ already contains $\mathfrak{H}_{10}$, so B_f does not appear.

Thus the types of B matrices that can occur are B_a for any size, B_d for $p_d = 1$, $q_d = 1$ (so $\mathfrak{G}(B_d) = \mathfrak{K}_2$), or matrices that represent isolated vertices, with a total of at most two loops.

We can consider B_d with $p_d = 1$, $q_d = 1$, and an isolated vertex with a loop, as being type A_a. Isolated vertices without loops can be viewed as type B_a with $q_a = 0$ (which we now allow). Thus all permissible forms of M can be constructed using only blocks of type A_a and B_a. If we take a block diagonal matrix that includes nonzero diagonal entries in three distinct blocks, the graph contains an induced $\mathfrak{H}_4$. Therefore we can only combine blocks such that at most two of the A_a blocks appear. To summarize, we can combine up to two matrices of type A_a with a arbitrary number of matrices of type B_a. Hence, $\mathfrak{G}(M)$ has the required form (2.1).

(2.1)⇔(2.2) is immediate and (2.2)⇒(1) follows from the proof of [5, Theorem 2], because the construction of a matrix $C \in \mathcal{S}(K_{s_1,s_2} \vee (K_{p_1} \dot\cup K_{q_1}) \vee \cdots \vee (K_{p_k} \dot\cup K_{q_k}))$ with $\operatorname{rank} C \leq 2$ actually shows $C \in \mathcal{S}(\mathfrak{K}_{s_1,s_2} \vee (\mathfrak{K}_{p_1} \dot\cup \mathfrak{K}_{q_1}) \vee \cdots \vee (\mathfrak{K}_{p_k} \dot\cup \mathfrak{K}_{q_k}))$. □

3. High minimum rank. In this section we extend the characterization of minimum rank equal to order for loopless loop graphs given in [16] to all loop graphs.

THEOREM 3.1. *For every loop graph* $\mathfrak{G}$*,* $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$ *if and only if* $\mathfrak{G}$ *has a unique spanning generalized cycle.*

Proof. If $\mathfrak{G}$ has a unique spanning generalized cycle, then $\det A \neq 0$ for all $A \in \mathcal{S}(\mathfrak{G})$, so $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$. Now suppose that $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$. If $\mathfrak{G}$ is a loop graph without loops, then $\mathrm{mr}(\mathfrak{G}) = \mathrm{mr}_0(G)$, where G denotes $\mathfrak{G}$ viewed as a simple graph and $\mathrm{mr}_0(G)$ is the zero diagonal minimum rank as defined in [16], and the result is established by Theorem 3.9 of the same paper. Thus, we are left to consider the case when $\mathfrak{G}$ contains at least one loop. Suppose there is a loop graph $\mathfrak{G}$ with $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$ that does not have a unique spanning generalized cycle. Let $\mathfrak{H}_* = (V_*, E_*)$ be a minimum counterexample in the sense that every loop graph $\mathfrak{G}$ on fewer than $|\mathfrak{H}_*|$ vertices having $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$ necessarily has a unique spanning generalized cycle, and every loop graph on $|\mathfrak{H}_*|$ vertices with fewer edges also has this property. Denote the order of $\mathfrak{H}_*$ by n_*. Next, observe that $\mathfrak{H}_*$ has at least two spanning generalized cycles, since at least one spanning generalized cycle is guaranteed by Remark 1.1.

Now let v be a vertex in $\mathfrak{H}_*$ such that $\ell := vv \in E(\mathfrak{H}_*)$. If ℓ is contained in every spanning generalized cycle of $\mathfrak{H}_*$, then by deleting ℓ and v, there is a one-to-one correspondence between the spanning generalized cycles of $\mathfrak{H}_*$ and those of $\mathfrak{H}_* - v$, so $\mathfrak{H}_* - v$ has at least two spanning generalized cycles. We obtain $A(v) \in \mathcal{S}(\mathfrak{H}_* - v)$ from $A \in \mathcal{S}(\mathfrak{H}_*)$ by deleting the row and the column corresponding to v. Moreover, $\det A(v) = \frac{1}{a_{vv}} \det A \neq 0$. Therefore, $\mathfrak{H}_* - v$ does not have a unique spanning generalized cycle and $\mathrm{mr}(\mathfrak{H}_* - v) = |\mathfrak{H}_* - v|$, violating the minimality of $\mathfrak{H}_*$. Similarly, if no spanning generalized cycle contains ℓ, then $\mathfrak{H}_*$ and $\mathfrak{H}_* - \ell$ have the same set of spanning generalized cycles. In this case, we obtain $\mathcal{S}(\mathfrak{H}_* - \ell)$ by setting the v,v-entry of each matrix in $\mathcal{S}(\mathfrak{H}_*)$ to zero. Since ℓ does not participate in any spanning generalized cycle, this action does not affect the determinant. Again, $\mathfrak{H}_* - \ell$ is a smaller counterexample. Thus, we are left to consider the case when $\mathfrak{H}_*$ has both a spanning generalized cycle $\mathcal{C}^{(1)}$ that contains ℓ and a spanning generalized cycle $\mathcal{C}^{(2)}$ that doesn't contain ℓ. Let $t = |E_*|$ and $Y = [y_{uw}]$ be a symmetric matrix of indeterminates $x_1, x_2, ..., x_t$ such that $\mathfrak{G}(Y) = \mathfrak{H}_*$ (so $uw \in E_*$ implies $y_{uw} = y_{wu} = x_i$ for some x_i); without loss of generality, let $y_{vv} = x_1$. Then the determinant of Y is a homogeneous polynomial of degree n_* in $x_1, x_2, ..., x_t$ and we can express it as

$$\det Y = x_1 p(x_2, ..., x_t) + q(x_2, ..., x_t)$$

Further, since $\ell \in \mathcal{C}^{(1)}$ and $\ell \notin \mathcal{C}^{(2)}$, neither $p(x_2, ..., x_t)$ nor $q(x_2, ..., x_t)$ is identically zero. Hence, $p(x_2, ..., x_t)q(x_2, ..., x_t) \not\equiv 0$. Thus, by [16, Lemma 3.4], there exist nonzero real numbers $c_2, ..., c_t$ such that $p(c_2, ..., c_t)q(c_2, ..., c_t) \neq 0$. Now define the matrix A by replacing $y_{uw} = x_i$ with c_i for $i = 2, ..., t$ and $y_{vv} = x_1$ with $\frac{-q(c_2,...,c_t)}{p(c_2,...,c_t)}$. Then $A \in \mathcal{S}(\mathfrak{H}_*)$ and $\det A = 0$ so $\mathrm{mr}(\mathfrak{H}_*) \leq \operatorname{rank} A \leq n_* - 1$, contradicting our assumption that $\mathrm{mr}(\mathfrak{H}_*) = n_*$. □

Remark 3.2. If $\mathfrak{G}$ does not have a unique generalized cycle and there is a vertex u of $\mathfrak{G}$ such that $\mathfrak{G}-u$ has a unique spanning generalized cycle, then $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|-1$.

Example 3.3. The converse of Remark 3.2 is false because $\mathfrak{P}_4^\ell$, the path on four vertices with a loop at each vertex, has $\mathrm{mr}(\mathfrak{P}_4^\ell) = 3$ but every induced subgraph on 3 vertices has minimum rank 2.

4. Schur complement reduction. In this section we use the Schur complement to develop a reduction lemma that allows the removal of two vertices, reducing the order of the graph. This technique was used in [22]. The next result is well known.

Lemma 4.1. [24, p. 217] *Suppose that $A \in \mathbb{R}^{k\times k}$ is invertible, $B \in \mathbb{R}^{(n-k)\times k}$, and $D \in \mathbb{R}^{(n-k)\times(n-k)}$. Then*

$$\mathrm{rank}\begin{bmatrix} A & B^\top \\ B & D \end{bmatrix} = \mathrm{rank}\begin{bmatrix} A & 0 \\ 0 & D - BA^{-1}B^\top \end{bmatrix} = \mathrm{rank}\, A + \mathrm{rank}(D - BA^{-1}B^\top)$$

For a loop graph $\mathfrak{G}$ that does not have an edge between vertices u and v (this includes the case of a loop when $u = v$), $\mathfrak{G}+uv$ denotes the loop graph obtained from $\mathfrak{G}$ by adding edge uv. Analogously, if $\mathfrak{G}$ does have edge uv , $\mathfrak{G}-uv$ denotes the loop graph obtained from $\mathfrak{G}$ by deleting edge uv (again the loop uu is permitted).

Lemma 4.2. (P_4 reduction) *Suppose that in the underlying simple graph G of a loop graph $\mathfrak{G}$, $P = (x, y, z, w)$ is an induced path and $\deg_G y = \deg_G z = 2$. Let $\mathfrak{G}'$ be the loop graph obtained from $\mathfrak{G}$ by deleting vertices y and z and adding an edge between x and w.*

(1) *If neither y nor z has a loop in $\mathfrak{G}$, then* $\mathrm{mr}(\mathfrak{G}) = \mathrm{mr}(\mathfrak{G}') + 2$.
(2) *If z has a loop in $\mathfrak{G}$ but y does not, then*

$$\mathrm{mr}(\mathfrak{G}) = \begin{cases} \mathrm{mr}(\mathfrak{G}' + xx) + 2 & \text{if } x \text{ does not have a loop;} \\ \min\{\mathrm{mr}(\mathfrak{G}'), \mathrm{mr}(\mathfrak{G}' - xx)\} + 2 & \text{if } x \text{ has a loop.} \end{cases}$$

Proof. We can describe all cases as $\mathrm{mr}(\mathfrak{G}) = \mathrm{mr}(\mathfrak{H})+2$ where $\mathfrak{H}$ is $\mathfrak{G}'$ except with a loop or no loop on x as specified. We establish the equality $\mathrm{mr}(\mathfrak{G}) = \mathrm{mr}(\mathfrak{H}) + 2$ by showing that $\mathrm{mr}(\mathfrak{G}) \geq \mathrm{mr}(\mathfrak{H})+2$ and $\mathrm{mr}(\mathfrak{G}) \leq \mathrm{mr}(\mathfrak{H})+2$. Let $n := |\mathfrak{G}|$ and define the $2\times(n-2)$ matrix $B := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}$. Order the vertices of $\mathfrak{G}$ so that y, z, x, w are the first four vertices (in that order), choose any order for the remaining vertices, and let $\alpha := \{y, z\}$.

For the lower bound on $\mathrm{mr}(\mathfrak{G})$, we choose $A = [a_{ij}] \in \mathcal{S}(\mathfrak{G})$ with $\mathrm{rank}\, A = \mathrm{mr}(\mathfrak{G})$ and partition A as $\begin{bmatrix} A[\alpha] & A[\alpha,\alpha) \\ A(\alpha,\alpha] & A(\alpha) \end{bmatrix}$. By Remark 1.3 applied to the forest

$\mathfrak{T} = (\{x, y, z, w\}, \{xy, zw\})$, we may assume that $A(\alpha, \alpha]^\top = A[\alpha, \alpha) = B$. Since y is adjacent to z in $\mathfrak{G}$ and in all cases y does not have loop in $\mathfrak{G}$, $A[\alpha]$ is invertible. We then define $C = A(\alpha) - A(\alpha, \alpha]A[\alpha]^{-1}A[\alpha, \alpha) = A(\alpha) - (A[\alpha]^{-1} \oplus 0)$. Note that $(A(\alpha))_{xw} = 0$ since x and w are not adjacent in $\mathfrak{G}$, so $C_{xw} = (A(\alpha))_{xw} - (A[\alpha]^{-1})_{yz} \neq 0$. In each case we show that the loop configuration is such that $C \in \mathcal{S}(\mathfrak{H})$. Then $\operatorname{rank} A = \operatorname{rank} C + 2$ by Lemma 4.1, so $\operatorname{mr}(\mathfrak{G}) = \operatorname{rank} A = \operatorname{rank} C + 2 \geq \operatorname{mr}(\mathfrak{H}) + 2$.

For the upper bound on $\operatorname{mr}(\mathfrak{G})$, we choose a matrix $C = [c_{ij}] \in \mathcal{S}(\mathfrak{H})$ with $\operatorname{rank} C = \operatorname{mr}(\mathfrak{H})$, noting that since x is adjacent to w in $\mathfrak{G}'$, so the entry c_{xw} is nonzero. We then construct a matrix $A \in \mathcal{S}(\mathfrak{G})$ defined by $A(\alpha, \alpha]^\top = A[\alpha, \alpha) = B$ and $A(\alpha) = C + A(\alpha, \alpha]A[\alpha]^{-1}A[\alpha, \alpha) = C + (A[\alpha]^{-1} \oplus 0)$. The choice of $A[\alpha]$ depends on the case, but in all cases $A[\alpha]$ is invertible and $(A[\alpha])_{yz} \neq 0$; $A[\alpha]$ is chosen so that $(A(\alpha))_{xw} = C_{xw} + (A[\alpha]^{-1})_{yz} = 0$. In each case we show that the loop configuration is such that $A \in \mathcal{S}(\mathfrak{G})$. Then $\operatorname{rank} A = \operatorname{rank} C + 2$ by Lemma 4.1, so $\operatorname{mr}(\mathfrak{G}) \leq \operatorname{rank} A = \operatorname{rank} C + 2 = \operatorname{mr}(\mathfrak{H}) + 2$.

Case (1): Neither y nor z has a loop in $\mathfrak{G}$. For the lower bound on $\operatorname{mr}(\mathfrak{G})$, $A[\alpha]$ has the form $\begin{bmatrix} 0 & a_{yz} \\ a_{yz} & 0 \end{bmatrix}$ and $A[\alpha]^{-1} = \begin{bmatrix} 0 & \frac{1}{a_{yz}} \\ \frac{1}{a_{yz}} & 0 \end{bmatrix}$, so $C \in \mathcal{S}(\mathfrak{G}')$. For the upper bound on $\operatorname{mr}(\mathfrak{G})$, define $A[\alpha] := -\begin{bmatrix} 0 & \frac{1}{c_{xw}} \\ \frac{1}{c_{xw}} & 0 \end{bmatrix}$, so $A \in \mathcal{S}(\mathfrak{G})$.

Case (2): z has a loop in $\mathfrak{G}$ but y does not. For the lower bound on $\operatorname{mr}(\mathfrak{G})$, $A[\alpha] = \begin{bmatrix} 0 & a_{yz} \\ a_{yz} & a_{zz} \end{bmatrix}$, which is invertible with $A[\alpha]^{-1} = \frac{1}{\det A[\alpha]} \begin{bmatrix} a_{zz} & -a_{yz} \\ -a_{yz} & 0 \end{bmatrix}$. If x has no loop, then the x, x-entry of C is $0 - \frac{a_{zz}}{\det A[\alpha]}$, which is nonzero; if x has a loop, then the x, x-entry of C is $a_{xx} - \frac{a_{zz}}{\det A[\alpha]}$, which can be zero or nonzero. Therefore,

$$\operatorname{mr}(\mathfrak{G}) \geq \begin{cases} \operatorname{mr}(\mathfrak{G}' + xx) + 2 & \text{when } x \text{ has no loop;} \\ \min\{\operatorname{mr}(\mathfrak{G}'), \operatorname{mr}(\mathfrak{G}' - xx)\} + 2 & \text{when } x \text{ has a loop.} \end{cases}$$

For the upper bound on $\operatorname{mr}(\mathfrak{G})$, when x has no loop, let $C = [c_{ij}] \in \mathcal{S}(\mathfrak{G}' + xx)$ be a matrix with $\operatorname{rank} C = \operatorname{mr}(\mathfrak{G} + xx)$. We define $A[\alpha] := -\begin{bmatrix} c_{xx} & c_{xw} \\ c_{xw} & 0 \end{bmatrix}^{-1}$. Then $A \in \mathcal{S}(\mathfrak{G})$, establishing the upper bound in this subcase. Now assume that x has a loop and let $C = [c_{ij}]$ be a matrix in $\mathcal{S}(\mathfrak{G}')$ or $\mathcal{S}(\mathfrak{G}' - xx)$ with $\operatorname{rank} C = \min\{\operatorname{mr}(\mathfrak{G}'), \operatorname{mr}(\mathfrak{G}' - xx)\}$. We define $A[\alpha]$ by

$$A[\alpha] := \begin{cases} -\begin{bmatrix} 1 & c_{xw} \\ c_{xw} & 0 \end{bmatrix}^{-1} & \text{when } c_{xx} = 0; \\ -\begin{bmatrix} 2c_{xx} & c_{xw} \\ c_{xw} & 0 \end{bmatrix}^{-1} & \text{when } c_{xx} \neq 0. \end{cases}$$

Then $A \in \mathcal{S}(\mathfrak{G})$, so $\operatorname{mr}(\mathfrak{G}) \leq \min\{\operatorname{mr}(\mathfrak{G}'), \operatorname{mr}(\mathfrak{G}' - xx)\} + 2$. □

5. Minimum rank for families of graphs. In this section we establish the minimum rank of a loop graph consisting of a simple path P_n, cycle C_n, or complete graph K_n with an arbitrary configuration of loops. We use the symbol $\mathfrak{P}_n$ (respectively, $\mathfrak{C}_n$) to denote P_n (respectively, C_n) with a given loop configuration, and $\mathfrak{K}_n^{\ell(s)}$ to denote the loop graph obtained from the simple complete graph on n vertices by adding a loop to each of s vertices (so $n - s$ vertices do not have loops); $\mathfrak{K}_n^{\ell(n)} = \mathfrak{K}_n$. When the vertices are numbered 1 to n, we say a vertex or loop is *odd* or *even* according as the number of its vertex is odd or even.

5.1. Path $\mathfrak{P}_n$. A path is a tree, so $\mathrm{M}(\mathfrak{P}_n) = \mathrm{Z}(\mathfrak{P}_n)$ [20]; thus $\mathrm{mr}(\mathfrak{P}_n)$ can be computed by using the zero forcing number. Here we give an explicit characterization. Given a path, a *numbering* of the vertices is defined by starting at one end with the number 1 and proceeding along the path, numbering the vertices consecutively (so $\mathfrak{P}_n$ has two numberings). Observe that for n odd, the parity of a vertex is the same in both numberings, whereas for n even the two numberings reverse the roles of odd and even in addition to reversing the order of the vertices.

Proposition 5.1. *For n odd,*

$$\mathrm{mr}(\mathfrak{P}_n) = \begin{cases} n & \textit{if } \mathfrak{P}_n \textit{ has is a unique odd loop;} \\ n-1 & \textit{otherwise.} \end{cases}$$

For n even,

$$\mathrm{mr}(\mathfrak{P}_n) = \begin{cases} n & \textit{if all odd loops of } \mathfrak{P}_n \textit{come after all even loops;} \\ n-1 & \textit{otherwise.} \end{cases}$$

Proof. Note that $n - 1 = \mathrm{mr}(P_n) \leq \mathrm{mr}(\mathfrak{P}_n)$, and by Theorem 3.1, $\mathrm{mr}(\mathfrak{P}_n) = n$ if and only if $\mathfrak{P}_n$ has a unique spanning generalized cycle. First suppose n is odd. Each odd loop vv can be associated with one spanning generalized cycle consisting of that loop and the edges (with endpoints) in perfect matching(s) of the component(s) of $\mathfrak{P}_n - v$, so $\mathfrak{P}_n = n$ if and only if $\mathfrak{P}_n$ has a unique odd loop. Now suppose n is even. Then $\mathfrak{P}_n$ has a spanning generalized cycle consisting of alternate edges, and has additional spanning generalized cycles(s) if and only if $\mathfrak{P}_n$ has an odd loop before an even loop. □

5.2. Cycle $\mathfrak{C}_n$. First note that $n - 2 = \mathrm{mr}(C_n) \leq \mathrm{mr}(\mathfrak{C}_n)$ (regardless of loop configuration). Given a cycle, a *numbering* of the vertices is defined by selecting one vertex to number 1 and proceeding around the cycle, numbering the vertices consecutively (a given cycle has many numberings). The property of having a numbering with a unique odd loop is used to characterize $\mathrm{mr}(\mathfrak{C}_n)$, but first we need a lemma.

Lemma 5.2. $\mathrm{mr}(\mathfrak{C}_n^{\ell}) = n - 2$.

Proof. The adjacency matrix A_{C_n} has eigenvalues $2\cos(\frac{2\pi k}{n})$ for $k = 1, \ldots, n$ [11]. For $n \neq 4$, $\cos(\frac{2\pi}{n}) = \cos\left(\frac{2\pi(n-1)}{n}\right) \neq 0$, so $A_{C_n} - 2\cos(\frac{2\pi}{n})I \in \mathcal{S}(\mathfrak{C}_n^\ell)$ and $\operatorname{rank}(A_{C_n} - 2\cos(\frac{2\pi}{n})I) = n-2 \geq \operatorname{mr}(\mathfrak{C}_n^\ell)$. For $n = 4$, $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 1 \end{bmatrix} \in \mathcal{S}(\mathfrak{C}_4^\ell)$ and $\operatorname{rank} A = 2$. □

OBSERVATION 5.3. *If n is even, then the underlying simple graph C_n is bipartite, and $\mathfrak{C}_n$ has a numbering with exactly one odd loop if and only if at least one of the two partite sets has exactly one loop.*

THEOREM 5.4.

$$\operatorname{mr}(\mathfrak{C}_n) = \begin{cases} n & \text{if } n \text{ is odd and } \mathfrak{C}_n \text{ has no loops;} \\ n-1 & \text{if } \mathfrak{C}_n \text{ has a numbering with exactly one odd loop;} \\ n-2 & \text{otherwise.} \end{cases}$$

If $\operatorname{mr}(\mathfrak{C}_n) = n-1$ then there exists a vertex v such that $\mathfrak{C}_n - v$ has a unique spanning generalized cycle. Furthermore, $\operatorname{M}(\mathfrak{C}_n) = \operatorname{Z}(\mathfrak{C}_n)$ unless n is odd and $\mathfrak{C}_n$ has no loops.

Proof. By Theorem 3.1, $\operatorname{mr}(\mathfrak{C}_n) = n$ if and only if $\mathfrak{C}_n$ has a unique spanning generalized cycle. If n is odd and $\mathfrak{C}_n$ is loopless, then $\mathfrak{C}_n$ has a unique spanning generalized cycle and $\operatorname{mr}(\mathfrak{C}_n) = n$. If n is odd and $\mathfrak{C}_n$ has at least one loop, then $\mathfrak{C}_n$ has at least two spanning generalized cycles (the cycle itself and a loop with a perfect matching on the remaining vertices), so $\operatorname{mr}(\mathfrak{C}_n) \leq n-1$. If n is even, then $\mathfrak{C}_n$ has at least three spanning generalized cycles (the cycle itself and two perfect matchings), so $\operatorname{mr}(\mathfrak{C}_n) \leq n-1$. If n is even and $\mathfrak{C}_n$ has no loops, then $\operatorname{mr}(\mathfrak{C}_n) = \operatorname{mr}_0(C_n) = n-2$ [16]. Henceforth, we assume $\mathfrak{C}_n$ has a loop, and thus $n-2 = \operatorname{mr}(C_n) \leq \operatorname{mr}(\mathfrak{C}_n) \leq n-1$.

Suppose $\mathfrak{C}_n$ has a numbering with a unique odd loop; without loss of generality this loop is at vertex 1. We apply Proposition 5.1 to $\mathfrak{P}_{n-1} := \mathfrak{C}_n - 2$ to show that $\operatorname{mr}(\mathfrak{C}_n - 2) = n-1$, implying $\operatorname{mr}(\mathfrak{C}_n) \geq n-1$. We use the numbering of $\mathfrak{P}_{n-1}$ determined by fixing 1 and renumbering everything else. If n is even, the vertices retain their parity under this renumbering and 1 is the only odd loop in $\mathfrak{P}_{n-1}$, which has odd order. If n is even then fixing 1 and renumbering the remaining vertices causes all other vertices to change parity. Since 1 is the only odd loop in $\mathfrak{C}_n$, there are no even loops in $\mathfrak{P}_{n-1}$, which has even order, so vacuously every odd loop is after every even loop.

Now assume that $\mathfrak{C}_n$ has a loop and no numbering has a unique odd loop. We show $\operatorname{mr}(\mathfrak{C}_n) = n-2$; note that this implies $\operatorname{M}(\mathfrak{C}_n) = \operatorname{Z}(\mathfrak{C}_n) = 2$, because any set of two consecutive vertices is a zero forcing set. The proof that $\operatorname{mr}(\mathfrak{C}_n) = n-2$ is by induction on the number of vertices using P_4 reduction (Lemma 4.2). A numbering on $\mathfrak{C}_n$ naturally induces a numbering on $\mathfrak{C}'_n$ by reducing every number greater than those assigned to y and z by two ($\mathfrak{C}'_n$ denotes the graph produced by the reduction); this does not change the parity of any vertex or loop. Since P_4 reduction reduces the order by two, we consider $n = 3$ and $n = 4$ as the base cases. The case $n = 3$ is clear, because $\text{mr}(\mathfrak{C}_3^\ell) = 1$ and $\mathfrak{C}_3^\ell$ is the only loop configuration with at least one loop and no numbering having exactly one odd loop. For $n = 4$, the possible loop configurations are all loops (i.e., $\mathfrak{C}_4^\ell$) or two nonadjacent loops; we denote the latter by $\mathfrak{C}_4^{(2)}$. By Lemma 5.2, $\text{mr}(\mathfrak{C}_4^\ell) = 2$. For $\mathfrak{C}_4^{(2)}$, define $A := A_{C_4} + \text{diag}(-1, 0, 1, 0) \in \mathcal{S}(\mathfrak{C}_4^{(2)})$; $\text{rank}\, A = 2$ so $\text{mr}(\mathfrak{C}_4^{(2)}) = 2$.

Now assume the theorem holds for all k with $3 \le k \le n-2$ and consider $\mathfrak{C}_n$, which by assumption has a loop and no numbering has a unique odd loop. If $\mathfrak{C}_n = \mathfrak{C}_n^\ell$, then $\text{mr}(\mathfrak{C}_n) = n-2$ by Lemma 5.2. If $\mathfrak{C}_n$ has two consecutive vertices without loops, then we apply P_4 reduction with y and z as loopless vertices; $\mathfrak{C}'_n$ inherits the property of not having a numbering with a unique odd loop, so we can apply the induction hypothesis. So assume $\mathfrak{C}_n$ has at least one vertex with no loop and does not have two consecutive vertices without loops (in addition to assuming $\mathfrak{C}_n$ has at least one loop and no numbering has a unique odd loop). We consider the cases n even and n odd separately.

Suppose first that n is even, so C_n is bipartite; denote the partite sets by X and Y. In $\mathfrak{C}_n$, neither X nor Y has exactly one loop and without loss of generality Y has a loopless vertex. Select a loopless vertex $y \in Y$ and preform P_4 reduction. Define $X' := X \setminus \{z\}$ and $Y' := Y \setminus \{y\}$. Note that Y' does not have exactly one loop. If X has exactly two loops, they are on vertices x and z, so X' has no loops in $\mathfrak{C}'_n - xx$. If X has more than two loops, then X' has at least two loops in $\mathfrak{C}'_n$. So in one of $\mathfrak{C}'_n - xx$ or $\mathfrak{C}'_n$, neither X' nor Y' has exactly one loop, and we can apply induction to conclude that $\text{mr}(\mathfrak{C}'_n - xx) = n - 4$ or $\text{mr}(\mathfrak{C}'_n) = n - 4$ and thus $\text{mr}(\mathfrak{C}_n) = n - 2$.

Finally suppose n is odd and examine the loop configuration of $\mathfrak{C}_n$. We consider maximal segments of consecutive vertices all having loops, which we call *loop segments*. Recall that $\mathfrak{C}_n$ has at least one loop and at least one vertex with no loop, does not have two consecutive vertices without loops, and no numbering has a unique odd loop. Because n is odd and $n \ge 5$, these properties imply that $\mathfrak{C}_n$ must have at least one of the following: (i) A loop segment with at least 4 vertices. (ii) Three or more separate loop segments with at least 2 vertices each. (iii) A loop segment with 3 vertices and a separate loop segment with at least 2 vertices. Choose y to be a loopless vertex adjacent to a loop segment with the greatest number of vertices, and let x denote the neighbor of y in this loop segment. Apply P_4 reduction to obtain $\mathfrak{C}'_n$. In each case, $\mathfrak{C}'_n$ has a loop segment with 4 or more vertices or has at least two loop segments with 2 or more vertices. Either of these is sufficient to imply every numbering has at least two odd loops, so $\text{mr}(\mathfrak{C}'_n) = n - 4$ and thus $\text{mr}(\mathfrak{C}_n) = n - 2$.

To establish the last statement it suffices to assume $\mathfrak{C}_n$ has a numbering with a unique odd loop and exhibit a zero forcing set of one vertex; without loss of generality the unique odd loop is at vertex 1. Then $\{2\}$ is a zero forcing set: Since 2 is blue, 3 has exactly one white neighbor, 4, so $3 \to 4$. We continue this process with $2k+1 \to 2k+2$ as the kth force, for $1 \leq k \leq \frac{n-2}{2}$. Thus all odd vertices except 1 are blue after $\left\lfloor \frac{n-2}{2} \right\rfloor$ forces. Then $1 \to 1$ n is even and $n \to 1$ if n is odd. Since there are now two consecutive blue vertices, we can completely color the cycle. Thus $1 \geq \mathrm{Z}(\mathfrak{C}_n) \geq \mathrm{M}(\mathfrak{C}_n) = 1$. □

5.3. Complete graph $\mathfrak{K}_n$ with deleted loops. The next result could be proved entirely from [16, Theorem 2.4] and Proposition 6.8 below, but instead we provide additional examples of optimal matrices.

Proposition 5.5.

$$\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) = \begin{cases} 3 & \textit{if } 3 \leq n-s; \\ 2 & \textit{if } 1 \leq n-s \leq 2 \leq n; \\ 1 & \textit{if } n-s=0 \textit{ and } 1 \leq n; \\ 0 & \textit{if } n-s=1=n. \end{cases}$$

Proof. Let $k := n - s$ and suppose first that $k \geq 3$. Then $\mathfrak{K}_3^{\ell(0)}$ (the loopless complete graph on 3 vertices) is an induced subgraph of $\mathfrak{K}_n^{\ell(s)}$, and $\mathrm{mr}(\mathfrak{K}_3^{\ell(0)}) = \mathrm{mr}_0(K_3) = 3$ by [16, Theorem 2.4]. Thus $\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) \geq 3$. For $\mathbf{v} \in \mathbb{R}^n$, let $(\mathbf{v})_i$ denote the ith coordinate of $\mathbf{v}$. Define the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbb{R}^n$ by

$$(\mathbf{v}_1)_i := \begin{cases} \sin \frac{i\pi}{2(k+1)}, \text{ if } i \leq k; \\ 1 \text{ if } i > k. \end{cases}$$

$$(\mathbf{v}_2)_i := \begin{cases} \cos \frac{i\pi}{2(k+1)}, \text{ if } i \leq k; \\ 1 \text{ if } i > k. \end{cases}$$

Also define $\mathbf{v}_3 = \mathbb{1}_n$ where $\mathbb{1}_n$ is the all ones n-vector.

Then we claim the matrix $A := \mathbf{v}_1\mathbf{v}_1^\top + \mathbf{v}_2\mathbf{v}_2^\top - \mathbf{v}_3\mathbf{v}_3^\top$ is a matrix in $\mathcal{S}(\mathfrak{K}_n^{\ell(s)})$ and $\operatorname{rank} A = 3$. Since A is the sum of three rank one matrices, $\operatorname{rank} A$ is less than or equal to 3. Therefore, it suffices to show $A \in \mathcal{S}(\mathfrak{K}_n^{\ell(s)})$. For $i, j \leq k$, $(A)_{ij} = \sin \frac{i\pi}{2(k+1)} \sin \frac{j\pi}{2(k+1)} + \cos \frac{i\pi}{2(k+1)} \cos \frac{j\pi}{2(k+1)} - 1 = \cos \frac{(i-j)\pi}{2(k+1)} - 1$, which is zero only when $i = j$. For $i > k, j \leq k$ (or $j > k, i \leq k$), $(A)_{ij} = \sin \frac{j\pi}{2(k+1)} + \cos \frac{j\pi}{2(k+1)} - 1 \neq 0$. For $i > k, j > k$, $(A)_{ij} = 1 + 1 - 1 = 1$.

In the case $n \geq 2 \geq k \geq 1$, it is clear that $\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) \geq 2$, and a rank 2 matrix may be constructed as follows (with J_m denoting the $m \times m$ all ones matrix): For

$k=1$: $\begin{bmatrix} J_{n-1} & \mathbb{1}_{n-1} \\ \mathbb{1}_{n-1}^{\top} & 0 \end{bmatrix}$. For $k=2$: $\begin{bmatrix} 2J_{n-2} & \mathbb{1}_{n-2} & \mathbb{1}_{n-2} \\ \mathbb{1}_{n-2}^{\top} & 0 & 1 \\ \mathbb{1}_{n-2}^{\top} & 1 & 0 \end{bmatrix}$. In the case $k=0, n \geq 1$, J_n has rank 1, and in the case $k=1=n$, the matrix $[0]$ has rank 0. □

6. Maximum rank and ranks in between.

In this section, we study the question of possible ranks for $A \in \mathcal{S}(\mathfrak{G})$. It is well known that for any simple graph G, the maximum possible rank is the order of G, and every rank between the minimum and maximum ranks is realized by some $A \in \mathcal{S}(G)$. However, this is not true in the case of loop graphs. Given a loop graph $\mathfrak{G}$, we say that $\mathfrak{G}$ *allows* rank r if there is a matrix $A \in \mathcal{S}(\mathfrak{G})$ such that $\operatorname{rank} A = r$, in which case A is said to *realize* rank r for $\mathfrak{G}$. The *maximum rank* of a loop graph $\mathfrak{G}$ is

$$\mathrm{MR}(\mathfrak{G}) = \max\{\operatorname{rank} A : A \in \mathcal{S}(\mathfrak{G})\}.$$

The proof of the next theorem is analogous to that of [16, Theorem 4.1] and is omitted.

Proposition 6.1. *Let $\mathfrak{G}$ be a loop graph and let m denote the maximum order of a generalized cycle of $\mathfrak{G}$. Then* $\mathrm{MR}(\mathfrak{G}) = m$.

Because the maximum order of a generalized cycle of a subgraph is less than or equal to the maximum order of a generalized cycle of a graph, the next corollary is immediate.

Corollary 6.2. *If $\mathfrak{H}$ is a subgraph of $\mathfrak{G}$, then* $\mathrm{MR}(\mathfrak{H}) \leq \mathrm{MR}(\mathfrak{G})$.

If $\mathfrak{B}$ is a (necessarily loopless) bipartite graph, then $\operatorname{rank} B$ is even for all $B \in \mathcal{S}(\mathfrak{B})$ [16], so it is possible for a loop graph $\mathfrak{G}$ to allow rank k, not allow rank $k+1$, and allow $k+2$. But it is not possible for $\mathfrak{G}$ to allow rank k, not allow rank $k+1$, not allow rank $k+2$, and allow $k+m$ for some $m \geq 3$, as shown in the next proposition.

Proposition 6.3. *Suppose* $\mathrm{mr}(\mathfrak{G}) \leq k \leq \mathrm{MR}(\mathfrak{G}) - 1$ *Then $\mathfrak{G}$ must allow* rank k *or* rank $k+1$.

Proof. Define $A^{(0)} = [a_{ij}^{(0)}] \in \mathcal{S}(\mathfrak{G})$ with $\operatorname{rank} A^{(0)} = \mathrm{mr}(\mathfrak{G})$ and $A = [a_{ij}] \in \mathcal{S}(\mathfrak{G})$ with $\operatorname{rank} A = \mathrm{MR}(\mathfrak{G})$. Let L be the list of pairs $(i,j), 1 \leq i \leq j \leq n$ ordered lexicographically, and let $L[k]$ denote the kth entry of L. Note that $|L| = \ell$ where $\ell := \frac{(n+1)n}{2}$. Assuming $A^{(k-1)}$ has been defined, define $A^{(k)}$ as follows: Let $L[k] = (s,t)$. If $s=t$, then define $B^{(k)} = [b_{ij}^{(k)}]$ by $b_{tt}^{(k)} = a_{tt} - a_{tt}^{(k-1)}$ and $b_{ij}^{(k)} = 0$ for $(i,j) \neq (t,t)$; note $\operatorname{rank} B^{(k)} \leq 1$. If $s \neq t$, then define $B^{(k)} = [b_{ij}^{(k)}]$ by $b_{st}^{(k)} = b_{ts}^{(k)} = a_{st} - a_{st}^{(k-1)}$ and $b_{ij}^{(k)} = 0$ for $(i,j) \neq (s,t)$ and $(i,j) \neq (t,s)$; note $\operatorname{rank} B^{(k)} \leq 2$. Finally, define $A^{(k)} := A^{(k-1)} + B^{(k)}$, so $\operatorname{rank} A^{(k-1)} - 2 \leq \operatorname{rank} A^{(k)} \leq A^{(k-1)} + 2$. Observe that each entry of $A^{(k)}$ is equal to the corresponding entry of $A^{(0)}$ or A, so $A^{(k)} \in \mathcal{S}(\mathfrak{G})$. Furthermore $A^{(\ell)} = A$, so we can move between $\mathrm{mr}(\mathfrak{G})$ and $\mathrm{MR}(\mathfrak{G})$ in

ℓ steps, changing the rank at each step by 0, ± 1, or ± 2. Thus we cannot avoid both k and $k+1$. ☐

If $\mathfrak{G}$ does not have a generalized cycle of order r, then $\mathfrak{G}$ does not allow rank r, because $S_r(A) = 0$ for all $A \in \mathcal{S}(\mathfrak{G})$ (see Remark 1.1). Thus it follows from Proposition 6.3 that for every k between $\mathrm{mr}(\mathfrak{G})$ and $\mathrm{MR}(\mathfrak{G}) - 1$, $\mathfrak{G}$ must have a generalized cycle of order k or order $k+1$. But this can be shown for all $k \leq \mathrm{MR}(\mathfrak{G})-1$ (not just those greater than the minimum rank) by a different method.

Remark 6.4. Let A be a real symmetric matrix. If all principal submatrices of A of order k and $k+1$ are singular, then $\operatorname{rank} A \leq k-1$. This was established in [8] by a technical proof, and in [4] it was observed that it follows from the fact that for any real symmetric matrix A, $\operatorname{rank} A$ is the maximum k such that A has a $k \times k$ nonsingular matrix [15, Corollary 8.9.2], and the fact that adding a row and column adds at most two to the rank.

Proposition 6.5. *Suppose $\mathfrak{G}$ has neither a generalized cycle of order k nor a generalized cycle of order $k+1$. Then $\mathfrak{G}$ does not have a generalized cycle of order m for all $m \geq k$, and $\mathrm{MR}(\mathfrak{G}) \leq k-1$.*

Proof. Since $\mathfrak{G}$ has neither a generalized cycle of order k nor a generalized cycle of order $k+1$, for $A \in \mathcal{S}(\mathfrak{G})$, all principal submatrices of A of order k and $k+1$ are singular, so $\operatorname{rank} A \leq k-1$. Thus $\mathrm{MR}(\mathfrak{G}) \leq k-1$, and by Proposition 6.1, $\mathfrak{G}$ has no generalized cycle of order $m \geq k$. ☐

Of course it is possible for the characteristic polynomial of a particular matrix to have several consecutive coefficients be zero and still have a nonzero determinant, but this must be caused by cancellation of terms, not by absence of generalized cycles. Proposition 6.5 is not true for directed loop graphs: a loopless directed n-cycle has an n-cycle and no other generalized cycles.

If $\mathfrak{G}$ does not have a generalized cycle of order r, then $\mathfrak{G}$ does not allow rank r. In a bipartite loop graph (which necessarily has no loops), all generalized cycles are even. The next example is a nonbipartite loop graph that has a gap in generalized cycles between minimum and maximum ranks, and thus necessarily has a gap in realizable ranks.

Example 6.6. Let $\mathfrak{G}$ and $\mathfrak{G}'$ be the loop graphs shown in Figure 6.1. Since $\mathfrak{G}$ has a 3-cycle, $\mathfrak{G}$ is not bipartite. We will show that $\mathrm{mr}(\mathfrak{G}) = 9$, $\mathrm{MR}(\mathfrak{G}) = 12$, but there is no generalized cycle of order 11, and thus rank 11 is not realizable by any matrix in $\mathcal{S}(\mathfrak{G})$. Note that $\mathfrak{G}$ is loopless and $\mathfrak{G}'$ is an induced subgraph of $\mathfrak{G}$. Since $\mathfrak{G}'$ has a unique spanning generalized cycle, $9 = \mathrm{mr}(\mathfrak{G}') \leq \mathrm{mr}(\mathfrak{G})$. On the other hand, the graph $\mathfrak{G}$ can be covered by three copies of $\mathfrak{C}_4^0$ and one $\mathfrak{C}_3^0$, so

$$\mathrm{mr}(\mathfrak{G}) \leq 3\,\mathrm{mr}(\mathfrak{C}_4^0) + \mathrm{mr}(\mathfrak{C}_3^0) = 3 \cdot 2 + 3 = 9.$$

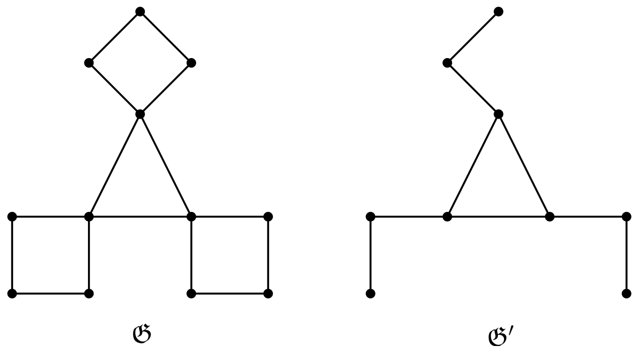

Fig. 6.1. *The graph $\mathfrak{G}$ for Example 6.6 and its induced subgraph $\mathfrak{G}'$.*

As a consequence, $\mathrm{mr}(\mathfrak{G}) = 9$. We can easily find generalized cycles of orders 9, 10, and 12 in $\mathfrak{G}$, implying $\mathrm{MR}(\mathfrak{G}) = 12$. If $\mathfrak{G}$ had a generalized cycle $\mathcal{C}$ of order 11, then $\mathcal{C}$ would contain an odd cycle or a loop, since 11 is odd. However, the triangle in the center is the only odd cycle. But by choosing the triangle, we see that the order of $\mathcal{C}$ must be less than or equal to 9. Hence we cannot find a generalized cycle of order 11, and rank 11 is not realizable by any matrix in $\mathcal{S}(\mathfrak{G})$. Finally, Proposition 6.3 ensures rank 10 is realizable. In summary, the realizable ranks are 9, 10, and 12.

If $\mathfrak{G}$ does not allow rank r for $\mathrm{mr}(\mathfrak{G}) < r < \mathrm{MR}(\mathfrak{G})$, does this imply the absence of generalized cycles of order r? The next example provides a negative answer.

Example 6.7. Let $\mathfrak{H}$ be the loop graph $\mathfrak{P}_3 = (x, y, z)$, with a loop on y. Let $\mathfrak{B}_n$ be the loop graph obtained from $\mathfrak{H}$ and $\mathfrak{K}_{n,n}$ by identifying vertex z with a vertex of $\mathfrak{K}_{n,n}$. It can be seen that this graph $\mathfrak{B}_n$ has a generalized cycle of every size ranging from 0 to $2n+2$, its order. Now we claim that the only realizable ranks are $\{4, 6, \ldots, 2n+2\}$. That is, no odd number between 4 and $2n+2$ can be realized.

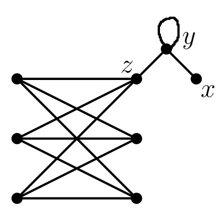

Fig. 6.2. *An illustration of $\mathfrak{B}_3$*

To see this, we set x and y to be the first and the second vertices and use Remark 1.3 to scale the matrix. Henceforth we may assume any matrix $A \in \mathcal{S}(\mathfrak{B}_n)$ has the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & t & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & & \vdots \\ 0 & 0 & 0 & 0 & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \cdots & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & M & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{bmatrix},$$

where t is a nonzero scalar and $M \in S(\mathfrak{K}_{n,n})$. By subtracting the first row/column from the third row/column, we obtain the matrix $B := \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix} \oplus M$, so $\operatorname{rank} B = \operatorname{rank} A = \operatorname{rank} M + 2$. Since we know $M \in S(\mathfrak{K}_{n,n})$ and the realizable ranks of M are $\{2, 4, \ldots, 2n\}$, the realizable ranks of $\mathfrak{B}_n$ are

$$\{2, 4, \ldots, 2n\} + 2 = \{4, 6, \ldots, 2n + 2\}.$$

We now consider adding a new vertex adjacent to all existing vertices. The ideas in the proof are similar to those in [16, Theorem 4.6], but since we expand it to include the case of a new vertex with a loop, we include the brief proof.

PROPOSITION 6.8. *Suppose $\mathfrak{H}$ is a loop graph of order n such that $\delta(\mathfrak{H}) \geq 1$, and that the graph $\mathfrak{G}$ is constructed from $\mathfrak{H}$ by joining a single vertex v (with or without a loop) to $\mathfrak{H}$. Suppose $\mathfrak{H}$ allows rank k. Then $\mathfrak{G}$ allows rank $k+1$, and if v has a loop then $\mathfrak{G}$ allows rank k.*

Proof. Given $A \in \mathcal{S}(\mathfrak{H})$ with $\operatorname{rank} A = k$, we can construct a matrix $\widetilde{A}$ in $\mathcal{S}(\mathfrak{G})$ with $\operatorname{rank} \widetilde{A} = k + 1$ as follows, and if the new vertex v has a loop, the rank k matrix B constructed is also in $\mathcal{S}(\mathfrak{G})$. Without loss of generality, let the new vertex be $n+1$. Since $\delta(\mathfrak{H}) \geq 1$, every row of A has a nonzero entry. By [16, Lemma 4.5], we can choose a real vector $\mathbf{x}$ such that every entry of $A\mathbf{x}$ is nonzero and $\mathbf{x}^\top A\mathbf{x} \neq 0$. Let $B := \begin{bmatrix} A & A\mathbf{x} \\ \mathbf{x}^\top A & \mathbf{x}^\top A\mathbf{x} \end{bmatrix}$. Then $\operatorname{rank} B = \operatorname{rank} A = k$, and if v has a loop $B \in \mathcal{S}(\mathfrak{G})$. We can change the entry $\mathbf{x}^\top A\mathbf{x}$ to either 0 or $2\mathbf{x}^\top A\mathbf{x}$ so that $\widetilde{A} = \begin{bmatrix} A & A\mathbf{x} \\ \mathbf{x}^\top A & 0 \end{bmatrix}$ or $\widetilde{A} = \begin{bmatrix} A & A\mathbf{x} \\ \mathbf{x}^\top A & 2\mathbf{x}^\top A\mathbf{x} \end{bmatrix}$. Then $\widetilde{A} \in \mathcal{S}(\mathfrak{G})$ and $\operatorname{rank} \widetilde{A} = k + 1$. ◻

COROLLARY 6.9. *$\mathfrak{K}_n^{\ell(s)}$ allows all ranks r such that $\operatorname{mr}(\mathfrak{K}_n^{\ell(s)}) \leq r \leq n = \operatorname{MR}(\mathfrak{K}_n^{\ell(s)})$.*

Proof. By Proposition 6.8, when we add a vertex with a loop, we may choose to leave the rank unchanged or increase it by one. Suppose first that $n - s \geq 3$.

Consider the induced subgraph obtained by taking all of the loopless vertices; this subgraph is $\mathfrak{K}_{n-s}^{\ell(0)}$. Since the subgraph has no loops, it must allow all ranks r such that $\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) = 3 \le r \le (n-s)$ by Corollary 4.7 in [16]. Then $\mathfrak{K}_n^{\ell(s)}$ can be obtained by joining s looped vertices to $\mathfrak{K}_{n-s}^{\ell(0)}$ without raising the rank, so $\mathfrak{K}_n^{\ell(s)}$ allows rank r for all r such that $\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) \le r \le (n-s)$. For r with $(n-s) \le r \le n$, construct an $(n-s)\times(n-s)$ matrix with full rank in $\mathcal{S}(\mathfrak{K}_{n-s}^{\ell(0)})$. By joining $r-(n-s)$ looped vertices while increasing rank by one at each step, we obtain a rank r matrix in $\mathcal{S}(\mathfrak{K}_{n-s}^{\ell(r-(n-s))})$. We then join $(n-r)$ additional looped vertices to obtain $\mathfrak{K}_n^{\ell(s)}$ without increasing the rank. Therefore, $\mathfrak{K}_n^{\ell(s)}$ allows all ranks r such that $\mathrm{mr}(\mathfrak{K}_n^{\ell(s)}) \le r \le n = \mathrm{MR}(\mathfrak{K}_n^{\ell(s)})$. The case $n-s \le 2$ is similar. □

OBSERVATION 6.10. *Since* $n-1 \le \mathrm{mr}(\mathfrak{P}_n) \le \mathrm{MR}(\mathfrak{P}_n) \le n$, $\mathfrak{P}_n$ *trivially allows all ranks* r *such that* $\mathrm{mr}(\mathfrak{P}_n) \le r \le \mathrm{MR}(\mathfrak{P}_n)$.

7. Additional topics and future research.

In this section we discuss extensions to loop graphs of additional results for simple graphs and pose open questions for future research.

7.1. Extreme minimum rank.

Recall that a loop graph has minimum rank equal to its order if and only if it has a unique spanning generalized cycle. It is well-known that for a simple graph G, $\mathrm{mr}(G) = |G| - 1$ if and only if G is a path.

QUESTION 7.1. *What loop graphs* $\mathfrak{G}$ *have* $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}| - 1$*?*

Results in Section 5 characterize the loop configurations of paths and cycles with minimum rank one less than order, but in general the question is open.

Minimum rank three has been characterized for loopless loop graphs (zero diagonal minimum rank) in [16] and it may be productive to investigate minimum rank three for other loop configurations (such as all loops).

QUESTION 7.2. *What loop graphs* $\mathfrak{G}$ *have* $\mathrm{mr}(\mathfrak{G}) = 3$*?*

However it is known that for simple graphs there is an infinite family of forbidden induced subgraphs for minimum rank three [18].

7.2. Minimum rank of additional families and small loop graphs.

The AIM Minimum Rank Catalog [1] lists the minimum rank of more than forty families of graphs. Extensions of these results to loop graphs could be investigated, including some very well-known graphs such as complete bipartite graphs.

Since the question of whether a loop graph has minimum rank equal to zero or one is easily answered, by applying the forbidden subgraph test one can determine for a loop graph whether minimum rank is equal to $0, 1, 2$, or is ≥ 3. For any loop graph of order n the unique spanning generalized cycle test determines whether the graph has minimum rank n or $\leq n-1$. These tests immediately determine the minimum rank of all loop graphs of order at most four. Furthermore, if the zero forcing number lower bound equals the unique spanning generalized cycle test upper bound, then the minimum rank is determined. These bounds have been implemented in the program [21], and work continues to add additional methods to this program, such as checking for graphs for which the minimum rank can be determined by other methods discussed in this paper, such as trees ($\mathrm{mr}(\mathfrak{T}) = |\mathfrak{T}| - \mathrm{Z}(\mathfrak{T})$) and cycles (Theorem 5.4), and applying cut-vertex reduction [23] and P_4 reduction (Lemma 4.2). This will give the software the capability to determine the minimum rank of most loop graphs of order five and possibly order six, at which point it may be feasible to complete the determination of minimum rank of loop graphs of order five (or six) by construction of matrices realizing the lower bounds, as was done for simple graphs of order at most seven in [13].

7.3. No useful Colin de Verdière type parameters. In this section we present an example that shows that a Colin de Verdière type parameter, i.e. a minor monotone lower bound on maximum nullity defined using the Strong Arnold Hypothesis, is unlikely to exist. Definitions of Colin de Verdière type parameters, minor monotonicity, and the Strong Arnold Hypothesis can be found in [3] or [14].

EXAMPLE 7.3. Let $\mathfrak{H}$ be the *party hat* graph in Figure 7.1. Then $\{1\}$ is a zero forcing set for $\mathfrak{H}$ with a forcing process $4 \to 5, 6 \to 2, 3 \to 3, 1 \to 4, 2 \to 6$. Thus $\mathrm{M}(\mathfrak{H}) \leq \mathrm{Z}(\mathfrak{H}) \leq 1$. But $\mathfrak{H}$ contains $\mathfrak{K}_3$ and $\mathrm{M}(\mathfrak{H}) = 1 < 2 = 3 - 1 = \mathrm{M}(\mathfrak{K}_3)$.

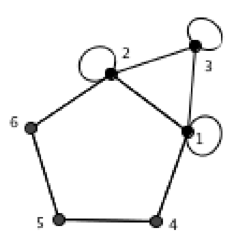

FIG. 7.1. *The party hat graph* $\mathfrak{H}$ *for Example 7.3.*

Since any matrix that has all entries nonzero, including a rank one matrix, satisfies the Strong Arnold Hypothesis, the Strong Arnold Hypothesis does not seem to imply minor monotonicity for loop graphs. Example 7.3 also implies that any minor monotone parameter β with $\beta \leq \mathrm{M}$ must have $\beta(\mathfrak{K}_3) \leq 1$, so any minor monotone parameter below M seems unlikely to be useful.

7.4. GCC is not true for loop graphs. The Graph Complement Conjecture (GCC) for simple graphs [14] is

$$\operatorname{mr}(G) + \operatorname{mr}(\overline{G}) \leq |G| + 2.$$

GCC is not true for loop graphs, as the next example shows.

Example 7.4. Consider the path on 4 vertices $\mathfrak{P}_4$ with loops on the two middle vertices, which is shown in Figure 7.2. Observe that $\mathfrak{P}_4$ is self complementary and $\operatorname{rank} \mathfrak{P}_4 = 4$ by Proposition 5.1, so

$$\operatorname{mr}(\mathfrak{P}_4) + \operatorname{mr}(\overline{\mathfrak{P}_4}) = 4 + 4 = 8 > 6 = |\mathfrak{P}_4| + 2.$$

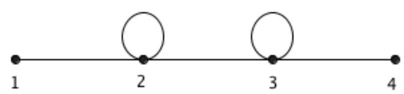

Fig. 7.2. *The graph $\mathfrak{P}_4$ for Example 7.4.*

However the question of whether there is a bound with a different additive constant remains open (see [9] for the analogous question for simple graphs).

Question 7.5. *Does there exist a positive integer d such that for all $\mathfrak{G}$,*

$$\operatorname{mr}(\mathfrak{G}) + \operatorname{mr}(\overline{\mathfrak{G}}) \leq |\mathfrak{G}| + d?$$

Example 7.4 shows that if such d exists then $d \geq 4$.

7.5. The δ Conjecture is not true for loop graphs. The δ Conjecture for simple graphs [14] is $\delta(G) \leq \operatorname{M}(G)$. The δ Conjecture is not true for loop graphs, because $\delta(\mathfrak{C}_3^0) = 2 > 0 = \operatorname{M}(\mathfrak{C}_3^0)$. Since in the loopless case, we are reducing the minimum number of nonzero entries per row, arguably the "δ Conjecture for loop graphs" should be $\delta(G) - 1 \leq \operatorname{M}(G)$. However, $\mathfrak{C}_3^0$ (or any odd cycle with no loops) is still a counterexample, and illustrates the importance of symmetry (there is a nonsymmetric matrix with the same nonzero pattern and rank two).

7.6. Minimum rank over other fields. In this section we discuss extension of our results in prior sections to fields other than the real numbers.

Low minimum rank over other fields. Barrett, van der Holst, and Loewy's characterization of minimum rank at most two (quoted here in Theorem 2.2) applies to all infinite fields of characteristic not two, and the proof of Theorem 2.3 remains valid for infinite fields of characteristic not two, characterizing loop graphs having minimum rank at most two over such fields.

Barrett, van der Holst, and Loewy also have characterizations for infinite fields of characteristic two [5] and for finite fields [6]. These results provide tools that may allow characterizing minimum rank at most two over finite fields or fields of characteristic two.

High minimum rank over other fields. For minimum rank equal to order of the loop graph, the situation is entirely different for fields of characteristic two, because a k-cycle with $k \geq 3$ does not contribute to the determinant due to the fact that a k-cycle contributes $2 \equiv 0 \pmod 2$ equal terms. For example, if $\operatorname{char} F = 2$, then $\operatorname{mr}^F(\mathfrak{C}^0_{2s+1}) = 2s$, despite the fact that $\mathfrak{C}^0_{2s+1}$ has a unique spanning generalized cycle.

In addition to assuming characteristic not two, the proof that a loopless loop graph has minimum rank equal to its order if and only if it has a unique spanning generalized cycle [16, Theorem 3.9] uses the fact that we can find nonzero field elements producing a nonzero value of a polynomial [16, Lemma 3.4]; this property is valid for infinite fields. But the proof of [16, Theorem 3.9] also uses the quadratic formula to extract a square root within the field [16, Lemma 3.5]. In the real numbers this is achieved by showing that the number whose square root is being extracted can be made positive. Thus the proof does not immediately extend to proper subfields of the real numbers, such as the rationals. But it does extend to algebraically closed fields of characteristic not two (which are necessarily infinite). The proof of our Theorem 3.1 generalizing this to all loop graphs remains valid for any infinite field of characteristic not two whenever the loopless base case is established. Thus Theorem 3.1 is valid for algebraically closed fields of characteristic not two, in addition to the real numbers.

Question 7.6. *Is there an example of loop graph $\mathfrak{G}$ that does not have a unique spanning generalized cycle and a (finite) field F with* $\operatorname{char} F \neq 2$ *such* $\operatorname{mr}^F(\mathfrak{G}) = |\mathfrak{G}|$*?*

Such an example, if one exists, is likely loopless, since if there is a loop that is in one spanning generalized cycle and not in another then $\operatorname{mr}^F(\mathfrak{G}) < |\mathfrak{G}|$ (because one can solve a linear equation).

Schur complement techniques over other fields. The use of the Schur complement in Lemma 4.1 is well known to be valid over any field. The proof of Lemma 4.2 (P_4 reduction) remains valid over any field with at least 3 elements; the only case where 3 are needed is in the construction of $A[\alpha]$ for the upper bound in the case $c_{xx} \neq 0$, where we need to avoid both 0 and $-c_{xx}$.

Maximum and realizable ranks over other fields. Much of the discussion in Section 6 relies on generalized cycles, so we assume $\operatorname{char} F \neq 2$. An infinite field suffices to ensure that any polynomial that is not identically zero can be made nonzero by a choice of values. With the exception of Corollary 6.9, all the results in Section 6 are valid for infinite fields of characteristic not two. Corollary 6.9 is valid for all fields of characteristic zero, because the proof of the loopless case in [16] constructs integer matrices.

REFERENCES

[1] AIM Minimum Rank Graph Catalog. `http://www.aimath.org/data/paper/AIM08/`.

[2] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioaba, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Lin. Alg. Appl.*, 428: 1628–1648, 2008.

[3] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. *J. Graph Theory*, 72: 146–177, 2013.

[4] W. Barrett, S. Butler, M. Catral, S. M. Fallat, H. T. Hall, L. Hogben, P. van den Driessche, M. Young. The principal rank characteristic sequence over various fields. Under review.

[5] W.W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two. *Elec. J. Lin. Alg.*, 11: 258-280, 2004.

[6] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two: The finite fields case. *Elec. J. Lin. Alg.*, 14: 32–42, 2005.

[7] C. Bozeman, AV. Ellsworth, L. Hogben, J.C.-H. Lin, G. Maurer, K. Nowak, A. Rodriguez, J. Strickland. Appendix to Minimum rank of graphs with loops. `https://orion.math.iastate.edu/lhogben/research/LoopGraphAppendix.pdf`.

[8] R. A. Brualdi, L. Deaett, D. D. Olesky, P. van den Driessche. The principal rank characteristic sequence of a real symmetric matrix. *Lin. Alg. Appl.*, 436: 2137–2155, 2012.

[9] R. Brualdi, L. Hogben, B. Shader. AIM Workshop on Spectra of Families of Matrices Described by Graphs, Digraphs and Sign Patterns, Final report: Mathematical Results, 2007. `http://aimath.org/pastworkshops/matrixspectrumrep.pdf`.

[10] G. Chen, F.J. Hall, Z. Li, B. Wei. On ranks of matrices associated with trees. *Graphs Combin.* 19: 323–334, 2003.

[11] D. M. Cvetcović, M. Doob, H. Sachs *Spectra of Graphs.* Academic Press, Inc., New York, 1980.

[12] L.M. DeAlba, T.L. Hardy, I.R. Hentzel, L. Hogben, A. Wangsness. Minimum rank and maximum eigenvalue multiplicity of symmetric tree sign patterns. *Lin. Alg. Appl.*, 418: 389–415, 2006.

[13] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, G. Tims. Techniques for determining the minimum rank of a small graph. *Lin. Alg. Appl.* 432: 2995–3001, 2010.

[14] S.M. Fallat, L. Hogben. Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs. In *Handbook of Linear Algebra*, 2nd ed., L. Hogben, Editor. CRC Press, Boca Raton, FL, 2014.

[15] C. Godsil, G. Royle. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[16] C. Grood, A. Harmse, L. Hogben, T.J. Hunter, B. Jacob, A. Klimas, S. McCathern. Minimum rank with zero diagonal. Under review.

[17] C. Grood, T.J. Hunter. Realizable Ranks with Zero Diagonal. Preprint.

[18] H.T. Hall. Minimum rank 3 is difficult to determine. Preprint.

[19] F. Harary. The determinant of the adjacency matrix of a graph. *SIAM Review*, 4: 202-210, 1962.

[20] L. Hogben. Minimum rank problems. *Lin. Alg. Appl.*, 432: 1961–1974, 2010.

[21] L. Hogben, K. Nowak, J.C.-H.Lin. Bounds on minimum rank of loop graphs (*Sage* program). Available from the authors.

[22] J.C.-H.Lin. The sieving process and lower bounds for the minimum rank problem. Under review.

[23] R.C. Mikkelson. Minimum rank of graphs that allow loops. Ph.D. Thesis, Iowa State University, 2008.

[24] F. Zhang. *Matrix Theory: Basic results and techniques*, 2nd ed. Springer, New York, 2011.