The enhanced principal rank characteristic sequence

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Abstract

The enhanced principal rank characteristic sequence (epr-sequence) of a symmetric $n\times n$ matrix is a sequence $\ell_1\ell_2\cdots\ell_n$ where ℓ_k is `A`, `S`, or `N` according as all, some, or none of its principal minors of order k are nonzero. Such sequences give more information than the (0,1) pr-sequences previously studied (where basically the kth entry is 0 or 1 according as none or at least one of its principal minors of order k is nonzero). Various techniques including the Schur complement are introduced to establish that certain subsequences such as `NAN` are forbidden in epr-sequences over fields of characteristic not two. Using probabilistic methods over fields of characteristic zero, it is shown that any sequence of `A`s and `S`s ending in `A` is attainable, and any sequence of `A`s and `S`s followed by one or more `N`s is attainable; additional families of attainable epr-sequences are constructed explicitly by other methods. For real symmetric matrices of orders 2, 3, 4, and 5, all attainable epr-sequences are listed with justifications.

Keywords. Principal rank characteristic sequence, enhanced principal rank characteristic sequence, minor, rank, symmetric matrix, Hermitian matrix, Schur complement

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1 Introduction

For a symmetric matrix over a field F or a complex Hermitian matrix, Brualdi et al. [3] and Barrett et al. [1] considered a principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. More precisely, the *principal rank characteristic sequence* of an $n \times n$ symmetric or complex Hermitian matrix B is the sequence $\mathrm{pr}(B) = r_0]r_1r_2 \cdots r_n$, where for $k = 0, 1, \ldots, n$, a 1 in the kth position indicates the existence of a principal submatrix of rank k and a 0 indicates no such submatrix exists. To obtain more information, we refine this sequence and instead of considering the presence or absence of such a principal submatrix, we consider three possibilities in the following definition.

Definition 1.1. The *enhanced principal rank characteristic sequence* of a symmetric matrix $B \in F^{n\times n}$ (or Hermitian matrix $B \in \mathbb{C}^{n\times n}$) is the sequence (epr-sequence) $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all } k \times k \text{ principal minors of the given order are nonzero;} \\ \mathtt{S} & \text{if some but not all } k \times k \text{ principal minors are nonzero;} \\ \mathtt{N} & \text{if none of the } k \times k \text{ principal minors are nonzero, i.e., all are zero.} \end{cases}$$

We are interested in which epr-sequences are *attainable* over a given field F, i.e., can be attained by some (symmetric or Hermitian) matrix over F, and also which sequences are *forbidden* over a given field, i.e., no such matrix has the sequence. We can now drop the convention of having a 0th term given by r_0 in the pr-sequence. In particular the relationship between the old and new naming conventions for the beginning of a sequence is as follows: $1]0 \leftrightarrow \mathtt{N}$, $1]1 \leftrightarrow \mathtt{S}$, and $0]1 \leftrightarrow \mathtt{A}$.

Brualdi et al. [3] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [5]. The study of epr-sequences provides additional information that may be helpful in work on the principal minor assignment problem, while remaining somewhat tractable. Furthermore, the enhanced principal rank characteristic sequence can be used to answer the following question [6, p. 112]: "For a real symmetric matrix, which lists of sizes, for which there exists a singular principal submatrix, can occur?" (See Corollary 4.7.)

In Section 2, we identify certain forbidden and certain attainable epr-sequences, with some results depending on the field; the Schur complement method for establishing forbidden subsequences is discussed in this section. In Section 3, we focus on epr-sequences attained by adjacency matrices of graphs. For fields of characteristic 0, we use probabilistic methods in Section 4 to establish that any sequence of As and Ss ending in A is attainable, and any sequence of As and Ss followed by one or more Ns is attainable. For real symmetric matrices, in Section 5 we determine which epr-sequences are attainable for orders 2, 3, 4, and 5.

For $B \in F^{n\times n}$, $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of B lying in rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Further, the complementary submatrix obtained from B by deleting the rows indexed by α and columns indexed by β is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while the complementary principal submatrix is denoted by $B(\alpha)$. The complement of α is denoted by α^c.

Following the notation in [1], we let $\overline{\ell_i \cdots \ell_j}$ indicate that the (complete) sequence may be repeated as many times as desired (or may be omitted entirely). All matrices are symmetric over a field F, or are complex Hermitian.

2 The enhanced principal rank characteristic sequence

We begin with some simple observations and applications of known results that are valid for epr-sequences of symmetric matrices over any field and for complex Hermitian matrices.

Observation 2.1. *If* $\mathrm{pr}(B) = r_0]r_1 \ldots r_n$ *and* $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ *is the enhanced pr-sequence for* B*, then* $\ell_k = \mathtt{N}$ *if and only if* $r_k = 0$*, and* $r_k = 1$ *if and only if* $\ell_k = \mathtt{A}$ *or* $\mathtt{S}$*, for* $k = 1, \ldots, n$.

There is only one submatrix of full order so it either has full rank or it does not, giving A or N as the last term in the epr-sequence. For the classes of matrices considered, the rank of the matrix is equal to the maximum rank of a *principal* submatrix; see, for example, [1, Theorem 1.1]. These statements lead to the following observation.

Observation 2.2. *An epr-sequence of a symmetric (or complex Hermitian) matrix* B *must end in* N *or* A*, and* $\operatorname{rank} B$ *is equal to the index of the last* A *or* S *in epr*(B).

The next result was proved over the real numbers in [3], and for any field in [1].

Theorem 2.3. [1, Theorem 2.1] *Suppose* $B \in F^{n\times n}$ *is symmetric or complex Hermitian,* $\mathrm{epr}(B) = \ell_1 \cdots \ell_n$*, and* $\ell_k = \ell_{k+1} = \mathtt{N}$ *for some* k*. Then* $\ell_i = \mathtt{N}$ *for all* $i \geq k$*. (That is, if an epr-sequence of a matrix ever has* NN*, then it must have* N*s from that point forward.)*

Jacobi's determinantal identity is used to relate the epr-sequence of a nonsingular matrix to that of its inverse. It is valid for symmetric matrices over any field and for complex Hermitian matrices. This implies that most epr-sequences that end in A now come in natural pairs. The situation for pr-sequences is more complicated, with the 0th term in the pr-sequence of the inverse depending on the existence of some zero principal minor of order $n - 1$ in the original matrix.

Theorem 2.4. (Inverse Theorem) *If* $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$ *then* $\mathrm{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2} \cdots \ell_1\mathtt{A}$ *(i.e., simply reverse the sequence except for the last* A*).*

2.1 Forbidden epr-sequences

The next result extends [3, Theorem 4.1] to epr-sequences.

Proposition 2.5. *The epr-sequence* $\mathtt{SN}\cdots\mathtt{A}\cdots$ *is forbidden for symmetric matrices over any field and for complex Hermitian matrices.*

Proof. Suppose $\mathrm{epr}(B) = \mathtt{SN}\cdots$. The S in the first position implies there is some $b_{ii} = 0$. The N in the second position implies $b_{ij} = 0$ for $j = 1, \ldots, n$. Thus every entry in row i is 0, and so for any k there is a singular $k \times k$ submatrix. □

The following theorem, which shows when a portion of an epr-sequence is inherited, is a useful tool when working with subsequences.

Theorem 2.6. *Suppose that $B \in F^{n\times n}$ is symmetric or complex Hermitian, $m \leq n$, and $1 \leq i \leq m$.*

1. *If $[\mathrm{epr}(B)]_i = \mathtt{N}$, then $[\mathrm{epr}(C)]_i = \mathtt{N}$ for all $m \times m$ principal submatrices C.*
2. *If $[\mathrm{epr}(B)]_i = \mathtt{A}$, then $[\mathrm{epr}(C)]_i = \mathtt{A}$ for all $m \times m$ principal submatrices C.*
3. *If $[\mathrm{epr}(B)]_m = \mathtt{S}$, then there exist $m \times m$ principal submatrices C_A and C_N of B such that $[\mathrm{epr}(C_A)]_m = \mathtt{A}$ and $[\mathrm{epr}(C_N)]_m = \mathtt{N}$.*
4. *If $i < m$ and $[\mathrm{epr}(B)]_i = \mathtt{S}$, then there exists an $m \times m$ principal submatrix C_S such that $[\mathrm{epr}(C_S)]_i = \mathtt{S}$.*

Proof. The inheritance of N and A simply follow by noting that a principal submatrix of a principal submatrix is a principal submatrix. The ability to choose ℓ_m in statement 3 follows by noting that there is some submatrix of full rank and there is also some submatrix that is not of full rank, so the appropriate one is chosen.

For the final statement, note that there are two submatrices of order i and that one has full rank and the other does not. Now suppose that the rows/columns of the submatrix with full rank are $p_1, p_2, \dots, p_i$ and that the rows/columns of the submatrix that does not have full rank are $q_1, q_2, \dots, q_i$ (and moreover without loss of generality they are ordered so that any common indices occur in the same spot on the two lists). Now consider the following possible sets of rows and columns.

$$\begin{gathered} p_1, p_2, p_3, \dots, p_i \\ q_1, p_2, p_3, \dots, p_i \\ q_1, q_2, p_3, \dots, p_i \\ q_1, q_2, q_3, \dots, p_i \\ \cdots \\ q_1, q_2, q_3, \dots, q_i \end{gathered}$$

Since the first list corresponds with a submatrix of full rank and the last list does not, then somewhere in between there are two consecutive rows where one list corresponds with a submatrix of full rank and the other list does not. The union of these two row index sets is of cardinality $i + 1$ (since they only differ in one position); thus adding the remaining $m - i - 1$ indices arbitrarily gives a principal submatrix of the correct order with the desired epr-sequence. □

Corollary 2.7. *No symmetric matrix over any field (or complex Hermitian matrix) can have* NSA *in its epr-sequence. Further, no symmetric matrix over any field (or complex Hermitian matrix) can have the epr-sequence $\cdots \mathtt{ASN} \cdots \mathtt{A} \cdots$.*

Proof. By Proposition 2.5, no epr-sequence has the form $\mathtt{SN}\cdots\mathtt{A}$, and thus by Theorem 2.4, no epr-sequence can end with NSA (because if it did then applying the inverse would result in a forbidden epr-sequence). Thus, no epr-sequence can contain NSA (because if it did then Theorem 2.6 could be applied to find a principal submatrix having epr-sequence ending in NSA, giving an impossible epr-sequence). The second statement follows by noting that if such a matrix exists, then there is an appropriate submatrix with an inverse having epr-sequence containing NSA, which is impossible. □

2.2 Forbidden initial epr-sequences and Schur complements

In this section, we rule out certain initial sequences for symmetric matrices over fields of characteristic not 2, and use a technique involving Schur complements to rule out other placements of subsequences.

Proposition 2.8. *No symmetric matrix over a field of characteristic not* 2 *has an epr-sequence starting* $\mathtt{NAN}\cdots$ *or* $\mathtt{NAS}\cdots$.

Proof. Suppose $B \in F^{n\times n}$, $\operatorname{char} F \neq 2$, and $\operatorname{epr}(B) = \mathtt{NA}\ell_3\cdots$. The N in the first position means that the main diagonal entries are all 0, while the A in the second position forces all of the off-diagonal entries to be nonzero. Therefore any 3×3 principal minor is $2pqr \neq 0$ where p, q, r are the three off-diagonal entries, so $\ell_3 = \mathtt{A}$. □

The hypotheses that the matrix is symmetric and $\operatorname{char} F \neq 2$ are important, as the matrices in the next example illustrate.

Example 2.9. The complex Hermitian matrix $B = \begin{bmatrix} 0 & i & 1 \\ -i & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and the symmetric matrix $C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \in {\mathbb{Z}_2}^{n\times n}$ both have epr-sequence NAN.

Corollary 2.10. *A symmetric matrix over a field of characteristic not* 2 *cannot have* SANA *or* NANA *in its epr-sequence.*

Proof. If one of these sequences is present, then by Theorems 2.6 and 2.4 there is an appropriate submatrix with an inverse that has an epr-sequence beginning with NAS or NAN, which is impossible by Proposition 2.8. □

Proposition 2.11. *Over a field of characteristic not* 2 *any epr-sequence of a symmetric matrix that starts* SAN *is of the form* $\mathtt{SAN}\overline{\mathtt{N}}$.

Proof. Let B be a symmetric matrix of order ≥ 4 with $\operatorname{epr}(B) = \mathtt{SAN}\cdots$. Since the first letter of $\operatorname{epr}(B)$ is S, there is at least one 0 term on the diagonal and without loss of generality we can assume it is in the $(1,1)$-position. Since the second letter is A, all the other entries in the first row/column must be nonzero (or else there is a 2×2 principal submatrix that does not have full rank). By a diagonal congruence, we may assume that these other entries in the first row/column are 1.

Next note that, since the third letter in $\mathrm{epr}(B)$ is $\mathtt{N}$, the determinant of the 3×3 principal submatrix $B[\{1,i,j\}]$ is 0, giving

$$0 = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & b_{ii} & b_{ij} \\ 1 & b_{ij} & b_{jj} \end{bmatrix} = 2b_{ij} - b_{ii} - b_{jj}.$$

So

$$b_{ij} = \frac{b_{ii} + b_{jj}}{2},$$

and thus every principal submatrix of B is completely determined by its diagonal. For example, when $n = 4$ such a matrix is of the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & \frac{a+a}{2} & \frac{a+b}{2} & \frac{a+c}{2} \\ 1 & \frac{b+a}{2} & \frac{b+b}{2} & \frac{b+c}{2} \\ 1 & \frac{c+a}{2} & \frac{c+b}{2} & \frac{c+c}{2} \end{bmatrix}$$

for some choice of a, b, c. Matrices of this form have rank 2, i.e., every row is a linear combination of rows one and two. This implies that there is no nonzero minor of order 4 or larger. □

Corollary 2.12. *Suppose B is a symmetric matrix over a field of characteristic not 2 that has $\mathrm{epr}(B) = \mathtt{SAN}\cdots$. Then B has at most two zeros on the diagonal.*

Proof. If $b_{ii} = b_{jj} = b_{kk} = 0$ then by Theorem 2.6, $\mathrm{epr}(B[\{i,j,k\}]) = \mathtt{NAN}$, which is impossible by Proposition 2.8. □

Recall that if C is a given $n\times n$ matrix, with a nonsingular principal submatrix $C[\alpha]$, then the matrix given by $C/C[\alpha] = C[\alpha^c] - C[\alpha^c, \alpha](C[\alpha])^{-1}C[\alpha, \alpha^c]$ is the *Schur complement* of $C[\alpha]$ in C. Schur complements of symmetric matrices have the following properties.

Proposition 2.13. *Suppose F is a field of characteristic not 2 and $C \in F^{n\times n}$ is a symmetric matrix of rank m. Let $C[\alpha]$ be a nonsingular principal submatrix of C with $|\alpha| = k \le m$, and let $B = C/C[\alpha]$. Then the following results hold.*

1. *B is an $(n-k)\times(n-k)$ symmetric matrix.*
2. *Assuming the indexing of B is inherited from C, any principal minor of B is given by*
$$\det B[\gamma] = \det C[\gamma \cup \alpha] / \det C[\alpha].$$
3. $\mathrm{rank}\, B = m - k$.
4. *Any nonsingular principal submatrix of C of order at most m is contained in a nonsingular principal submatrix of order m.*

Proof. The first three statements are all basic properties involving Schur complements, so we omit their verification. For the fourth statement, which is surely known, we offer the following short argument. Suppose $C[\alpha]$ is a full rank principal submatrix of order $k \le m$. Then by property (3), the rank of B is $m-k$. Hence there exists an $(m-k)\times(m-k)$ principal submatrix of B of full rank. In this case, using property (2), it follows that there exists an $m \times m$ principal submatrix of C of full rank that contains α, as desired. □

We note that results of Johnson et al. [6, Section 5 (S)] imply that for a real symmetric matrix, no epr-sequence can end in NAN. Here we present a brief independent proof of a more general result.

Theorem 2.14. *Neither the epr-sequences* NAN *nor* NAS *can occur as a subsequence of the epr-sequence of a symmetric matrix over a field of characteristic not* 2*.*

Proof. Suppose that there exists a real symmetric matrix C with subsequence NAN occurring in its epr-sequence in positions $k+1, k+2, k+3$, respectively. By Proposition 2.8, NAN is not at the start of the epr-sequence for C, and by Theorem 2.3, in the place directly to the left of the first N there is either an A or an S. Then there exists a $k \times k$ principal submatrix of C, say $C[\alpha]$, that is nonsingular. Let $B = C/C[\alpha]$. In this case, using Proposition 2.13.2, the epr-sequence associated with B starts with NAN, which contradicts Proposition 2.8. Hence no such C exists.

If a sequence contained NAS, where the S entry is in the kth position of the epr-sequence for C, then C must have a singular $k \times k$ principal submatrix and further this matrix has nonsingular $(k-1)\times(k-1)$ principal submatrices and only singular $(k-2)\times(k-2)$ principal matrices. Therefore it contains a principal submatrix that contains the subsequence NAN. But this is impossible by the above argument. □

Theorem 2.15. *In the epr-sequence of a symmetric matrix over a field of characteristic not* 2*, the subsequence* ANS *can occur only as the initial subsequence.*

Proof. Suppose that a symmetric matrix B has ANS occurring in positions $k, k+1$, and $k+2$. By Theorem 2.6, B contains some principal submatrix C of order $k+3$ whose epr-sequence $\ell_1 \cdots \ell_{k+3}$ also has $\ell_k \ell_{k+1} \ell_{k+2} = \texttt{ANS}$. Since Corollary 2.7 excludes NSA, and S is not allowed as the last entry of any epr-sequence, $\ell_{k+3} = \texttt{N}$. Because C is singular and contains a nonsingular $(k+2)\times(k+2)$ principal submatrix, the rank of C is $k+2$, and hence by Proposition 2.13(4), every order k principal submatrix is contained in an order $k+2$ nonsingular principal submatrix of C.

Since $\ell_k = \texttt{A}$, any $k \times k$ principal submatrix $C[\alpha]$ of C is nonsingular, so we can take its 3×3 Schur complement $C/C[\alpha]$. Consider $\operatorname{epr}(C/C[\alpha]) = \ell_1' \ell_2' \ell_3'$. By Proposition 2.13(3), $\operatorname{rank}(C/C[\alpha]) = k+2-k = 2$ so ℓ_2' is S or A, and $\ell_3' = \texttt{N}$. Choose a single index i of $C/C[\alpha]$. By Proposition 2.13(2), $\det((C/C[\alpha])[\{i\}]) = \det C[\alpha \cup \{i\}]/\det C[\alpha]$ Since $C[\alpha \cup \{i\}]$ is $(k+1)\times(k+1)$ and $\ell_{k+1} = \texttt{N}$, $\det((C/C[\alpha])[\{i\}]) = 0$, i.e., $\ell_1' = \texttt{N}$. Thus $\operatorname{epr}(C/C[\alpha]) = \texttt{NAN}$ or NSN, but NAN is prohibited by Theorem 2.14, so $\operatorname{epr}(C/C[\alpha]) = \texttt{NSN}$. So we can choose $\{i,j\}$ such that $(C/C[\alpha])[\{i,j\}]$ is singular. Then by Proposition 2.13(2) $\det(C/C[\alpha])[\{i,j\}] = \det C[\alpha \cup \{i,j\}]/\det C[\alpha]$ so $C[\alpha \cup \{i,j\}]$ is singular. Thus $C[\alpha]$ is also contained in a singular $(k+2)\times(k+2)$ principal submatrix of C.

Partition the index set $\{1,\ldots,k+3\}$ into a pair of sets $X = \{i : C(\{i\}) \text{ is singular}\}$ and $Y = \{i : C(\{i\}) \text{ is nonsingular}\}$. If either X or Y had a three-element subset U, then $C(U)$ would be an order k principal submatrix of C that was not contained in both a nonsingular and a singular principal submatrix of order $k+2$. It follows that $|X| = |Y| = 2$, and so $k = 1$. □

2.3 Attainable epr-sequences

We now consider methods for constructing families of matrices attaining given epr-sequences. For order n, the identity matrix is denoted by I_n and the all 1s matrix by J_n, while $\mathbb{1}_n$ denotes the all 1s vector of length n. For a field F, $a \in F$ and $n \geq 2$, define the matrices $L_n(a) := \begin{bmatrix} I_{n-1} & \mathbb{1}_{n-1} \\ \mathbb{1}_{n-1}{}^T & a \end{bmatrix}$.

Observation 2.16. *For any field:*

- $\operatorname{epr}(I_n) = \mathtt{A}\overline{\mathtt{A}}$.
- *For* $n \geq 2$, $\operatorname{epr}(J_n) = \mathtt{A}\overline{\mathtt{N}}$.
- *For* $n \geq 2$, $\operatorname{epr}(I_{n-2} \oplus J_2) = \mathtt{A}\overline{\mathtt{S}}\mathtt{N}$.
- $\operatorname{epr}(0_n) = \mathtt{N}\overline{\mathtt{N}}$.
- *For* $n \geq 2$, $\operatorname{epr}(I_1 \oplus 0_{n-1}) = \mathtt{S}\overline{\mathtt{N}}$.
- *For* $n \geq 2$, $\operatorname{epr}(I_{n-2} \oplus L_2(0)) = \mathtt{S}\overline{\mathtt{S}}\mathtt{A}$.
- *For* $n \geq 2$, $\operatorname{epr}(I_{n-1} \oplus 0_1) = \mathtt{S}\overline{\mathtt{S}}\mathtt{N}$.

The next result follows from [3, Theorem 2.2] and symmetry.

Proposition 2.17. *For a field of characteristic* 0, $n \geq 2$ *and* $1 \leq k \leq n$, $\operatorname{epr}(J_n - kI_n) = \overline{\mathtt{A}}\,\mathtt{N}\,\overline{\mathtt{A}}$ *with the* $\mathtt{N}$ *in the* k*th position.*

Proposition 2.18. *For a field* F *of characteristic* 0, $n \geq 2$, $\operatorname{epr}(L_n(k-1)) = \overline{\mathtt{A}}\,\mathtt{S}\,\overline{\mathtt{A}}\,\mathtt{A}$ *for* $1 \leq k < n$, *with the* $\mathtt{S}$ *in the* k*th position.*

Proof. Suppose $\operatorname{epr}(L_n(k-1)) = \ell_1 \cdots \ell_n$. For $1 \leq m \leq n$, every $m \times m$ principal submatrix is of the form $L_m(k-1)$ or I_m, and note that $\det L_m(k-1) = (k-1) - (m-1) = k - m$. Thus $\ell_m = \mathtt{A}$ for $m \neq k$ and $m = n$, and $\ell_k = \mathtt{S}$. □

It was observed in [3] that given a matrix and its pr-sequence, a matrix that has this pr-sequence extended by an additional 0 can be found by doing a simple copy of the last row and column [3, Theorem 2.6]. However, it *cannot* be guaranteed that $\mathtt{N}$ can be added to an attainable epr-sequence to obtain another attainable epr-sequence. Over a field of characteristic not 2, any epr-sequence ending $\mathtt{NA}$ cannot be extended by adding $\mathtt{N}$ because $\mathtt{NAN}$ is prohibited. The problem is that a simple row and column copy may destroy the delicate property of having all minors of order $i > 1$ be nonsingular. But singularity can be preserved.

Observation 2.19. *Let* $B \in F^{n \times n}$ *have epr-sequence* $\ell_1 \ell_2 \cdots \ell_n$.

1. *Form a matrix* B' *from* B *by copying the last row down and then the last column across. Then the epr-sequence of* B' *is* $\ell_1 \ell'_2 \cdots \ell'_n \mathtt{N}$ *with* $\ell'_i = \mathtt{N}$ *if* $\ell_i = \mathtt{N}$ *and* $\ell'_i = \mathtt{S}$ *otherwise for* $2 \leq i \leq n$.

2. *Form a matrix B'' from B by taking the direct sum with $[0]$. Then the epr-sequence of B'' is $\ell_1''\ell_2''\cdots\ell_n''\mathtt{N}$ with $\ell_i''=\mathtt{N}$ if $\ell_i=\mathtt{N}$ and $\ell_i''=\mathtt{S}$ otherwise for $1\le i\le n$.*

The attainability of the following sequences is established by applying Observation 2.19 to the sequences $\mathtt{S\overline{S}N}$ in Observation 2.16 and $\mathtt{\overline{A}N\overline{A}A}$ in Proposition 2.17.

Corollary 2.20.

1. *For any field, the epr-sequence $\mathtt{S\overline{S}N\overline{N}}$ is attainable.*

2. *For a field of characteristic 0, $n\ge 3$ and $1\le k\le n$, the epr-sequence $\mathtt{\overline{S}N\overline{S}SN\overline{N}}$ with the first $\mathtt{N}$ in the kth position is attainable.*

In [1, Theorem 2.3] it was shown that

$$\operatorname{supp}(B_1\oplus B_2)=(\operatorname{supp}(B_1)+\operatorname{supp}(B_2))\cup\operatorname{supp}(B_1)\cup\operatorname{supp}(B_2) \tag{1}$$

where $\operatorname{supp}(B)=\{i:(\operatorname{pr}(B))_i=1\}$, and for sets X and Y, $X+Y=\{x+y:x\in X,y\in Y\}$. Here we define $\mathtt{AS}(B):=\{i:(\operatorname{epr}(B))_i=\mathtt{A}\text{ or }(\operatorname{epr}(B))_i=\mathtt{S}\}$, $\mathtt{AS}_0(B):=\mathtt{AS}(B)\cup\{0\}$, and $\mathtt{NS}(B):=\{i:(\operatorname{epr}(B))_i=\mathtt{N}\text{ or }(\operatorname{epr}(B))_i=\mathtt{S}\}$. With this notation, (1) becomes $\mathtt{AS}_0(B_1\oplus B_2)=\mathtt{AS}_0(B_1)+\mathtt{AS}_0(B_2)$ (adding a zero into the set avoids the need to take the union, because $\mathtt{AS}(B_i)\subseteq\mathtt{AS}_0(B_1)+\mathtt{AS}_0(B_2), i=1,2$). The next theorem extends this to obtain the epr-sequence of a direct sum of two matrices. Define $[m]:=\{0,1,\dots,m\}$, and note that for any set S, $S+\emptyset=\emptyset$.

Theorem 2.21. (Reducible Matrix Theorem) *Let $B_i\in F^{n_i\times n_i}, i=1,2$ be symmetric matrices over a field F or complex Hermitian matrices and let $\operatorname{epr}(B_1\oplus B_2)=\ell_1\ell_2\cdots\ell_n$. Then*

$$\mathtt{AS}_0(B_1\oplus B_2)=\mathtt{AS}_0(B_1)+\mathtt{AS}_0(B_2),\qquad \mathtt{AS}(B_1\oplus B_2)=\mathtt{AS}_0(B_1\oplus B_2)\setminus\{0\}. \tag{2}$$

$$\mathtt{NS}(B_1\oplus B_2)=\big(\mathtt{NS}(B_1)+[n_2]\big)\cup\big([n_1]+\mathtt{NS}(B_2)\big) \tag{3}$$

$$\ell_i=\begin{cases}\mathtt{A} & \text{if } i\in\mathtt{AS}(B_1\oplus B_2)\setminus\mathtt{NS}(B_1\oplus B_2);\\ \mathtt{S} & \text{if } i\in\mathtt{AS}(B_1\oplus B_2)\cap\mathtt{NS}(B_1\oplus B_2);\\ \mathtt{N} & \text{if } i\in\mathtt{NS}(B_1\oplus B_2)\setminus\mathtt{AS}(B_1\oplus B_2).\end{cases} \tag{4}$$

Proof. As noted earlier, (2) follows from [1, Theorem 2.3]. For (3): $\mathtt{NS}(B)$ is the set of indices k such that B has a singular $k\times k$ principal submatrix. A singular $k\times k$ principal submatrix of $B_1\oplus B_2$ can be obtained by taking a singular piece in B_1 and the rest in B_2 (and by including $0\in[m]$, the piece in B_2 may be empty) or vice versa. Then (4) follows from (2) and (3). □

Corollary 2.22.

- *Over any field, $n\ge 3$ and $1\le s\le n-2$, $\operatorname{epr}(I_{s-1}\oplus L_{n-s+1}(1))=\mathtt{A}\,\overline{\mathtt{S}}\,\overline{\mathtt{A}}\,\mathtt{A}$ where there are $s\ge 1$ copies of $\mathtt{S}$.*

- *Over a field of characteristic 0, $n\ge 2$, $1\le k$, and $1\le s\le n-1$, $\operatorname{epr}(I_{s-1}\oplus(J_{n-s+1}-kI_{n-s+1}))=\overline{\mathtt{A}}\,\overline{\mathtt{S}}\,\overline{\mathtt{A}}\,\mathtt{A}$ where the first $\mathtt{S}$ is in position k and there are s copies of $\mathtt{S}$.*

Over a field of characteristic 0, we can use Hankel matrices of binomial coefficients to generate epr-sequences of the form $\mathtt{S}\overline{\mathtt{A}}\,\overline{\mathtt{N}}$ (another proof that $\mathtt{S}\overline{\mathtt{A}}\,\overline{\mathtt{N}}$ is attainable over any field of characteristic 0 is given in Section 4). Define $H_n^{(k)} = [h_{ij}^{(k)}]$ where $h_{i,j}^{(k)} = \binom{i+j+k-3}{k}$.

Example 2.23. Observe that

$$H_5^{(1)} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}, \quad H_5^{(2)} = \begin{bmatrix} 0 & 1 & 3 & 6 & 10 \\ 1 & 3 & 6 & 10 & 15 \\ 3 & 6 & 10 & 15 & 21 \\ 6 & 10 & 15 & 21 & 28 \\ 10 & 15 & 21 & 28 & 36 \end{bmatrix} \quad H_5^{(3)} = \begin{bmatrix} 0 & 1 & 4 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \\ 4 & 10 & 20 & 35 & 56 \\ 10 & 20 & 35 & 56 & 84 \\ 20 & 35 & 56 & 84 & 120 \end{bmatrix}.$$

It is straightforward to verify that $\operatorname{epr}(H_5^{(1)}) = \mathtt{SANNN}$, $\operatorname{epr}(H_5^{(2)}) = \mathtt{SAANN}$ and $\operatorname{epr}(H_5^{(3)}) = \mathtt{SAAAN}$. Then $\operatorname{epr}(H_4^{(1)}) = \mathtt{SANN}$, $\operatorname{epr}(H_4^{(2)}) = \mathtt{SAAN}$, and $\operatorname{epr}(H_3^{(1)}) = \mathtt{SAN}$ follow from Theorem 2.6 (or by direct verification).

3 Graphs and epr-sequences

The $(0, 1)$ adjacency matrix of a graph G is denoted by $A(G)$. The complete graph, the star graph (centered at 1), the path graph and the cycle graph, all on n vertices, are denoted by $K_n, K_{1,n}, P_n$ and C_n, respectively. Adjacency matrices of graphs provide numerous examples of attainable epr-sequences. Note that the epr-sequence of $A(G)$ for any graph G always begins with $\mathtt{N}$.

Observation 3.1.

- *Over a field of characteristic* 0, $\operatorname{epr}(A(K_n)) = \mathtt{N}\overline{\mathtt{A}}$.
- *For* $n \geq 3$, $\operatorname{epr}(A(K_{1,n-1})) = \mathtt{NS}\overline{\mathtt{N}}$.
- *For* $n \geq 3$, $\operatorname{epr}(A(P_n)) = \overline{\mathtt{NS}}\mathtt{N}$ *if* n *is odd;* $\operatorname{epr}(A(P_n)) = \overline{\mathtt{NS}}\mathtt{NA}$ *if* n *is even.*
- *Over a field of characteristic not* 2*, with* $n \geq 4$, $\operatorname{epr}(A(C_n)) = \overline{\mathtt{NS}}\mathtt{NN}$ *if* $n = 4k$*;* $\operatorname{epr}(A(C_n)) = \overline{\mathtt{NS}}\mathtt{NAA}$ *if* $n = 4k + 1$ *or* $n = 4k + 3$*;* $\operatorname{epr}(A(C_n)) = \overline{\mathtt{NS}}\mathtt{NA}$ *if* $n = 4k + 2$.

The next corollary follows from Observation 3.1 and Observation 2.19.

Corollary 3.2. *For* $n \geq 3$ *and* $1 \leq k \leq n$*, the epr-sequence* $\overline{\mathtt{NS}}\,\overline{\mathtt{N}}\,\mathtt{N}$ *is attainable over a field of characteristic* 0.

For symmetric matrices with zero diagonal we can view the matrix as a weighted adjacency matrix, and associate a (simple) graph to the matrix. The graph $\mathcal{G}(B)$ of a symmetric matrix $B \in F^{n \times n}$ with zero diagonal is the simple graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : b_{ij} \neq 0 \text{ and } i \neq j\}$. Here are some general observations relating terms in an epr-sequence of a symmetric matrix $B \in F^{n \times n}$ with zero diagonal and its associated graph $\mathcal{G}(B)$.

Observation 3.3. *Suppose $B \in F^{n\times n}$ is a symmetric matrix with zero diagonal, and let* $\operatorname{epr}(B) = \mathtt{N}\ell_2 \cdots \ell_n$.

1. *If $\ell_2 = \mathtt{A}$, then $\mathcal{G}(B) = K_n$.*
2. *If $\ell_k = \mathtt{A}$ for some k, then $\mathcal{G}(B)$ has no isolated vertex.*
3. *If $\ell_3 = \mathtt{N}$ and $\operatorname{char} F \neq 2$, then $\mathcal{G}(B)$ is triangle-free (otherwise, B has a 3×3 principal minor equal to $\det \begin{bmatrix} 0 & p & q \\ p & 0 & r \\ q & r & 0 \end{bmatrix} = 2pqr \neq 0$).*

4 Probabilistic techniques for fields of characteristic zero

In this section we use probabilistic methods to establish that over a field of characteristic 0, any epr-sequence that does not contain an $\mathtt{N}$ is attainable (Theorem 4.4), as is any epr-sequence that has all copies of $\mathtt{N}$ consecutively at the end of the sequence (Theorem 4.6).

Proposition 4.1. *Suppose $\operatorname{char} F = 0$ and let $B \in F^{n\times n}$ be symmetric. Assume $\operatorname{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ and $r = \operatorname{rank} B$. Construct a matrix B' from B by adjoining a new last row formed by taking a random linear combination of r independent rows. Then adjoin the same linear combination of the columns. Denote the epr-sequence of B' by $\ell'_1\ell'_2 \cdots \ell'_n\ell'_{n+1}$. Then $\ell'_i = \mathtt{N}$ for $i = r+1, \ldots, n+1$, and for $1 \leq i \leq r$ with high probability $\ell'_i = \mathtt{A}$ if $\ell_i = \mathtt{A}$ and $\ell'_i = \mathtt{S}$ if $\ell_i = \mathtt{S}$ or $\ell_i = \mathtt{N}$.*

Proof. Since the maximum number of linearly independent rows in B' is r, $\ell'_i = \mathtt{N}$ for $i = r+1, \ldots, n+1$. Suppose $k \leq r$. If $\ell_k = \mathtt{S}$, then clearly $\ell'_k = \mathtt{S}$. So it remains to show that with high probability $\ell_k = \mathtt{A}$ implies $\ell'_k = \mathtt{A}$ and $\ell_k = \mathtt{N}$ implies $\ell'_k = \mathtt{S}$.

Let $C = B[T]$ be a $(k-1)\times(k-1)$ principal submatrix of B with $\operatorname{rank} C \geq k-2$. Define $C' := B'[T \cup \{n+1\}]$. We claim that with high probability $\operatorname{rank} C' = k$. If $\operatorname{rank} C = k-1$, then the new row restricted to the first $k-1$ entries is in the span of the rows of C, but with high probability the (k,k) diagonal entry is wrong for adding a new row and column without increasing the rank, so $\operatorname{rank} C' = k$. If $\operatorname{rank} C = k-2$, with high probability the new row is not in the span of the rows of C, so $\operatorname{rank} C' = \operatorname{rank} C + 2 = k$.

Suppose $\ell_k = \mathtt{A}$. Then for every $(k-1)\times(k-1)$ principal submatrix C, it is possible to add a row and column from B and obtain a nonsingular matrix, so $\operatorname{rank} C \geq k-2$, and with high probability $\operatorname{rank} C' = k$. Thus, with high probability $\ell'_k = \mathtt{A}$.

Suppose $\ell_k = \mathtt{N}$. Then $\ell_{k-1} \neq \mathtt{N}$ by Theorem 2.3, so there exists a $(k-1)\times(k-1)$ principal submatrix $C = B[T]$ with $\operatorname{rank} C = k-1$. Then with high probability $\operatorname{rank} C' = k$, and so with high probability $\ell'_k = \mathtt{S}$. □

Lemma 4.2. *Suppose $\operatorname{char} F = 0$ and let $B \in F^{n\times n}$ be symmetric. Assume $\operatorname{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$ with $\ell_k \in \{\mathtt{A}, \mathtt{S}\}$ for $k = 1, \ldots, n-1$. Then there exists a matrix $B' \in F^{(n+1)\times(n+1)}$ such that $\operatorname{epr}(B') = \ell_1 \cdots \ell_{n-1}\mathtt{AA}$.*

Proof. Form a symmetric matrix B' from B by appending a last row and column of entries chosen as random rational numbers. Let $\text{epr}(B') = \ell_1\ell_2' \cdots \ell_n'\ell_{n+1}'$. Clearly if $\ell_k = \mathtt{S}$ then $\ell_k' = \mathtt{S}$. We show that if $\ell_k = \mathtt{A}$ then $\ell_k' = \mathtt{A}$, and $\ell_{n+1}' = \mathtt{A}$, both with high probability. Suppose $\ell_k = \mathtt{A}$. Then for every $(k-1) \times (k-1)$ principal submatrix, it is possible to append a row and column and obtain a nonsingular matrix. Then with high probability for any $(k-1) \times (k-1)$ principal submatrix of B, appending the relevant part of row and column $n+1$ in B' results in a nonsingular matrix. Similarly, with high probability, B' is nonsingular. □

Lemma 4.3. *Suppose* $\operatorname{char} F = 0$ *and let* $B \in F^{n\times n}$ *be symmetric. Assume* $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$ *with* $\ell_k \in \{\mathtt{A}, \mathtt{S}\}$ *for* $k = 1, \ldots, n$*. Then there exists a matrix* $B' \in F^{(n+1)\times(n+1)}$ *such that* $\text{epr}(B') = \ell_1 \cdots \ell_{n-1}\mathtt{SA}$.

Proof. Since $\ell_{n-1} \in \{\mathtt{A}, \mathtt{S}\}$, there exists an index i such that $B(\{i\})$ is nonsingular; without loss of generality $i = 1$, and we abbreviate this submatrix by $B(1)$. Then because any $(n-1)$-vector is in the range of $B(1)$, B can be partitioned as $B = \begin{bmatrix} c & \mathbf{v}^T B(1) \\ B(1)\mathbf{v} & B(1) \end{bmatrix}$ with $c \neq \mathbf{v}^T B(1)\mathbf{v}$ because B is nonsingular. Define

$$B' := \begin{bmatrix} c & \mathbf{v}^T B(1) & c' \\ B(1)\mathbf{v} & B(1) & B(1)\mathbf{w} \\ c' & \mathbf{w}^T B(1) & \mathbf{w}^T B(1)\mathbf{w} \end{bmatrix}$$

where $B(1)\mathbf{w}$ is a random vector and c' is random. Then with high probability B' is nonsingular and no ℓ_k has been altered for $i \leq n-1$. Observe that $B'(1)$ is singular. □

Theorem 4.4. *Any epr-sequence that does not contain* N *and ends in* A *is attainable over every field of characteristic* 0*.*

Proof. The proof is by induction. The sequences A, AA, and SA are all attainable. Assume all epr-sequences of length $\leq n$ consisting of A and S and ending in A are attainable. Consider the sequence $\ell_1 \cdots \ell_n\mathtt{A}$. The sequence $\mathtt{S}\overline{\mathtt{S}}\mathtt{A}$ is attainable (Observation 2.16), so assume there exists $i \leq n$ such that $\ell_i = \mathtt{A}$ and let k be the largest index such that $\ell_k = \mathtt{A}$. By the induction hypothesis there is a symmetric matrix B such that $\text{epr}(B) = \ell_1 \cdots \ell_k$.

If $k = n$ then $\ell_1 \cdots \ell_n\mathtt{A}$ is attainable by Lemma 4.2. So assume $k < n$ and $\ell_{k+1} = \cdots = \ell_n = \mathtt{S}$. By the induction hypothesis $\ell_1 \cdots \ell_{k-1}\mathtt{A}$ is attainable. Then by applying Lemma 4.2 followed by Lemma 4.3 $n-k$ times, $\ell_1 \cdots \ell_{k-1}\mathtt{AS} \cdots \mathtt{SA} = \ell_1 \cdots \ell_n\mathtt{A}$ is attained. □

Lemma 4.5. *Suppose* $\operatorname{char} F = 0$ *and let* $B \in F^{n\times n}$ *be symmetric. Assume* $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$ *with* $\ell_k \in \{\mathtt{A}, \mathtt{S}\}$ *for* $k = 1, \ldots, n$*. Then there exists a matrix* $B' \in F^{(n+1)\times(n+1)}$ *such that* $\text{epr}(B') = \ell_1 \cdots \ell_{n-1}\mathtt{SN}$.

Proof. Since $\ell_{n-1} \in \{\mathtt{A}, \mathtt{S}\}$, there exists an index i such that $B(\{i\})$ is nonsingular; without loss of generality $i = 1$, and we abbreviate this submatrix by $B(1)$. Then because any $(n-1)$-vector is in the range of $B(1)$, B can be partitioned as $B = \begin{bmatrix} c & \mathbf{v}^T B(1) \\ B(1)\mathbf{v} & B(1) \end{bmatrix}$ with $c \neq \mathbf{v}^T B(1)\mathbf{v}$ because B is nonsingular. Define

$$B' := \begin{bmatrix} c & \mathbf{v}^T B(1) & \mathbf{v}^T B(1)\mathbf{w} \\ B(1)\mathbf{v} & B(1) & B(1)\mathbf{w} \\ \mathbf{w}^T B(1)\mathbf{v} & \mathbf{w}^T B(1) & \mathbf{w}^T B(1)\mathbf{w} \end{bmatrix}$$

where $B(1)\mathbf{w}$ is a random vector. Then with high probability no ℓ_k has been altered for $i \leq n-1$. Observe that B' and $B'(1)$ are singular. □

Theorem 4.6. *Any epr-sequence* $\ell_1 \ell_2 \cdots \ell_n \mathtt{N} \cdots \mathtt{N}$ *with* $\ell_k \in \{\mathtt{A}, \mathtt{S}\}$ *for* $k = 1, \ldots, n$ *and* $t \geq 1$ *copies of* $\mathtt{N}$ *is attainable over a field of characteristic* 0.

Proof. By Theorem 4.4 $\ell_1 \cdots \ell_{n-1}\mathtt{A}$ is attainable. If $\ell_n = \mathtt{A}$, then apply Proposition 4.1 t times to $\ell_1 \cdots \ell_{n-1}\mathtt{A}$ to obtain $\ell_1 \cdots \ell_{n-1}\mathtt{A}\mathtt{N} \cdots \mathtt{N} = \ell_1 \cdots \ell_n \mathtt{N} \cdots \mathtt{N}$. If $\ell_n = \mathtt{S}$, then apply Lemma 4.5 to $\ell_1 \cdots \ell_{n-1}\mathtt{A}$ to obtain $\ell_1 \cdots \ell_{n-1}\mathtt{S}\mathtt{N} = \ell_1 \cdots \ell_n \mathtt{N}$, and then apply Proposition 4.1 $t-1$ times to $\ell_1 \cdots \ell_n \mathtt{N}$ to obtain $\ell_1 \cdots \ell_n \mathtt{N} \cdots \mathtt{N}$. □

Theorems 4.4 and 4.6 can be used to answer the following question of Johnson et al. [6, p. 112]: Which subsets T of $\{1, \ldots, n\}$ can occur as the list of sizes k for which there exists a $k \times k$ singular principal submatrix of B?

Corollary 4.7. *For any subset* T *of* $\{1, \ldots, n\}$ *there exists an* $n \times n$ *real symmetric matrix such that* T *is the list of sizes* k *for which there exists a* $k \times k$ *singular principal submatrix of* B.

Proof. A real symmetric matrix B realizes such a list of sizes T if and only if $\operatorname{epr}(B) = \ell_1 \cdots \ell_n$ and for $k = 1, \ldots, n$, $k \in T$ if and only if $\ell_k = \mathtt{N}$ or $\mathtt{S}$. So given T, use Theorem 4.4 or 4.6 to construct matrix B with $\operatorname{epr}(B) = \ell_1 \cdots \ell_n$ having the following properties:

- $\ell_k = \mathtt{S}$ if $k \in T$ and $\ell_k = \mathtt{A}$ if $k \notin T$ for $k = 1, \ldots n-1$.
- $\ell_n = \mathtt{N}$ if $n \in T$ and $\ell_n = \mathtt{A}$ if $n \notin T$. □

The need to prove Theorems 4.4 and 4.6 also illustrates the additional information provided by the enhanced principal rank characteristic sequence, as opposed to the principal rank characteristic sequence (the only pr-sequences covered by these theorems are of the form $\overline{1}\,\overline{0}$, which is attained by $I_k \oplus 0_{n-k}$).

5 The enhanced principal rank characteristic sequence over $\mathbb{R}$

The next result was established in the proof of [3, Proposition 8.1].

Theorem 5.1. [3, Proposition 8.1] *Any attainable epr-sequence over* $\mathbb{R}$ *that begins with* `AN` *is attainable by a matrix with every entry equal to* 1 *or* -1.

Therefore by a computer search we can find all such sequences, and the following table gives all attainable epr-sequences beginning with `AN` up through $n = 10$.

Table 1: All attainable epr-sequences beginning with `AN` up through $n = 10$.

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
AN	ANA	ANAA	ANAAA	ANAAAA	ANAAAAA	ANAAAAAA	ANAAAAAAA	ANAAAAAAAA
	ANN	ANNN	ANNNN	ANNNNN	ANNNNNN	ANNNNNNN	ANNNNNNNN	ANNNNNNNNN
		ANSN	ANSNA	ANSNAA	ANSNNNN	ANSNNNNN	ANSNNNNNN	ANSNNNNNNN
			ANSNN	ANSNNN	ANSNSNA	ANSNSNNN	ANSNSNNNN	ANSNSNNNNN
			ANSSA	ANSNSN	ANSNSNN	ANSNSNSN	ANSNSNSNA	ANSNSNSNNN
			ANSSN	ANSSAA	ANSNSSN	ANSNSSNN	ANSNSNSNN	ANSNSNSNSN
				ANSSNN	ANSSNNN	ANSSNNNN	ANSNSSNNN	ANSNSSNNNN
				ANSSSA	ANSSSAA	ANSSSAAA	ANSSNNNNN	ANSSNNNNNN
				ANSSSN	ANSSSNA	ANSSSNAA	ANSSSAAAA	ANSSSNNNNN
					ANSSSNN	ANSSSNNN	ANSSSNNNN	ANSSSNSNNN
					ANSSSSA	ANSSSNSN	ANSSSNSNA	ANSSSNSNSN
					ANSSSSN	ANSSSSAA	ANSSSNSNN	ANSSSNSSNN
						ANSSSSAN	ANSSSNSSA	ANSSSNSSSN
						ANSSSSNN	ANSSSNSSN	ANSSSSAAAA
						ANSSSSSA	ANSSSSAAA	ANSSSSAAAN
						ANSSSSSN	ANSSSSNNN	ANSSSSNNNN
							ANSSSSSAA	ANSSSSSAAA
							ANSSSSSAN	ANSSSSSNAA
							ANSSSSSNA	ANSSSSSNNN
							ANSSSSSNN	ANSSSSSNSN
							ANSSSSSSA	ANSSSSSSAA
							ANSSSSSSN	ANSSSSSSAN
								ANSSSSSSNN
								ANSSSSSSSA
								ANSSSSSSSN

5.1 Epr-sequences of order at most 4 over $\mathbb{R}$

Recall that an epr-sequence must end in either `A` or `N`. For order n, this gives $2 \times 3^{n-1}$ sequences. There are two epr-sequences of order 1 that end in `A` or `N`, namely `A` attained by I_1, and `N` attained by 0_1. There are six epr-sequences of order 2 that end in `A` or `N`, and they can all be attained; an integer example for each is given in Table 2.

There are 18 epr-sequences of order 3 that end in `A` or `N`. By applying the results established previously, the following epr-sequences for matrices of order 3 over $\mathbb{R}$ can be eliminated: `NSA`, `NAN`, `SNA`, `NNA` (Corollary 2.7, Theorem 2.14, Proposition 2.5, and Theorem 2.3). Each of the remaining 14 epr-sequences can be realized by an integer matrix; see Table 3.

Table 2: All epr-sequences for order 2 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

epr-Sequence	Real matrix	Result
AA	I_2	Observation 2.16
AN	J_2	Observation 2.16
NA	$A(K_2) = J_2 - I_2$	Proposition 2.17
NN	0_2	Observation 2.16
SA	$L_2(0)$	Proposition 2.18
SN	$I_1 \oplus 0_1$	Observation 2.16

Table 3: All epr-sequences for order 3 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

epr-Sequence	Real matrix	Result
AAA	I_3	Observation 2.16
AAN	$J_3 - 3I_3$	Proposition 2.17
ANA	$J_3 - 2I_3$	Proposition 2.17
ANN	J_3	Observation 2.16
ASA	$L_3(1)$	Proposition 2.18
ASN	$I_1 \oplus J_2$	Observation 2.16
NAA	$A(K_3) = J_3 - I_3$	Proposition 2.17
NNN	0_3	Observation 2.16
NSN	$A(K_{1,2})$	Observation 3.1
SAA	$L_3(0)$	Proposition 2.18
SAN	$H_3^{(1)}$	Example 2.23
SNN	$I_1 \oplus 0_2$	Observation 2.16
SSA	$I_1 \oplus L_2(0)$	Observation 2.16
SSN	$I_2 \oplus 0_1$	Observation 2.16

We now determine all attainable sequences of order 4 over $\mathbb{R}$. The next result follows from Theorem 5.1 and Table 1.

Corollary 5.2. *The only attainable order* 4 *epr-sequences that begin* `AN` *are* `ANAA`, `ANNN`, *and* `ANSN`.

There are 54 order 4 sequences that end in `N` or `A`. Of these 54 sequences, we eliminate those that contain `NNS` or `NNA` (Theorem 2.3), leaving 47 possible sequences. The subsequences `NSA`, `NAS`, and `NAN` are ruled out by Corollary 2.7 and Theorem 2.14, leaving 37 possible sequences. The sequence `SANA` is ruled out by Proposition 2.11, the sequence `ASNA` is ruled out by Corollary 2.7, and the sequence `SNAA` is ruled out by Proposition 2.5. The remaining 34 epr-sequences are all attainable over $\mathbb{R}$; see Table 4. The next example gives a realizing matrix for the particular sequence `NAAN`. For a given epr-sequence, the notation M_{epr} denotes a specific matrix realizing this epr-sequence.

Example 5.3. For $M_{\texttt{NAAN}} := \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}$, $\text{epr}(M_{\texttt{NAAN}}) = \texttt{NAAN}$.

Table 4: All epr-sequences for order 4 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

epr-Sequence	Real matrix	Result
AAAA	I_4	Observation 2.16
AAAN	$J_4 - 4I_4$	Proposition 2.17
AANA	$J_4 - 3I_4$	Proposition 2.17
AANN		Theorem 4.6
AASA	$L_4(0)^{-1}$	Theorem 2.4, Proposition 2.18
AASN		Theorem 4.6
ANAA	$J_4 - 2I_4$	Proposition 2.17
ANNN	J_4	Observation 2.16
ANSN		Corollary 5.2
ASAA	$L_4(1)$	Proposition 2.18
ASAN		Theorem 4.6
ASNN		Theorem 4.6
ASSA	$I_1 \oplus L_3(1)$	Corollary 2.22
ASSN		Theorem 4.6
NAAA	$A(K_4) = J_4 - I_4$	Proposition 2.17
NAAN	$M_{\texttt{NAAN}}$	Example 5.3
NNNN	0_4	Observation 2.16
NSNA	$A(P_4)$	Observation 3.1
NSNN	$A(K_{1,3})$	Observation 3.1
NSSA	$A(G_1)$	G_1 = paw graph
NSSN	$(J_3 - I_3) \oplus 0_1$	Corollary 2.20
SAAA	$L_4(0)$	Proposition 2.18
SAAN	$H_4^{(2)}$	Example 2.23
SANN	$H_4^{(1)}$	Example 2.23
SASA		Theorem 4.4
SASN		Theorem 4.6
SNNN	$I_1 \oplus 0_3$	Observation 2.16
SNSN	$(J_3 - 2I_3) \oplus 0_1$	Corollary 2.20
SSAA	$I_1 \oplus A(K_3)$	Corollary 2.22
SSAN		Theorem 4.6
SSNA	$A(G_1)^{-1}$	Theorem 2.4, G_1 = paw graph
SSNN	$I_2 \oplus 0_2$	Corollary 2.20
SSSA	$I_2 \oplus L_2(0)$	Observation 2.16
SSSN	$I_3 \oplus 0_1$	Observation 2.16

The list of reasons that epr-sequences are unattainable (and thus not listed in Table 4) is summarized in [4], and similarly for order 5. Note that the justifications for attainability given in Table 4 do not provide explicit matrices for some of these sequences. In each case where no matrix is listed, we have constructed such matrices using essentially the method cited[1] and the documentation is available in [4].

5.2 Epr-sequences of order 5 over $\mathbb{R}$

The next result follows from Theorem 5.1 and Table 1.

Corollary 5.4. *The only attainable order* 5 *epr-sequences that begin* AN *are* ANAAA, ANNNN, ANSNA, ANSNN, ANSSA, *and* ANSSN.

[1] In the case of random linear combinations, nonrandom combinations with the same independence properties were used.

Proposition 5.5. *The epr-sequences* NSSNA *and* NAANA *are forbidden as initial subsequences of epr-sequences for real symmetric matrices.*

Proof. We use X to denote A or S, for the purpose of connecting epr-sequences to pr-sequences. Because 1]01101 is forbidden as an initial sequence of a pr-sequence for real symmetric matrices [3, Theorem 6.4], all epr-sequences of the form NXXNA are forbidden as the initial sequence of an epr-sequence for real symmetric matrices. In particular, NSSNA and NAANA are forbidden. □

There are 162 order 5 sequences that end in N or A. Of these 162 sequences, we eliminate those that contain NNS or NNA (Theorem 2.3); there are 33 such sequences, leaving 129 remaining possible sequences. Of these, those containing the subsequences NSA, NAS, and NAN are ruled out by Corollary 2.7 and Theorem 2.14; there are 39 additional such sequences, leaving 90 possible sequences. The subsequences SANA and SANS are ruled out by Proposition 2.11; there are 4 additional such sequences, leaving 86 possible sequences. The sequences AASNA, ASNAA, and SASNA are ruled out by Corollary 2.7, the sequences SNAAA, SNAAN, SNSNA, and SNSSA are ruled out by Proposition 2.5, and the sequence AANSN is is ruled out by Theorem 2.15, leaving 78 remaining possible sequences. The sequences NSSNA and NAANA are eliminated by Proposition 5.5, leaving 76 remaining possible sequences. Finally, ANAAN is ruled out by exhaustive search (Corollary 5.4). The remaining 75 epr-sequences are all attainable over $\mathbb{R}$; see Table 5. The next example gives some of the realizing matrices used to establish that these epr-sequences are attainable.

Example 5.6.

$$M_{\texttt{ASNSN}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}, \; M_{\texttt{ASSNA}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}, \; M_{\texttt{NAAAN}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix},$$

$$M_{\texttt{NAANN}} = \begin{bmatrix} 0 & 1 & 1 & 3 & 3 \\ 1 & 0 & -3 & -4 & -1 \\ 1 & -3 & 0 & -1 & -4 \\ 3 & -4 & -1 & 0 & -3 \\ 3 & -1 & -4 & -3 & 0 \end{bmatrix}, \; M_{\texttt{NAASA}} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & -1 & 2 & 0 \end{bmatrix}, \; M_{\texttt{NAASN}} = \begin{bmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix},$$

$$M_{\texttt{NSSAN}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}, \; M_{\texttt{SSNAA}} = \begin{bmatrix} 0 & 3 & 3 & 0 & 0 \\ 3 & 2 & 1 & 3 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}, \; M_{\texttt{SSSNA}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 5: All epr-sequences for order 5 that can be attained by real symmetric matrices. The sequences are listed in lexicographic order.

epr-Sequence	Real matrix	Result
AAAAA	I_5	Observation 2.16
AAAAN	$J_5 - 5I_5$	Proposition 2.17
AAANA	$J_5 - 4I_5$	Proposition 2.17
AAANN		Theorem 4.6
AAASA	$L_5(3)$	Proposition 2.18
AAASN		Theorem 4.6
AANAA	$J_5 - 3I_5$	Proposition 2.17
AANNN		Theorem 4.6
AASAA	$L_5(2)$	Proposition 2.18
AASAN		Theorem 4.6
AASNN		Theorem 4.6
AASSA	$I_1 \oplus (J_4 - 3I_4)$	Corollary 2.22
AASSN		Theorem 4.6
ANAAA	$J_5 - 2I_5$	Proposition 2.17
ANNNN	J_5	Observation 2.16
ANSNA		Corollary 5.4
ANSNN		Corollary 5.4
ANSSA		Corollary 5.4
ANSSN		Corollary 5.4
ASAAA	$L_5(1)$	Proposition 2.18
ASAAN		Theorem 4.6
ASANN		Theorem 4.6
ASASA		Theorem 4.4
ASASN		Theorem 4.6
ASNNN		Theorem 4.6
ASNSN	$M_{\mathtt{ASNSN}}$	Example 5.6
ASSAA	$I_1 \oplus L_4(1)$	Corollary 2.22
ASSAN		Theorem 4.6
ASSNA	$M_{\mathtt{ASSNA}}$	Example 5.6
ASSNN		Theorem 4.6
ASSSA	$I_2 \oplus (J_3 - 2I_3)$	Corollary 2.22
ASSSN	$I_3 \oplus J_2$	Observation 2.16
NAAAA	$A(K_5) = J_5 - I_5$	Proposition 2.17
NAAAN	$M_{\mathtt{NAAAN}}$	Example 5.6
NAANN	$M_{\mathtt{NAANN}}$	Example 5.6
NAASA	$M_{\mathtt{NAASA}}$	Example 5.6
NAASN	$M_{\mathtt{NAASN}}$	Example 5.6
NNNNN	0_5	Observation 2.16
NSNAA	$A(C_5)$	Observation 3.1
NSNNN	$A(K_{1,4})$	Observation 3.1
NSNSN	$A(P_5)$	Observation 3.1
NSSAA	$A(G_2)$	G_2 where G_2 is the bowtie graph
NSSAN	$M_{\mathtt{NSSAN}}$	Example 5.6
NSSNN	$(J_3 - I_3) \oplus 0_2$	Corollary 2.20
NSSSA	$A(G_3)$	G_3 is the house graph
NSSSN	$(J_4 - I_4) \oplus 0_1$	Corollary 2.20
SAAAA	$L_5(0)$	Proposition 2.18
SAAAN	$H_5^{(3)}$	Example 2.23
SAANA	${M_{\mathtt{NAASA}}}^{-1}$	Example 5.6 and Theorem 2.4
SAANN	$H_5^{(2)}$	Example 2.23
SAASA		Theorem 4.4
SAASN		Theorem 4.6
SANNN	$H_5^{(1)}$	Example 2.23

Table 5 (continued): All epr-sequences for order 5 that can be attained by real symmetric matrices.

epr-Sequence	Real matrix	Result
SASAA		Theorem 4.4
SASAN		Theorem 4.6
SASNN		Theorem 4.6
SASSA		Theorem 4.4
SASSN		Theorem 4.6
SNNNN	$I_1 \oplus 0_4$	Observation 2.16
SNSNN	$(J_3 - 2I_3) \oplus 0_2$	Corollary 2.20
SNSSN	$(J_4 - 2I_4) \oplus 0_1$	Corollary 2.20
SSAAA	$I_1 \oplus (J_4 - I_4)$	Corollary 2.22
SSAAN		Theorem 4.6
SSANN		Theorem 4.6
SSASA		Theorem 4.4
SSASN		Theorem 4.6
SSNAA	$M_{\texttt{SSNAA}}$	Example 5.6
SSNNN	$I_2 \oplus 0_3$	Corollary 2.20
SSNSN	$(J_4 - 3I_4) \oplus 0_1$	Corollary 2.20
SSSAA	$I_2 \oplus (J_3 - I_3)$	Corollary 2.22
SSSAN		Theorem 4.6
SSSNA	$M_{\texttt{SSSNA}}$	Example 5.6
SSSNN	$I_3 \oplus 0_2$	Corollary 2.20
SSSSA	$I_3 \oplus L_2(0)$	Observation 2.16
SSSSN	$I_4 \oplus 0_1$	Observation 2.16

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