Relationships between the Completion Problems for Various Classes of Matrices

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1 Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not (all entries specified is also allowed). A *completion* of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\mathrm{N} \times \mathrm{N}$, where $\mathrm{N} = \{1, . . . , n\}$. In this paper a pattern is assumed to include all diagonal positions.

A *symmetric* pattern is a pattern with the property that (i, j) is in the pattern if and only if (j, i) is also in the pattern; symmetric patterns are also called positionally or combinatorially symmetric. An *asymmetric* pattern is a pattern with the property that if (i, j) is in the pattern, then (j, i) is not in the pattern.

A partial matrix *specifies the pattern* if its specified entries are exactly those listed in the pattern. For a class X of real matrices, we say a pattern *has X-completion* if every partial X-matrix specifying the pattern can be completed to an X-matrix. The matrix completion problem (for patterns) for the class of X-matrices is to determine which patterns have X-completion.

Matrix completion problems have been studied for many classes of matrices, including positive definite matrices [6], P-matrices [11], [4], P_0-matrices [2], M-matrices [8], M_0-matrices [9], inverse M-matrices [12], [7] and

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many other subclasses of P- and P_0-matrices [5], [9], [1]. Recently, completion problems for several classes have been studied at the same time, and relationships are being found between the completion problems for different classes. In this paper we survey techniques that apply to many classes, and examine relationships between the solutions to the matrix completion problems for certain pairs of classes. We also describe a new result for the weakly sign symmetric P_0-completion problem and apply the relationship results to extend this result to the classes of weakly sign symmetric P-matrices and sign symmetric P-matrices. The latter result resolves the issue of the completability of the symmetric n-cycle for sign symmetric P- matrices, a problem described as difficult in [5].

For $\alpha \subseteq N$, the *principal submatrix* $A(\alpha)$ is obtained from the $n \times n$ matrix A by deleting all rows and columns not in α. A *principal minor* of A is the determinant of a principal submatrix of A. The matrix $A \in \mathbf{R}^{n\times n}$ is a P-(respectively, P_0-, $P_{0,1}$-) matrix if every principal minor is positive (non-negative, non-negative and all diagonal elements of A are positive).

Additional classes of matrices are obtained by imposing various restrictions on the signs of the entries. The conditions we will discuss here are:

- *weakly sign symmetric,* which requires $a_{ij}\, a_{ji} \geq 0$ for each pair i,j
- *sign symmetric,* which requires $a_{ij}\, a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each pair i,j
- *nonnegative,* which requires $a_{ij} \geq 0$ for all i,j
- *positive,* which requires $a_{ij} > 0$ for all i,j

Let X be any of the classes: P-matrices, weakly sign symmetric P-matrices, sign symmetric P-matrices, nonnegative P-matrices, positive P-matrices, $P_{0,1}$-matrices, weakly sign symmetric $P_{0,1}$-matrices, sign symmetric $P_{0,1}$-matrices, nonnegative $P_{0,1}$-matrices, P_0-matrices, weakly sign symmetric P_0-matrices, sign symmetric P_0-matrices, nonnegative P_0-matrices. Since every principal submatrix of an X-matrix is an X-matrix, we define a partial matrix A to be a *partial X- matrix* if every fully specified principal submatrix is an X-matrix, and for X one of the nonnegative or positive classes, in addition all specified entries must be nonnegative or positive.

Digraphs are used to study matrix completion problems. A *digraph* $G = (V_G, E_G)$ is a finite set of positive integers V_G, whose members are called *vertices*, and a set E_G of ordered pairs (v,u) of distinct vertices, called *arcs*. For a pattern Q for $n \times n$ matrices (that includes all diagonal positions), the digraph of Q is the digraph $G = (N,E)$ where $E = \{(v,u) \mid (v,u) \in Q \text{ and } v \neq u\}$. A partial matrix A that specifies a pattern Q is also referred to as specifying the digraph G of Q, and G is said to *have X-completion* if Q does, i.e., if every partial X-matrix specifying G can be completed to an X-matrix. The *order* of a digraph is the number of vertices.

A *subdigraph* of the digraph $G = (V_G, E_G)$ is a digraph $H = (V_H, E_H)$, where $V_H \subseteq V_G$ and $E_H \subseteq E_G$ (note that $(v,u) \in E_H$ requires $v, u \in V_H$ since H is a digraph). If $W \subseteq V_G$, the *subdigraph of G induced by W,* $<W>$, is the digraph (W, E_W) with $E_W = E_G \cap (W \times W)$. A subdigraph induced by a subset of vertices is also called an *induced subdigraph*.

A *path* (respectively, *semipath*) in a digraph $G = (V,E)$ is sequence of vertices $v_1, v_2, ..., v_{k-1}, v_k$ in V such that for $i = 1, ..., k-1$ the arc $(v_i, v_{i+1}) \in E$ (respectively, $(v_i, v_{i+1}) \in E$ or $(v_{i+1}, v_i) \in E$) and all vertices are distinct except possibly $v_1 = v_k$. Clearly, a path is a semipath, although the converse is false. The *length* of the (semi)path $v_1, v_2, ..., v_{k-1}, v_k$ is $k-1$. A *cycle* is a path in which the first vertex equals the last vertex. A digraph whose vertex set consists of the k vertices $v_1, v_2, ..., v_k$ and whose arcset consists of exactly the arcs in the two cycles $v_1, v_2, ..., v_k, v_1$ and $v_1, v_k, v_{k-1}, ..., v_1$ is a *symmetric k-cycle.*

A digraph is *connected* if there is a semipath from any vertex to any other vertex (a digraph of order 1 is connected); otherwise it is *disconnected*. A *component* of a digraph is a maximal connected subdigraph. A *cut-vertex* of a connected digraph is a vertex whose deletion disconnects the digraph; more generally, a cut-vertex is a vertex whose deletion disconnects the component containing the vertex. A digraph is *nonseparable* if it is connected and has no cut-vertices. A *block* of a digraph is a subdigraph that is nonseparable and is maximal with respect to this property. A (sub)digraph is called a *clique* if it contains all possible arcs between its vertices. A digraph is *block-clique* if every block is a clique. Block-clique digraphs are called 1-chordal in [5]. A digraph is *strongly connected* if there is a path from any vertex to any other vertex. Clearly, a strongly connected digraph is connected, although the converse is false.

2 Multi-class Methods and Relationship Theorems

As the number of classes for which the matrix completion problem for patterns has been studied has grown, several new techniques that apply to many classes have been discovered. Two important examples of such techniques appeared in [5]. One is the asymmetric completion of a partial matrix specifying a block-clique digraph.

Theorem 1 [5]. *Let X be any of the classes (weakly sign symmetric, sign symmetric, nonnegative, positive) P-matrices, (weakly sign symmetric, sign symmetric, nonnegative)* $P_{0,1}$*-matrices, (weakly sign symmetric, sign symmetric, nonnegative)* P_0*-matrices. Let G be a block-clique digraph. Then G has X-completion, i.e., any partial X-matrix specifying G can be completed to an X-matrix.*

This extends to:

Theorem 2 [9]. *Let X be any of the classes (weakly sign symmetric, sign symmetric, nonnegative, positive) P-matrices, (weakly sign symmetric, nonnegative)* $P_{0,1}$*- matrices, (weakly sign symmetric, nonnegative)* P_0*-matrices. A digraph G has X-completion if and only if every nonseparable strongly connected induced subdigraph of G has X-completion.*

The technique of Theorem 2 has been exploited widely ([1], [2], [3], [9]) and now is a standard method of attack for digraph classification.

Another important method introduced in [5] is the inductive completion of symmetric *n*-cycles.

Theorem 3 [5]. *If A is a partial positive P-matrix, the graph of whose specified entries is a symmetric n-cycle, then A can be completed to a positive P-matrix.*

Unlike Theorems 1 and 2, which were proved for a large number of classes, the base step of the inductive proof of Theorem 3 is heavily class dependent. However, it was noted in [5] that the same inductive argument would apply to other classes, including (weakly) sign symmetric *P*-matrices, if one could find an appropriate starting point. That is, find a *k* such that any partial (weakly) sign symmetric *P*-matrix having a symmetric *k*-cycle as the graph of its specified entries can be completed to a (weakly) sign symmetric *P*-matrix. Such a *k* has now been found, namely $k = 6$ [3]. In [3] it is also shown that the result is not true for $k = 5$, thus completely answering the question of which symmetric *n*-cycles have (weakly) sign symmetric *P*-completion. This is discussed further in Section 3 below. This illustrates how the base for the induction varies with the class. In contrast, for the class of nonnegative P_0-matrices, the proof in [1] begins with $n = 5$.

We now turn our attention to the idea of inferring that a pattern has *X*-completion from the fact that it has *Y*-completion, as opposed to showing that the same technique applies to both the classes *X* and *Y*. If *X* and *Y* are classes of matrices with $X \subseteq Y$, in general it is not possible to conclude either that a pattern that has *Y*-completion must have *X*-completion (because the completion to a *Y*-matrix may not be an *X*-matrix) or that a pattern that has *X*-completion must have Y-completion (because there may be a partial Y-matrix that is not a partial X-matrix). However, in cases where there is a natural relationship between the classes X and Y, it is sometimes possible to conclude that certain (or all) patterns that have Y-completion have X-completion or vice versa. Such a result is called a *relationship theorem*. Relationship theorems were first studied in [10], and we survey several relationship theorems here, as well as proving one new one (Theorem 7 below). The first of these theorems exploits certain properties of the natural relationship between the classes of P- matrices and P_0-matrices.

The classes of matrices X and X_0 are referred to as a *pair of* Π/Π_0*-classes* if

1. Any partial X-matrix is a partial X_0-matrix.
2. For any X_0-matrix A and $\varepsilon > 0$, $A + \varepsilon I$ is a X-matrix.
3. For any partial X-matrix A, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is a partial X-matrix (where $\tilde{I}$ is the partial identity matrix specifying the same pattern as A).

If A is a P_0-matrix and $\varepsilon > 0$, then $A + \varepsilon I$ is a P-matrix. For any partial P-matrix A, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is also a partial P-matrix, because the determinant is a continuous function of the entries of the matrix. Hence the classes P- matrices and P_0-matrices are a pair of Π/Π_0-classes, as are subclasses defined by one of the conditions (weakly) sign symmetric or nonnegative.

Theorem 4 [10]. *For a pair of Π/Π_0-classes, if a pattern has Π_0-completion then it must also have Π-completion.*

Corollary 5 [10].

- *Any pattern that has P_0-completion also has P-completion.*
- *Any pattern that has weakly sign symmetric P_0-completion also has weakly sign symmetric P-completion.*
- *Any pattern that has sign symmetric P_0-completion also has sign symmetric P-completion.*
- *Any pattern that has nonnegative P_0-completion also has nonnegative P-completion.*
- *Any pattern that has $P_{0,1}$-completion also has P-completion.*
- *Any pattern that has weakly sign symmetric $P_{0,1}$-completion also has weakly sign symmetric P-completion.*
- *Any pattern that has sign symmetric $P_{0,1}$-completion also has sign symmetric P-completion.*
- *Any pattern that has nonnegative $P_{0,1}$-completion also has nonnegative P-completion.*

Although not a direct corollary of Theorem 4, the analogous result is also true for the classes of $P_{0,1}$-matrices and P_0-matrices. This result is obvious for patterns that include all diagonal positions, but is true more generally [10].

Theorem 6.

- *Any pattern that has P_0-completion also has $P_{0,1}$-completion.*
- *Any pattern that has weakly sign symmetric P_0-completion also has weakly sign symmetric $P_{0,1}$-completion.*
- *Any pattern that has sign symmetric P_0-completion also has sign symmetric $P_{0,1}$-completion.*
- *Any pattern that has nonnegative P_0-completion also has nonnegative $P_{0,1}$-completion.*

The following relationship result is new.

Theorem 7. *Let Q be a pattern that has nonnegative P-completion. Then Q has positive P-completion.*

Proof. Let A be a partial positive P-matrix specifying Q. The matrix A is a partial nonnegative P-matrix specifying Q, and so can be completed to nonnegative P-matrix $\hat{A}$. The only reason $\hat{A}$ might not be a positive P-matrix is if some entries (that were originally unspecified) are zero. Since there are only finitely many principal minors of $\hat{A}$ and these are continuous functions of the entries of $\hat{A}$, we can slightly perturb zero entries while maintaining all principal minors positive. Thus $\hat{A}$ can be converted into a positive P-matrix that completes A.

For the next result we need to distinguish the case of a symmetrically placed pair of entries in a partial matrix, where one member of the pair, a_{ij}, is specified (the *specified twin*) and the other member of the pair, x_{ij}, is unspecified (the *unspecified twin*).

Lemma 8 [3]. *Let Q be a pattern that has weakly sign symmetric P-completion, where for any partial weakly sign symmetric P-matrix A specifying Q, there is a completion in which zero is assigned to any unspecified twin whose specified twin is zero. Then Q has sign symmetric P-completion.*

Since a partial matrix specifying a symmetric pattern does not have any specified/unspecified twins we have the following corollary.

Corollary 9 [3]. *Any symmetric pattern that has weakly sign symmetric P-completion also has sign symmetric P-completion.*

Note that Theorem 7 and Lemma 8 (and hence Corollary 9) rely on the positivity of the determinants, and the analogous result for P_0-matrices is false: The symmetric 6-cycle has weakly sign symmetric P_0-completion (Theorem 11 below) but the symmetric 6-cycle does not have sign symmetric P_0-completion (see the example in Section 3 below, taken from [5]).

Although these relationship results are all of the type that any pattern that has Y-completion must also have X-completion where $X \subseteq Y$, it is possible to have that any pattern that has X-completion must also have Y-completion (where again $X \subseteq Y$). An example of the latter situation would be X is the class of sign symmetric P_0-matrices and Y is the class of weakly sign symmetric P_0-matrices. Clearly $X \subseteq Y$. A pattern (that includes all diagonal positions) has sign symmetric P_0-completion if and only if the pattern is block-clique [5]. It is known [5] that any pattern of this type has weakly sign symmetric P_0-completion. Thus in this case any pattern that has X-completion must also have Y-completion. In this direction we have the following relationship result:

Lemma 10 [3]. *Any asymmetric pattern that has sign symmetric P-completion has weakly sign symmetric P-completion.*

3 Weakly sign symmetric P- (P_0-) completion for the symmetric *n*-cycle

In this section we state the result of the weakly sign symmetric P_0-completion of the symmetric 6-cycle and apply the techniques of Section 2 to extend completion to n-cycles with $n \geq 6$ for the classes of weakly sign symmetric P_0-matrices and (weakly) sign symmetric P-matrices.

Theorem 11 [3]. *A pattern whose digraph is a symmetric 6-cycle has weakly sign symmetric P_0-completion. That is, any partial weakly sign symmetric P_0-matrix, the digraph of whose specified entries is a symmetric 6-cycle, can be completed to a weakly sign symmetric P_0-matrix.*

Note that the analogous result is not true for sign symmetric P_0-matrices as Example 3.4 of [5] shows: $A = \begin{bmatrix} 1 & 1 & x_{13} & x_{14} & x_{15} & -1 \\ 1 & 1 & 1 & x_{24} & x_{25} & x_{26} \\ x_{31} & 1 & 1 & 1 & x_{35} & x_{36} \\ x_{41} & x_{42} & 1 & 1 & 1 & x_{46} \\ x_{51} & x_{52} & x_{53} & 1 & 1 & 1 \\ -1 & x_{62} & x_{63} & x_{64} & 1 & 1 \end{bmatrix}$ cannot be completed to a sign symmetric P_0-matrix, because

Det $A(\{4,5,6\}) = -1 + x_{46} + x_{64} - x_{46}x_{64}$, so $x_{46}, x_{64} > 0$.

Det $A(\{3,4,6\}) = -x_{36}x_{63} + x_{46}x_{63} + x_{36}x_{64} - x_{46}x_{64}$ so $x_{36}, x_{63} > 0$.

Det $A(\{2,3,6\}) = -x_{26}x_{62} + x_{36}x_{62} + x_{26}x_{63} - x_{36}x_{63}$ so $x_{26}, x_{62} > 0$.

But then Det $A(\{1,2,6\}) = -1 - x_{26} - x_{62} - x_{26}x_{62} < 0$

The weakly sign symmetric P_0-completion of A given by the proof of Theorem 11 is $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$.

The inductive technique of [5] and the nonzero loop digraph technique of [1] can be used to establish the result for $n > 6$:

Theorem 12 [3]. *Any pattern whose digraph is a symmetric n-cycle with $n \geq 6$ has weakly sign symmetric P_0-completion.*

Theorem 6 yields the corresponding result for weakly sign symmetric $P_{0,1}$-matrices:

Corollary 13. *Any pattern whose digraph is a symmetric n-cycle with $n \geq 6$ has weakly sign symmetric $P_{0,1}$-completion.*

Corollary 5 and Corollary 9 yield the corresponding results for (weakly) sign symmetric *P*-matrices:

Corollary 14 [3]. *Any pattern whose digraph is a symmetric n-cycle with $n \geq 6$ has (weakly) sign symmetric P-completion. That is, any partial weakly sign* *symmetric P-matrix, the digraph of whose specified entries is a symmetric n-cycle, can be completed to a weakly sign symmetric P-matrix, and any partial sign symmetric P-matrix, the digraph of whose specified entries is a symmetric n-cycle, can be completed to a sign symmetric P-matrix.*

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