PATH COVER NUMBER, MAXIMUM NULLITY, AND ZERO FORCING NUMBER OF ORIENTED GRAPHS AND OTHER SIMPLE DIGRAPHS

ADAM BERLINER*, CORA BROWN†, JOSHUA CARLSON‡, NATHANAEL COX*, LESLIE HOGBEN§, JASON HU¶, KATRINA JACOBS‖, KATHRYN MANTERNACH**, TRAVIS PETERS††, NATHAN WARNBERG‡, AND MICHAEL YOUNG‡

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Abstract. An oriented graph is a simple digraph obtained from a simple graph by choosing exactly one of the two arcs (u,v) or (v,u) to replace each edge $\{u,v\}$. A simple digraph describes the zero-nonzero pattern of off-diagonal entries of a family of (not necessarily symmetric) matrices. Minimum rank of a simple digraph is the minimum rank of this family of matrices; maximum nullity is defined analogously. The simple digraph zero forcing number and path cover number are related parameters. We establish bounds on the range of possible values of all these parameters for oriented graphs, establish connections between the values of these parameters for a simple graph G, for various orientations $\vec{G}$ and for the doubly directed digraph $\overleftrightarrow{G}$, and establish an upper bound on the number of arcs in a simple digraph in terms of the zero forcing number.

Keywords. zero forcing number, maximum nullity, minimum rank, path cover number, simple digraph, oriented graph

AMS subject classifications. 05C50, 05C20, 15A03

1. Introduction.

The maximum nullity and zero forcing number of simple digraphs are studied in [10] and [5]. We study connections between these parameters and path cover number, and we study all these parameters for special types of digraphs derived from graphs, including oriented graphs and doubly directed graphs. Section 2 considers oriented graphs. We establish a bound on the difference of values of the parameters path cover number, maximum nullity, and zero forcing number for

*Department of Mathematics, Statistics, and Computer Science, St. Olaf College, Northfield, MN 55057 (berliner@stolaf.edu), (coxn@stolaf.edu). Research of N. Cox supported by NSF DMS 0750986

†Department of Mathematics, Carleton College, Northfield, MN 55057 (brownc@carleton.edu). Research supported by NSF DMS 0750986.

‡Department of Mathematics, Iowa State University, Ames, IA 50011 (jmsdg7@iastate.edu), (warnberg@iastate.edu), (myoung@iastate.edu). Research of J. Carlson supported by ISU Holl funds. Research of M. Young supported by NSF DMS 0946431.

§Department of Mathematics, Iowa State University, Ames, IA 50011, (lhogben@iastate.edu) and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).

¶Department of Mathematics, University of California, Berkeley, CA 94720-3840 (jason.hu@berkeley.edu). Research supported by NSF DMS 0750986.

‖Department of Mathematics, Pomona College, 610 North College Avenue, Claremont, CA 91711 (klj02011@mymail.pomona.edu). Research supported by NSF DMS 0750986.

**Department of Mathematics and Computer Science, Central College, Pella, IA 50219 (manternachk1@central.edu). Research supported by ISU Holl funds.

††Natural and Mathematical Science Division, Culver-Stockton College, Canton, MO 63435 (tpeters319@gmail.com).

two orientations of one graph, and determine the range of values of these parameters for orientations of paths and cycles, and some of the possible values for tournaments. We establish connections between these parameters for a simple graph and its doubly directed digraph in Section 3. In Section 4, we establish an upper bound on the number of arcs of a simple digraph in terms of the zero forcing number. We also show that several results for simple graphs fail for oriented graphs, including the Graph Complement Conjecture and Sinkovic's theorem that maximum nullity is at most path cover number for outerplanar graphs.

All graphs and digraphs are taken to be simple. We use $G = (V(G), E(G))$ to denote a graph and $\Gamma = (V(\Gamma), E(\Gamma))$ to denote a digraph, often using V and E when G or Γ is clear. For a digraph Γ and $R \subseteq V$, the *induced subdigraph* $\Gamma[R]$ is the digraph with vertex set R and arc set $\{(v, w) \in E : v, w \in R\}$; an analogous definition is used for graphs. The subdigraph induced by the complement $\overline{R}$ is also denoted by $\Gamma - R$, or in the case R is a single vertex v, by $\Gamma - v$. A digraph $\Gamma = (V, E)$ is *transitive* if for all $u, v, w \in V$, $(u, v), (v, w) \in E$ implies $(u, w) \in E$.

For a digraph $\Gamma = (V, E)$ having $v, u \in V$ and $(v, u) \in E$, u is an *out-neighbor* of v and v is an *in-neighbor* of u. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of out-neighbors of v in Γ, and analogously for *in-degree*, denoted by $\deg^-(v)$. Define $\delta^+(\Gamma) = \min\{\deg^+(v) : v \in V\}$ and $\delta^-(\Gamma) = \min\{\deg^-(v) : v \in V\}$. For a digraph Γ, the *reversal* Γ^{T} is obtained from Γ by reversing all the arcs.

Let G be a graph. A *path* in G is a subgraph $P = (\{v_1, \ldots, v_k\}, E(P))$ where $E(P) = \{\{v_i, v_{i+1}\} :\leq i \leq k - 1\}$; this path is often denoted by $(v_1, \ldots, v_k)$ and its length is $k - 1$. We say that a path in G is an *induced path* if it is an induced subgraph of G. A *path cover* of a graph G is a set of vertex disjoint induced paths that includes all vertices of G.

Now suppose Γ is a digraph. A *path* in Γ is a subdigraph $P = (\{v_1, \ldots, v_k\}, E(P))$ where $E(P) = \{(v_i, v_{i+1}) : 1 \leq i \leq k - 1\}$; this path is often denoted by $(v_1, \ldots, v_k)$, its length is $k - 1$, and the arcs of $E(P)$ are called *path arcs*. If $(v_1, \ldots, v_k)$ is a path in Γ, v_1 is called the *initial vertex* and v_k is the *terminal vertex*. We say vertex u has *access* to v in Γ if there is a path from u to v. A path $(v_1, \ldots, v_k)$ in Γ is an *induced path* if E does not contain any arc of the form (v_i, v_j) with $j > i + 1$ or $i > j + 1$. We note this does not necessarily imply that the path subdigraph is induced, because any of the arcs in $\{(v_{i+1}, v_i) : 1 \leq i \leq k - 1\}$ are permitted. A path $(v_1, \ldots, v_k)$ in Γ is *Hessenberg* if E does not contain any arc of the form (v_i, v_j) with $j > i + 1$. Any induced path is Hessenberg but not vice versa. A *path cover* for Γ is a set of vertex disjoint Hessenberg paths that includes all vertices of Γ [10].

For graphs G and digraphs Γ, the *path cover number* $\mathrm{P}(G)$ or $\mathrm{P}(\Gamma)$ is the minimum number of paths in a path cover (induced for a graph, Hessenberg for a digraph) and a *minimum path cover* is a path cover with this minimum number of paths.

Zero forcing was introduced in [1] for (simple) graphs. We define zero forcing for (simple) digraphs as in [10]. Let Γ be a digraph with each vertex colored either white or blue[1]. The *color change rule* is: If u is a blue vertex of Γ, and exactly one out-neighbor v of u is white, then change the color of v to blue. In this situation, we say that u *forces* v and write $u \to v$. Given a coloring of Γ, the *final coloring* is the result of applying the color change rule until no more changes are possible. A *zero forcing set* for Γ is a subset of vertices B such that, if initially the vertices of B are colored blue and the remaining vertices are white, the final coloring of Γ is all blue. The *zero forcing number* $\mathrm{Z}(\Gamma)$ is the minimum of $|B|$ over all zero forcing sets $B \subseteq V(\Gamma)$.

For a given zero forcing set B for Γ, we create a *chronological list of forces* by constructing the final coloring, listing the forces in the order in which they were performed. Although for a given set of vertices B the final coloring is unique, B need not have a unique chronological list of forces. Suppose Γ is a digraph and $\mathcal{F}$ is a chronological list of forces for a zero forcing set B. A *forcing chain* is an ordered set of vertices $(w_1, w_2, \ldots, w_k)$ where $w_j \to w_{j+1}$ is a force in $\mathcal{F}$ for $1 \leq j \leq k-1$. A *maximal forcing chain* is a forcing chain that is not a proper subset of another forcing chain. The following results will be used.

LEMMA 1.1. [10] *Suppose* Γ *is a digraph and* $\mathcal{F}$ *is a chronological list of forces of a zero forcing set* B*. Then, every maximal forcing chain is a Hessenberg path that starts with a vertex in* B*.*

For a fixed chronological list of forces $\mathcal{F}$ of a zero forcing set B of Γ, the *chain set* is the set of all maximal forcing chains. By Lemma 1.1, the chain set of $\mathcal{F}$ is a path cover, called a *zero forcing path cover* and the maximal forcing chains are also called *forcing paths*.

PROPOSITION 1.2. [10] *For any digraph* Γ, $\mathrm{P}(\Gamma) \leq \mathrm{Z}(\Gamma)$.

A *cycle* of length $k \geq 3$ in a graph G or digraph Γ is a sub(di)graph consisting of a path $(v_1, \ldots, v_k)$ and the additional edge or arc $\{v_k, v_1\}$ or (v_k, v_1).

LEMMA 1.3. *Suppose* $P = (v_1, \ldots, v_k)$ *is a Hessenberg path in a digraph* Γ*. Then* P *is an induced path or* $\Gamma[V(P)]$ *contains a (digraph) cycle of length at least 3.*

Proof. Suppose P is not an induced path. Then Γ must contain an arc of the form (v_i, v_j) with $j > i+1$ or $i > j+1$. Since P is Hessenberg, Γ does not contain an arc of the form (v_i, v_j) with $j > i+1$. Thus Γ must contain an arc of the form (v_i, v_j) with $i > j+1$. Then $(v_j, v_{j+1}, \ldots, v_i, v_j)$ is a (digraph) cycle in $\Gamma[V(P)]$. □

Let F be a field. For a square matrix $A = [a_{ij}] \in F^{n\times n}$, the *digraph of* A, denoted $\Gamma(A) = (V, E)$, is the (simple) digraph described by the off-diagonal zero-nonzero pattern of the entries: the set of vertices is $V = \{1, 2, \ldots, n\}$ and the set of arcs is $E = \{(i, j) : a_{ij} \neq 0, i \neq j\}$. Note that the value of the diagonal entries of A does not affect $\Gamma(A)$.

Conversely, given any simple digraph Γ (along with an ordering of the vertices),

[1]The early literature uses the color black rather than blue.

we may associate with Γ a family of matrices $\mathcal{M}^F(\Gamma) = \{A \in F^{n\times n} : \Gamma(A) = \Gamma\}$. The *minimum rank* over F of a digraph Γ is $\mathrm{mr}^F(\Gamma) = \min\{\mathrm{rank}\, A : A \in \mathcal{M}^F(\Gamma)\}$ and the *maximum nullity* over F of Γ is $\mathrm{M}^F(\Gamma) = \max\{\mathrm{null}\ A : A \in \mathcal{M}^F(\Gamma)\}$. It is immediate that that $\mathrm{mr}^F(\Gamma) + \mathrm{M}^F(\Gamma) = n$.

Similarly, symmetric matrices and undirected graphs are associated. For a symmetric matrix $A = [a_{ij}] \in F^{n\times n}$, the *graph of* A is the (simple) graph $\mathcal{G}(A) = (V, E)$ with $V = \{1, 2, \ldots, n\}$ and $E = \{\{i, j\} : i \neq j \text{ and } a_{ij} \neq 0\}$. The family of symmetric matrices associated with G is $\mathcal{S}^F(G) = \{A \in F^{n\times n} : A^T = A, \mathcal{G}(A) = G\}$, and minimum rank and maximum nullity are similarly defined for undirected graphs.

For the much of this paper, we let $F = \mathbb{R}$ and we write $\mathcal{S}(G), \mathcal{M}(\Gamma), \mathrm{M}(\Gamma)$, and $\mathrm{mr}(\Gamma)$ rather than $\mathcal{S}^{\mathbb{R}}(G), \mathcal{M}^{\mathbb{R}}(\Gamma), \mathrm{M}^{\mathbb{R}}(\Gamma)$, and $\mathrm{mr}^{\mathbb{R}}(\Gamma)$, etc. If a graph or digraph parameter that depends on matrices does not change regardless of the field F, then we say that that parameter is *field independent*; in this case $\mathrm{M}^F(\Gamma) = \mathrm{M}(\Gamma)$ for every field F.

Remark 1.4. Clearly $\mathrm{mr}^F(\Gamma^{\mathrm{T}}) = \mathrm{mr}^F(\Gamma)$, and $\mathrm{Z}(\Gamma^{\mathrm{T}}) = \mathrm{Z}(\Gamma)$ is known [5]. Because the reversal of a Hessenberg path is a Hessenberg path, $\mathrm{P}(\Gamma^{\mathrm{T}}) = \mathrm{P}(\Gamma)$.

2. Oriented graphs.

In this section we establish results for minimum rank, maximum nullity, zero forcing number, and path cover number of oriented graphs. Given a graph G, an orientation $\vec{G}$ of G is a digraph obtained by replacing each edge $\{u, v\}$ by exactly one of the arcs (u, v) and (v, u) (so a graph G has $2^{|E(G)|}$ orientations, some of which may be isomorphic to each other).

2.1. Range over orientations.

We consider the range of values of $\beta(\vec{G})$ over all possible orientations, for the parameters $\beta = \mathrm{mr}, \mathrm{M}, \mathrm{Z}, \mathrm{P}$.

Theorem 2.1. *Suppose β is a positive-integer-valued digraph parameter with the following properties for every oriented graph $\vec{G}$:*

(1) $\beta(\vec{G}^{\mathrm{T}}) = \beta(\vec{G})$.
(2) If $(u, v) \in E(\vec{G})$ and $\vec{G}_0$ is obtained from $\vec{G}$ by replacing (u, v) by (v, u) (i.e., reversing the orientation of one arc), then $|\beta(\vec{G}_0) - \beta(\vec{G})| \leq 1$.

Then for any two orientations $\vec{G}_1$ and $\vec{G}_2$ of the same graph G,

$$|\beta(\vec{G}_2) - \beta(\vec{G}_1)| \leq \left\lfloor \frac{E(G)}{2} \right\rfloor.$$

Furthermore, every integer between $\beta(\vec{G}_2)$ and $\beta(\vec{G}_1)$ is attained as $\beta(\vec{G})$ for some orientation $\vec{G}$ of G.

Proof. Without loss of generality, $\beta(\vec{G}_2) \geq \beta(\vec{G}_1)$. Let $e = |E(G)|$. Because $\vec{G}_1$ and $\vec{G}_2$ share the same underlying graph, it is possible to obtain $\vec{G}_2$ from $\vec{G}_1$ by reversing some of the arcs of $\vec{G}_1$. Let ℓ be the number of arcs we need to reverse to obtain $\vec{G}_2$ from $\vec{G}_1$. By hypothesis, reversing the direction of one arc changes the value of β by at most one, so $\beta(\vec{G}_2) - \beta(\vec{G}_1) \leq \ell$. The number of arcs that must be reversed to obtain $\vec{G}_2^{\mathrm{T}}$ from $\vec{G}_1$ is $e-\ell$, so $\beta(\vec{G}_2^{\mathrm{T}})-\beta(\vec{G}_1)\le e-\ell$. By hypothesis, $\beta(\vec{G}_2^{\mathrm{T}})=\beta(\vec{G}_2)$, so $\beta(\vec{G}_2)-\beta(\vec{G}_1)\le\lfloor\frac{e}{2}\rfloor$. The last statement follows from hypothesis (2) and the fact that we can go from $\vec{G}_1$ to $\vec{G}_2$ by reversing one arc at a time. □

Corollary 2.2. *If $\vec{G}_1$ and $\vec{G}_2$ are both orientations of the graph G, then:*

$$|\operatorname{mr}(\vec{G}_2)-\operatorname{mr}(\vec{G}_1)|\le\lfloor\frac{E(G)}{2}\rfloor,$$

$$|\operatorname{M}(\vec{G}_2)-\operatorname{M}(\vec{G}_1)|\le\lfloor\frac{E(G)}{2}\rfloor,$$

$$|\operatorname{Z}(\vec{G}_2)-\operatorname{Z}(\vec{G}_1)|\le\lfloor\frac{E(G)}{2}\rfloor,\ \textit{and}$$

$$|\operatorname{P}(\vec{G}_2)-\operatorname{P}(\vec{G}_1)|\le\lfloor\frac{E(G)}{2}\rfloor.$$

Furthermore, every integer between $\beta(\vec{G}_2)$ and $\beta(\vec{G}_1)$ is attained as $\beta(\vec{G})$ for some orientation $\vec{G}$ of G when β is any of the parameters $\operatorname{mr},\operatorname{M},\operatorname{Z},\operatorname{P}$.

Proof. The first hypothesis of Theorem 2.1, $\beta(\vec{G}^{\mathrm{T}})=\beta(\vec{G})$, is established for these parameters in Remark 1.4. To show these parameters satisfy the second hypothesis of Theorem 2.1, suppose arc (u,v) of $\vec{G}$ is reversed to obtain $\vec{G}_0$ from $\vec{G}$. In each case, the process is reversible, so it suffices to prove $\beta(\vec{G}_0)\le\beta(\vec{G})+1$.

For minimum rank, suppose $\Gamma(A)=\vec{G}$ and $\operatorname{rank}A=\operatorname{mr}(\vec{G})$. Define B by $b_{uu}=b_{vv}=b_{uv}=b_{vu}=-a_{uv}$ and $b_{ij}=0$ for all other entries of B. Then $\Gamma(A+B)=\vec{G}_0$ and $\operatorname{rank}(A+B)\le\operatorname{rank}A+1$. Thus $\operatorname{mr}(\vec{G}_0)\le\operatorname{mr}(\vec{G})+1$. The statement for maximum nullity is equivalent.

For zero forcing number, choose a minimum zero forcing set B and chronological list of forces $\mathcal{F}$ of $\vec{G}$. If the force $u\to v$ is in $\mathcal{F}$, then $B\cup\{v\}$ is a zero forcing set for $\vec{G}_0$. If $u\not\to v$ and for some w, $v\to w$ is in $\mathcal{F}$, then $B\cup\{u\}$ is a zero forcing set for $\vec{G}_0$. If v does not perform a force and $u\to v$ is not in $\mathcal{F}$, then B is a zero forcing set for $\vec{G}_0$. Thus, $\operatorname{Z}(\vec{G}_0)\le\operatorname{Z}(\vec{G})+1$.

For path cover number, suppose $\mathcal{P}=\{P^{(1)},P^{(2)},\dots,P^{(k)}\}$ is a path cover of $\vec{G}$ and $|\mathcal{P}|=\operatorname{P}(\vec{G})$. If (u,v) is not an arc in one of the paths in $\mathcal{P}$, then $\mathcal{P}$ is a path cover for $\vec{G}_0$ and $\operatorname{P}(\vec{G}_0)\le\operatorname{P}(\vec{G})$. So suppose (u,v) is an arc in some path $P^{(\ell)}$. Then we construct a path cover for $\vec{G}_0$ by replacing $P^{(\ell)}$ by the two paths resulting from deleting the arc (u,v). Thus, $\operatorname{P}(\vec{G}_0)\le\operatorname{P}(\vec{G})+1$. □

2.2. Hierarchal orientation. We establish a method for finding an orientation $\vec{G}$ of a graph G for which $\operatorname{P}(\vec{G})=\operatorname{P}(G)$. Let $\mathcal{P}=\{P^{(1)},P^{(2)},\dots,P^{(k)}\}$ be any path cover of a graph G. A *rooted path cover* of G, $\mathcal{R}=\{R^{(1)},R^{(2)},\dots,R^{(k)}\}$, is obtained from $\mathcal{P}$ by choosing one endpoint as the *root* of $P^{(i)}$ for each $i=1,\dots,k$. $\mathcal{R}$ is a *minimum rooted path cover* if $|\mathcal{R}|=\operatorname{P}(G)$. In the case $\mathcal{P}$ is a zero forcing path cover of a zero forcing set B, then the root of $P^{(i)}$ is automatically chosen to be the unique element of B that is a vertex of $P^{(i)}$. A rooted path cover obtained from $\mathcal{P}$ naturally orders $V(P^{(i)})$, starting with the root, and we denote this order by $R^{(i)} = (r_1^{(i)}, r_2^{(i)}, \dots, r_{s_i}^{(i)})$ where $s_i - 1$ is the length of $P^{(i)}$. Observe that if a rooted path cover is formed from a zero forcing path cover of a zero forcing set, the ordering within each rooted path coincides with the forcing order in that path.

Definition 2.3. Given a rooted path cover $\mathcal{R}$ of a graph G, the *hierarchal orientation* $\overrightarrow{G_{\mathcal{R}}}$ of G resulting from $\mathcal{R}$ is defined by orienting G as follows:

(1) Orient $R^{(i)}$ as $r_1^{(i)} \to r_2^{(i)} \to \cdots \to r_{s_i}^{(i)}$; that is, replace the edge $\{r_j^{(i)}, r_{j+1}^{(i)}\}$ by the arc $(r_j^{(i)}, r_{j+1}^{(i)})$ for $j = 1, \dots, s_i - 1$.

(2) For any edge between $R^{(i)}$ and $R^{(j)}$ with $i < j$, orient as $i \to j$; that is, if $i < j$, replace the edge $\{r_{\ell_i}^{(i)}, r_{\ell_j}^{(j)}\}$ by the arc $\left(r_{\ell_i}^{(i)}, r_{\ell_j}^{(j)}\right)$.

Since by definition, the paths in a path cover of a graph are induced, all the edges of G have been oriented by these two rules.

Observation 2.4. *For any rooted path cover $\mathcal{R}$ of G, $\mathcal{R}$ is a path cover of $\overrightarrow{G_{\mathcal{R}}}$ (with each path originating at its root), so* $\mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) \le |\mathcal{R}|$.

Proposition 2.5. An oriented graph $\vec{G}$ is the hierarchal orientation $\overrightarrow{G_{\mathcal{R}}}$ of G for some rooted path cover $\mathcal{R}$ of G (not necessarily minimum) if and only if $\vec{G}$ does not contain a digraph cycle.

Proof. Suppose $\mathcal{R} = \{R^{(1)}, \dots, R^{(k)}\}$ is rooted path cover of G. Since each path $R^{(i)}$ is induced, in order for $\overrightarrow{G_{\mathcal{R}}}$ to have a digraph cycle, $V(\overrightarrow{G_{\mathcal{R}}})$ would have to include vertices from at least two paths $R^{(i)}$ and $R^{(j)}$, with $i < j$. But by definition of $\overrightarrow{G_{\mathcal{R}}}$, there are no arcs from vertices in $R^{(j)}$ to vertices in $R^{(i)}$.

Suppose $\vec{G}$ does not contain a digraph cycle. Then we may order the vertices $\{v_1, \dots, v_n\}$, v_j does not have access to v_i whenever $j > i$. Then if $V(R^{(i)}) = \{v_i\}$, $\mathcal{R} = \{R^{(1)}, \dots, R^{(n)}\}$ is a rooted path cover and $\vec{G} = \overrightarrow{G_{\mathcal{R}}}$. □

Theorem 2.6. *Suppose $\mathcal{R} = \{R^{(1)}, \dots, R^{(k)}\}$ is a rooted path cover of G and $\overrightarrow{G_{\mathcal{R}}}$ is the hierarchal orientation of G resulting from $\mathcal{R}$. Then any path cover for $\overrightarrow{G_{\mathcal{R}}}$ is a path cover for G. If $\mathcal{R}$ is a minimum rooted path cover, then* $\mathrm{P}(G) = \mathrm{P}(\overrightarrow{G_{\mathcal{R}}})$.

Proof. Let P be a Hessenberg path in $\overrightarrow{G_{\mathcal{R}}}$. By Proposition 2.5, $\overrightarrow{G_{\mathcal{R}}}$ does not contain a digraph cycle, so by Lemma 1.3, P is an induced path. Thus, any path cover for $\overrightarrow{G_{\mathcal{R}}}$ is a path cover for G, and this implies $\mathrm{P}(G) \le \mathrm{P}(\overrightarrow{G_{\mathcal{R}}})$. If $\mathcal{R}$ is a minimum rooted path cover of G, then $\mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) \le |\mathcal{R}| = \mathrm{P}(G) \le \mathrm{P}(\overrightarrow{G_{\mathcal{R}}})$, so $\mathrm{P}(G) = \mathrm{P}(\overrightarrow{G_{\mathcal{R}}})$. □

Example 2.7. Not every orientation $\vec{G}$ having $\mathrm{P}(\vec{G}) = \mathrm{P}(G)$ is a hierarchal orientation. The oriented graph $\vec{G}$ shown in Figure 2.1 has $\mathrm{P}(\vec{G}) = 2 = \mathrm{P}(G)$, but $\vec{G}$ is not a hierarchal orientation because $\vec{G}$ contains a digraph cycle.

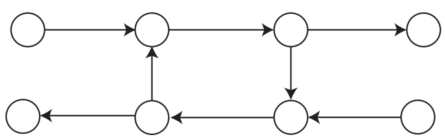

FIG. 2.1. *An oriented graph $\vec{G}$ that is not a hierarchal orientation but has* $\mathrm{P}(\vec{G}) = \mathrm{P}(G)$.

Although for any graph G we can find an orientation so that $\mathrm{P}(\vec{G}) = \mathrm{P}(G)$, this is not always the case for zero forcing number or maximum nullity.

EXAMPLE 2.8. Consider K_4, the complete graph on four vertices. It is well known that $\mathrm{M}(K_4) = \mathrm{Z}(K_4) = 3$, whereas we show that for any orientation $\vec{K_4}$ of K_4, $2 \geq \mathrm{Z}(\vec{K_4}) \geq \mathrm{M}(\vec{K_4})$. If $\vec{K_4}$ contains a directed 3-cycle, then any one vertex on the 3-cycle and the remaining vertex form a zero forcing set. If $\vec{K_4}$ has no directed 3-cycle, then we may order the vertices $\{u_1, u_2, u_3, u_4\}$ where u_j does not have access to u_i whenever $j > i$. Then, $\{u_1, u_3\}$ is a zero forcing set.

OBSERVATION 2.9. *If* $\mathcal{R} = \{R^{(1)}, \ldots, R^{(k)}\}$ *is a rooted path cover for* G, *then the set of roots* $\{r_1^{(1)}, \ldots, r_1^{(k)}\}$ *is a zero forcing set of the digraph* $\overrightarrow{G_{\mathcal{R}}}$, *as zero forcing can be done in path order along* $R^{(k)}$, *followed by* $R^{(k-1)}$, *etc.*

THEOREM 2.10. *Suppose* G *is a graph and* $\mathcal{R}$ *is a minimum rooted path cover of* G. *Then* $\mathrm{Z}(\overrightarrow{G_{\mathcal{R}}}) = \mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) = \mathrm{P}(G)$.

Proof. From Theorem 2.6, Proposition 1.2, Observation 2.9, and the hypotheses, $\mathrm{P}(G) = \mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) \leq \mathrm{Z}(\overrightarrow{G_{\mathcal{R}}}) \leq |\mathcal{R}| = \mathrm{P}(G)$. □

Whenever $\mathrm{P}(G) = \mathrm{Z}(G)$, we can use a minimum rooted path cover to find an orientation of G realizing $\mathrm{Z}(G)$ as its zero forcing number.

COROLLARY 2.11. *Suppose* G *is a graph such that* $\mathrm{P}(G) = \mathrm{Z}(G)$ *and* $\mathcal{R}$ *is a minimum rooted path cover of* G. *Then* $\mathrm{Z}(\overrightarrow{G_{\mathcal{R}}}) = \mathrm{Z}(G)$.

Because $\mathrm{P}(T) = \mathrm{Z}(T)$ for every (simple undirected) tree T [1], we have the following corollary.

COROLLARY 2.12. *If* T *is a tree, then there exists an orientation* $\vec{T}$ *of* T *such that* $\mathrm{Z}(\vec{T}) = \mathrm{Z}(T)$.

If we allow a path cover that is not a minimum path cover, it is not difficult to find a graph and rooted path cover $\mathcal{R}$ with $\mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) < \mathrm{Z}(\overrightarrow{G_{\mathcal{R}}})$ (in fact, $\mathrm{P}(\overrightarrow{G_{\mathcal{R}}}) < \mathrm{M}(\overrightarrow{G_{\mathcal{R}}})$).

EXAMPLE 2.13. Let G be the double triangle graph shown in Figure 2.2(a) and consider the rooted path cover of G defined by $\mathcal{R} = \{R^{(1)}, R^{(2)}, R^{(3)}\}$ where $V(R^{(1)}) = \{1\}, V(R^{(2)}) = \{3\}$, and $V(R^{(3)}) = \{2, 4\}$ with 2 as the root of $R^{(3)}$. The hierarchal orientation $\overrightarrow{G_{\mathcal{R}}}$ is shown in Figure 2.2(b).

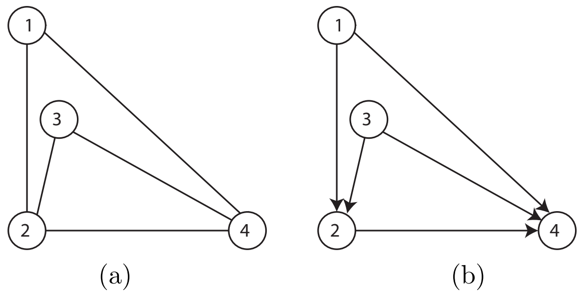

FIG. 2.2. *An hierarchal orientation* $\vec{G_{\mathcal{R}}}$ *having* $\mathrm{P}(\vec{G_{\mathcal{R}}}) < \mathrm{M}(\vec{G_{\mathcal{R}}}) = \mathrm{Z}(\vec{G_{\mathcal{R}}})$.

Then $\mathrm{P}(\vec{G_{\mathcal{R}}}) = 2$ because paths $(1,2)$ and $(3,4)$ cover all vertices, and the vertices 1 and 3 must each be initial vertices of any path they are in. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $\Gamma(A) = \vec{G_{\mathcal{R}}}$ and nullity $A = 3$. The set $\{1,2,3\}$ is a zero forcing set for $\vec{G_{\mathcal{R}}}$ so $3 \leq \mathrm{M}(\vec{G_{\mathcal{R}}}) \leq \mathrm{Z}(\vec{G_{\mathcal{R}}}) \leq 3$.

Every graph G we have examined has $\mathrm{M}(\vec{G_{\mathcal{R}}}) = \mathrm{P}(\vec{G_{\mathcal{R}}})$ for minimum rooted path covers $\mathcal{R}$, but these examples all involve a small number of vertices.

QUESTION 2.14. *Does* $\mathrm{M}(\vec{G_{\mathcal{R}}}) = \mathrm{P}(\vec{G_{\mathcal{R}}})$ *if* $\mathcal{R}$ *is a minimum rooted path cover of* G*?*

2.3. Tournaments. A *tournament* is an orientation of of the complete graph K_n. In this section we consider the possible values of path cover number, maximum nullity, and zero forcing number for tournaments.

EXAMPLE 2.15. We create an orientation of K_n by labeling the vertices $\{1,\ldots,n\}$ and by orienting the edges $\{u,v\}$ as (u,v) if and only if $v < u-1$ or $v = u+1$. The resulting orientation is called the *Hessenberg tournament* of order n, denoted $\vec{K}_n^{(H)}$. This is the Hessenberg path on n vertices containing all possible arcs except those of the form $(u+1,u)$ for $1 \leq u \leq n-1$. Since the zero forcing number of any Hessenberg path is one, $\mathrm{P}(\vec{K}_n^{(H)}) = \mathrm{M}(\vec{K}_n^{(H)}) = \mathrm{Z}(\vec{K}_n^{(H)}) = 1$. Observe that $\vec{K}_n^{(H)}$ is self-complementary.

EXAMPLE 2.16. Label the vertices of K_n by $\{1,\ldots,n\}$ and orient the edges $\{u,v\}$ as (u,v) if and only if $u < v$. The resulting orientation is the *transitive tournament*, denoted $\vec{K}_n^{(T)}$. We show that $\mathrm{P}(\vec{K}_n^{(T)}) = \mathrm{Z}(\vec{K}_n^{(T)}) = \mathrm{M}(\vec{K}_n^{(T)}) = \lceil \frac{n}{2} \rceil$ for any n. Let A be the adjacency matrix of $\vec{K}_n^{(T)}$, and let $D = \operatorname{diag}(0,1,0,1,\ldots)$. Then $\Gamma(A+D) = \vec{K}_n^{(T)}$ and $\operatorname{nullity}(A+D) = \lceil \frac{n}{2} \rceil$ because $A+D$ has $\lfloor \frac{n}{2} \rfloor$ duplicate rows and if n is odd, an additional row of zeros. The set of odd numbered vertices, $B = \{1,3,\ldots,\}$, is a zero forcing set. Thus, $\lceil \frac{n}{2} \rceil \leq \mathrm{M}(\vec{K}_n) \leq \mathrm{Z}(\vec{K}_n) \leq \lceil \frac{n}{2} \rceil$. Furthermore, from the definition of $\vec{K}_n^{(T)}$, no more than 2 vertices can be on the same Hessenberg path.

PROPOSITION 2.17. *For any tournament* $\vec{K}_n$, $1 \le \mathrm{P}(\vec{K}_n) \le \lceil \frac{n}{2} \rceil$, *and for every integer* k *with* $1 \le k \le \lceil \frac{n}{2} \rceil$, *there is an orientation* $\vec{K}_n$ *having* $\mathrm{P}(\vec{K}_n) = k$.
For every integer k *with* $1 \le k \le \lceil \frac{n}{2} \rceil$, *there is an orientation* $\vec{K}_n$ *having* $\mathrm{Z}(\vec{K}_n) = k$.

Proof. For both P and Z, $\vec{K}_n^{(H)}$ (Example 2.15) realizes the lower bound and $\vec{K}_n^{(T)}$ (Example 2.16) realizes the upper bound. For the upper bound on attainable path cover numbers, partition the vertices of $\vec{K}_n$ into $\lceil \frac{n}{2} \rceil$ sets of size two or one. Each pair of vertices and the arc between them forms a path. The assertion that all values for P and Z between 1 and $\lceil \frac{n}{2} \rceil$ are possible follows from Corollary 2.2. □

For $n \le 7$, the transitive tournament $\vec{K}_n^{(T)}$ achieves the highest zero forcing number; i.e., for all orientations $\vec{K}_n$, $\mathrm{Z}(\vec{K}_n) \le \lceil \frac{n}{2} \rceil$ (this has been verified using the program [12] written in *Sage*). But for $n = 8$, there exists a tournament having maximum nullity greater than that of the transitive tournament, as in the next example.

EXAMPLE 2.18.

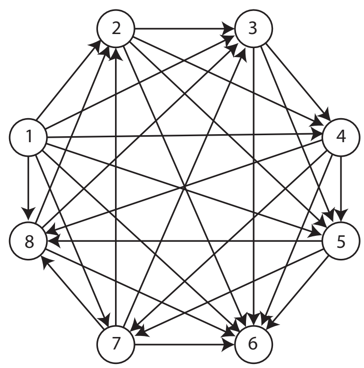

FIG. 2.3. *A tournament* $\vec{K}_8$ *having* $M(\vec{K}_8) = \mathrm{Z}(\vec{K}_8) = 5 > \lceil \frac{8}{2} \rceil$.

Let $\vec{K}_8$ be the tournament shown in Figure 2.3. Observe that $\{1, 2, 3, 4, 8\}$ is a zero forcing set for $\vec{K}_8$ so $\mathrm{Z}(\vec{K}_8) \le 5$. The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

has rank $A = 3$ and $\Gamma(A) = \vec{K}_8$, so $5 \le \mathrm{M}(\vec{K}_8)$. Thus, $\mathrm{Z}(\vec{K}_8) = \mathrm{M}(\vec{K}_8) = 5 > 4 = \lceil \frac{8}{2} \rceil$. We also show that $\mathrm{P}(\vec{K}_8) = 3$: Since $\{(2, 4, 8), (3, 5, 7), (1, 6)\}$ is a path cover,

$\mathrm{P}(\vec{K}_8) \leq 3$. There are no induced paths of length greater than two in $\vec{K}_8$, so by Lemma 1.3 any path of length three or more must have a cycle. Thus vertices 1 and 6 must be in paths of length at most two. If they are in separate paths in a path cover $\mathcal{P}$, then $|\mathcal{P}| \geq 3$. So assume $(1, 6)$ is a path in $\mathcal{P}$. Since $\vec{K}_8 - \{1, 6\}$ is not a (Hessenberg) path, $|\mathcal{P}| \geq 3$.

There are also examples of tournaments $\vec{K}_n$ for which $M(\vec{K}_n) < \mathrm{Z}(\vec{K}_n)$.

PROPOSITION 2.19. *The tournament* $\vec{K}_7$ *shown in Figure 2.4 has* $\mathrm{P}(\vec{K}_7) = 2$, $\mathrm{M}(\vec{K}_7) = 3$, *and* $\mathrm{Z}(\vec{K}_7) = 4$.

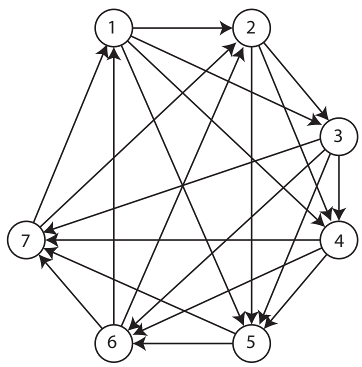

FIG. 2.4. *A tournament* $\vec{K}_7$ *having* $M(\vec{K}_7) = 3 < 4 = \mathrm{Z}(\vec{K}_7)$.

Proof. Because $\{(4, 6, 1, 3), (2, 5, 7)\}$ is a path cover for $\vec{K}_7$, and $\vec{K}_7$ is not a Hessenberg path, $\mathrm{P}(\vec{K}_7) = 2$.

Next we show $\mathrm{M}(\vec{K}_7) \leq 3$. Suppose $\Gamma(A) = \vec{K}_7$. The nonzero pattern of A is

$$\begin{bmatrix} ? & * & * & * & * & 0 & 0 \\ 0 & ? & * & * & * & 0 & 0 \\ 0 & 0 & ? & * & * & * & * \\ 0 & 0 & 0 & ? & * & * & * \\ 0 & 0 & 0 & 0 & ? & * & * \\ * & * & 0 & 0 & 0 & ? & * \\ * & * & 0 & 0 & 0 & 0 & ? \end{bmatrix}$$

where $*$ denotes a nonzero entry and ? may have any real value. Rows 1, 5, and 7 are necessarily linearly independent, by considering columns $2, 3$, and 6.

For A to achieve nullity 4, $\operatorname{rank} A = 3$ and thus all the remaining rows must be in the span of rows $1, 5$, and 7. We show this is impossible, implying that $\mathrm{mr}(\vec{K}_7) \geq 4$ and $\mathrm{M}(\vec{K}_7) \leq 3$. Once that is done, construct a matrix A with $\Gamma(A) = \vec{K}_7$ and $\operatorname{rank} A = 4$ by setting all nonzero off-diagonal entries to 1 and setting the diagonal entries as $a_{ii} = 0$ for i odd and $a_{ii} = 1$ for i even, so $\mathrm{M}(\vec{K}_7) = 3$.

If $a_{11} = 0$, then row 3 cannot be expressed as a linear combination of rows 1, 5, and 7: By considering column 1, the coefficient of row 7 must be zero, which implies that then the coefficient of row 1 must be zero by considering column 2, but by considering column 4, row 3 is not a multiple of row 5. Thus $a_{11} \neq 0$.

If $a_{77} \neq 0$, then row 2 cannot be expressed as a linear combination of rows 1, 5, and 7: By considering column 6 the coefficient of row 5 must be zero, which implies that the coefficient of row 7 must be zero by considering column 7, but by considering column 1, row 2 is not a multiple of row 1 (because $a_{11} \neq 0$). Thus $a_{77} = 0$.

If $a_{55} = 0$, then row 4 cannot be expressed as a linear combination of rows 1, 5, and 7: By considering column 3 the coefficient of row 1 must be zero, which implies that the coefficient of row 7 must be zero by considering column 1. But row 4 is not a multiple of row 5 (because $a_{55} = 0$). Thus $a_{55} \neq 0$.

Now row 6 cannot be expressed as a linear combination of rows 1, 5, and 7: By considering column 3 the coefficient of row 1 must be zero. By considering column 5 the coefficient of row 5 must be zero. Now by considering column 7, row 6 is not a scalar multiple of row 7. Therefore row 6 is not a linear combination of rows 1, 5, and 7, $\operatorname{rank} A \geq 4$, and $\operatorname{mr}(\vec{K}_7) \geq 4$.

Finally we show that $\mathrm{Z}(\vec{K}_7) = 4$. Observe that any zero forcing set must contain a vertex from the set $\{1, 2\}$: If 1 and 2 are initially colored white, the only vertices that can force them are 6 and 7, but 1 and 2 are both out-neighbors of 6 and of 7. Observe that any zero forcing set must contain a vertex from the set $\{6, 7\}$: If 6 and 7 are initially colored white, the only vertices that can force them are 3, 4, and 5, but 6 and 7 are both out-neighbors of 3, of 4, and of 5. Observe that any zero forcing set must contain a vertex from the set $\{3, 4\}$: If 3 and 4 are initially colored white, the only vertices that can force them are 1 and 2, but 3 and 4 are both out-neighbors of 1 and of 2. Observe that any zero forcing set must contain a vertex from the set $\{4, 5\}$: If 4 and 5 are initially colored white, the only vertices that can force them are 1, 2, and 3, but 4 and 5 are both out-neighbors of 1, of 2, and of 3. Hence, a zero forcing set must contain at least four vertices, unless vertex 4 is the only vertex from $\{3, 4\}$ and $\{4, 5\}$ selected. However, by inspection the sets $\{1, 4, 6\}, \{1, 4, 7\}, \{2, 4, 6\}, \{2, 4, 7\}$ are not zero forcing sets. The set $\{1, 2, 4, 6\}$ is a zero forcing set for $\vec{K}_7$, and so $\mathrm{Z}(\vec{K}_7) = 4$. □

2.4. Orientations of paths. In this section we consider the possible values of path cover number, maximum nullity, and zero forcing number for orientations of paths.

Example 2.20. Starting with the path P_n, label the vertices in path order by $\{1, \ldots, n\}$ and orient the edge $\{i, i+1\}$ as arc $(i, i+1)$ for $i = 1, \ldots, n-1$. The resulting orientation is the *path orientation* of P_n, denoted $\vec{P}_n^{(H)}$. Then $\mathrm{P}(\vec{P}_n^{(H)}) = \mathrm{M}(\vec{P}_n^{(H)}) = \mathrm{Z}(\vec{P}_n^{(H)}) = 1$.

Example 2.21. Starting with the path P_n, label the vertices in path order by $\{1, \ldots, n\}$ and orient the edges as follows: Orient $\{1, 2\}$ as $(1, 2)$. For $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, orient $\{2i+1, 2i\}$ and $\{2i+1, 2i+2\}$ as $(2i+1, 2i)$ and $(2i+1, 2i+2)$. If n is odd, orient $\{n-1, n\}$ as $(n, n-1)$. The resulting orientation is the *alternating orientation* of P_n, denoted $\vec{P}_n^{(A)}$. Note that all odd-numbered vertices have in-degree zero. So $\mathrm{P}(\vec{P}_n^{(A)}) = \mathrm{M}(\vec{P}_n^{(A)}) = \mathrm{Z}(\vec{P}_n^{(A)}) = \lceil \frac{n}{2} \rceil$ because the odd vertices form a minimum zero forcing set, there is no directed path of length greater than one, and the adjacency matrix A of $\vec{P}_n^{(A)}$ has rank $\lfloor \frac{n}{2} \rfloor$.

Proposition 2.22. *For any oriented path* $\vec{P}_n$, $1 \le \mathrm{P}(\vec{P}_n) \le \lceil \frac{n}{2} \rceil$, $1 \le \mathrm{M}(\vec{P}_n) \le \lceil \frac{n}{2} \rceil$, *and* $1 \le \mathrm{Z}(\vec{P}_n) \le \lceil \frac{n}{2} \rceil$. *For every integer* k *with* $1 \le k \le \lceil \frac{n}{2} \rceil$, *there are (possibly three different) orientations* $\vec{P}_n$ *having* $\mathrm{P}(\vec{P}_n) = k$, $\mathrm{M}(\vec{P}_n) = k$, *and* $\mathrm{Z}(\vec{P}_n) = k$.

Proof. The proof of the second statement follows from Examples 2.20 and 2.21 and Corollary 2.2. To complete the proof we show that $\mathrm{Z}(\vec{P}_n) \le \lceil \frac{n}{2} \rceil$ for every orientation $\vec{P}_n$. Apply Corollary 2.2 to the given orientation $\vec{P}_n$ and $\vec{P}_n^{(H)}$. Then $\mathrm{Z}(\vec{P}_n) - \mathrm{Z}(\vec{P}_n^{(H)}) \le \lfloor \frac{n-1}{2} \rfloor$, so $\mathrm{Z}(\vec{P}_n) \le \lfloor \frac{n-1}{2} \rfloor + 1$. If n is odd, $\lfloor \frac{n-1}{2} \rfloor + 1 = \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$. If n is even, $\lfloor \frac{n-1}{2} \rfloor + 1 = (\frac{n}{2} - 1) + 1 = \lceil \frac{n}{2} \rceil$. Therefore $\mathrm{Z}(\vec{P}_n) \le \lceil \frac{n}{2} \rceil$. □

2.5. Orientations of cycles. In this section we consider the possible values of path cover number, maximum nullity, and zero forcing number for orientations of cycles of length at least 4 (since a cycle of length 3 is a complete graph).

Example 2.23. Starting with the cycle C_n, label the vertices in cycle order by $\{1, \dots, n\}$ and orient the edge $\{i, i+1\}$ as arc $(i, i+1)$ for $i = 1, \dots, n$ (where $n+1$ is interpreted as 1). The resulting orientation is the *cycle orientation* of C_n, denoted $\vec{C}_n^{(H)}$. Then $\mathrm{P}(\vec{C}_n^{(H)}) = \mathrm{M}(\vec{C}_n^{(H)}) = \mathrm{Z}(\vec{C}_n^{(H)}) = 1$.

Example 2.24. Starting with C_n, label the vertices in cycle order by $\{1, \dots, n\}$ and orient the edges as follows: Orient $\{1, 2\}$ and $\{1, n\}$ as $(1, 2)$ and $(1, n)$. For $i = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$, orient $\{2i+1, 2i\}$ and $\{2i+1, 2i+2\}$ as arcs $(2i+1, 2i)$ and $(2i+1, 2i+2)$. If n is odd, orient the edge $\{n-1, n\}$ as $(n, n-1)$. The resulting orientation is the *alternating orientation* of C_n, denoted $\vec{C}_n^{(A)}$. If n is odd there is one path of length 2, so $\mathrm{P}(\vec{C}_n^{(A)}) = \lfloor \frac{n}{2} \rfloor$. Let S be the set of odd numbered vertices (with the exception of vertex n if n is odd), so every vertex in S has in-degree zero and $|S| = \lfloor \frac{n}{2} \rfloor$. Clearly $S \subseteq B$ for any zero forcing set, and every vertex in S has two out-neighbors not in S, so every zero forcing set must have cardinality at least $\lfloor \frac{n}{2} \rfloor + 1$. Since $S \cup \{2\}$ is a zero forcing set, $\mathrm{Z}(\vec{C}_n^{(A)}) = \lfloor \frac{n}{2} \rfloor + 1$. We construct a matrix $A \in \mathcal{M}(\vec{C}_n^{(A)})$ of nullity $\lfloor \frac{n}{2} \rfloor + 1$, showing that $\mathrm{M}(\vec{C}_n^{(A)}) = \lfloor \frac{n}{2} \rfloor + 1$. Any matrix in $\mathcal{M}(\vec{C}_n^{(A)})$ has two nonzero off-diagonal entries in every odd row (except n if n is odd) and no nonzero off-diagonal entries in every even row. Define a matrix $A = [a_{ij}]$ with $\Gamma(A) = \vec{C}_n^{(A)}$ by setting $a_{ii} = 0$ except $a_{nn} = -1$ if n is odd, and in each odd row the first nonzero entry is 1 and the second is -1. Then $\operatorname{null} A = \lfloor \frac{n}{2} \rfloor + 1$.

Proposition 2.25. *Let* $\vec{C}_n$ *be any orientation of* C_n $(n \ge 4)$. *Then* $1 \le \mathrm{P}(\vec{C}_n) \le \lfloor \frac{n}{2} \rfloor$ *and for every integer* k *with* $1 \le k \le \lfloor \frac{n}{2} \rfloor$, *there is an orientation* $\vec{C}_n$ *having* $\mathrm{P}(\vec{C}_n) = k$. *For any orientation of a cycle* $\vec{C}_n$, $1 \le \mathrm{M}(\vec{C}_n) \le \mathrm{Z}(\vec{C}_n) \le \lfloor \frac{n}{2} \rfloor + 1$ *and for every integer* k *with* $1 \le k \le \lfloor \frac{n}{2} \rfloor + 1$, *there are (possibly two different) orientations*

$\vec{C}_n$ *having* $\mathrm{M}(\vec{C}_n) = k$ *and* $\mathrm{Z}(\vec{C}_n) = k$.

Proof. The proof of the second part of each statement follows from Examples 2.20 and 2.21 and Corollary 2.2. To complete the proof we show that $\mathrm{P}(\vec{C}_n) \le \lfloor \frac{n}{2} \rfloor$ and $\mathrm{Z}(\vec{C}_n) \le \lfloor \frac{n}{2} \rfloor + 1$ for all orientations $\vec{C}_n$, by exhibiting a path cover and zero forcing set of cardinality not exceeding this bound.

For path cover number: If n is even, choose two adjacent vertices, cover them with one path, and delete them, leaving a path on $n-2$ vertices which is an even number. By Proposition 2.22, there is a path cover of these $n-2$ vertices with $\frac{n}{2}-1$ paths, so there is a path cover of $\vec{C}_n$ having $\frac{n}{2} = \lfloor \frac{n}{2} \rfloor$ paths. If n is odd, then for any orientation $\vec{C}_n$, there is a path on 3 vertices. Cover these vertices with that path and delete them, leaving a path on $n-3$ vertices (again an even number), which can be covered by $\frac{n-3}{2}$ paths, and there is a path cover of $\vec{C}_n$ having $\frac{n-3}{2}+1 = \lfloor \frac{n}{2} \rfloor$ paths.

For zero forcing number: Delete any one vertex v, leaving a path on $n-1$ vertices, which has a zero forcing set B with $|B| = \lceil \frac{n-1}{2} \rceil$ by Proposition 2.22. Then the set $B' := B \cup \{v\}$ is a zero forcing set for $\vec{C}_n$ and $|B'| = \lceil \frac{n-1}{2} \rceil + 1$. If n is even, $\lceil \frac{n-1}{2} \rceil + 1 = \frac{n}{2} + 1 = \lfloor \frac{n}{2} \rfloor + 1$. If n is odd, $\lceil \frac{n-1}{2} \rceil + 1 = \frac{n-1}{2} + 1 = \lfloor \frac{n}{2} \rfloor + 1$. □

3. Doubly directed graphs. Given a graph G, the *doubly directed graph* $\overleftrightarrow{G}$ of G is the digraph obtained by replacing each edge $\{u, v\}$ by both of the arcs (u, v) and (v, u). In this section we establish results for minimum rank, maximum nullity, zero forcing number, and path cover number of doubly directed graphs.

Proposition 3.1. $\mathrm{P}(G) = \mathrm{P}(\overleftrightarrow{G})$ *for any graph* G.

Proof. $\mathrm{P}(G)$ is the minimum number of induced paths of G and $\mathrm{P}(\overleftrightarrow{G})$ is the minimum number of Hessenberg paths in $\overleftrightarrow{G}$. It is enough to show that all Hessenberg paths in $\overleftrightarrow{G}$ are induced. Suppose P is a Hessenberg path in $\overleftrightarrow{G}$ that is not induced. Then there exists some arc $(v_i, v_j) \in E(\overleftrightarrow{G})$, where $i > j+1$. But because the digraph is doubly directed, $(v_j, v_i) \in E(\overleftrightarrow{G})$, which contradicts the definition of a Hessenberg path. Therefore, all Hessenberg paths must be induced. Thus, $\mathrm{P}(G) = \mathrm{P}(\overleftrightarrow{G})$. □

Proposition 3.2. $\mathrm{Z}(G) = \mathrm{Z}(\overleftrightarrow{G})$ *for any graph* G.

Proof. The color change rule for graphs is that a blue vertex v can force a white vertex w if w is the only white neighbor of v. The color change rule for digraphs is that a blue vertex V can force a white vertex w if w is the only white out-neighbor of v. If G is a graph then for any vertex $v \in V(G)$, w is a neighbor of v in G if and only if w is an out-neighbor of v in $\overleftrightarrow{G}$. This means that v forces w in G if and only if v forces w in $\overleftrightarrow{G}$. Thus B is a zero forcing set in G if and only if B is a zero forcing set in $\overleftrightarrow{G}$ and $\mathrm{Z}(G) = \mathrm{Z}(\overleftrightarrow{G})$. □

Observation 3.3. *For any graph* G, $\mathrm{M}(G) \le \mathrm{M}(\overleftrightarrow{G})$, *since* $\mathcal{S}(G) \subseteq \mathcal{M}(\overleftrightarrow{G})$.

Corollary 3.4. *If* G *is a graph such that* $\mathrm{M}(G) = \mathrm{Z}(G)$, *then* $\mathrm{M}(G) = \mathrm{M}(\overleftrightarrow{G})$.

Proof. $\mathrm{Z}(G) = \mathrm{Z}(\overleftrightarrow{G}) \geq \mathrm{M}(\overleftrightarrow{G}) \geq \mathrm{M}(G) = \mathrm{Z}(G)$. □

It was established in [1] that for every tree T, $\mathrm{P}(T) = \mathrm{M}(T) = \mathrm{Z}(T)$, giving the following corollary.

COROLLARY 3.5. *If T is a tree, then* $\mathrm{P}(\overleftrightarrow{T}) = \mathrm{M}(\overleftrightarrow{T}) = \mathrm{Z}(\overleftrightarrow{T})$.

As the following example shows, it is possible to have $\mathrm{M}(\overleftrightarrow{G}) > \mathrm{M}(G)$.

EXAMPLE 3.6. The complete tripartite graph on three sets of three vertices $K_{3,3,3}$ has $V_1 = \{1, 2, 3\}, V_2 = \{4, 5, 6\}, V_3 = \{7, 8.9\}, V(K_{3,3,3}) = V_1 \dot\cup V_2 \dot\cup V_3$ and $E(K_{3,3,3})$ equal to the set of all edges with one vertex in V_i and the other in V_j $(i \neq j)$. It is well-known that $\mathrm{mr}(K_{3,3,3}) = 3$. Let J_3 be the 3×3 matrix with all entries equal to 1, 0_3 be the 3×3 matrix with all entries equal to 0, and let $A = \begin{bmatrix} 0_3 & J_3 & -J_3 \\ J_3 & 0_3 & J_3 \\ J_3 & J_3 & 0_3 \end{bmatrix}$. Then $\Gamma(A) = \overleftrightarrow{K_{3,3,3}}$ and rank $A = 2$. Thus, $\mathrm{M}(\overleftrightarrow{K_{3,3,3}}) = 7 > 6 = \mathrm{M}(K_{3,3,3})$.

The pentasun H_5 graph shown in Figure 3.1 has $\mathrm{M}(H_5) = 2 < 3 = \mathrm{P}(H_5)$ [4], establishing the noncomparability of M and P (because there are many examples of graphs G with $\mathrm{P}(G) < \mathrm{M}(G)$). The same is true for the doubly directed pentasun.

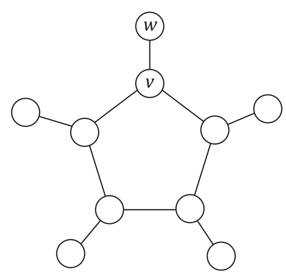

FIG. 3.1. *The pentasun H_5.*

EXAMPLE 3.7. Theorem 2.8 of [5] describes the cut-vertex reduction method for calculating M for directed graphs with a cut-vertex. We compute $\mathrm{M}(\overleftrightarrow{H_5}) = 2$ by applying the cut-vertex reduction method to vertex v, using the notation found in [5]. Because $\mathrm{M}(H_5 - w) = \mathrm{Z}(H_5 - w) = 2$ and $\mathrm{M}(H_5 - \{v, w\}) = \mathrm{Z}(H_5 - \{v, w\}) = 2$, $\mathrm{M}(\overleftrightarrow{H_5} - w) = \mathrm{Z}(\overleftrightarrow{H_5} - w) = 2$ and $\mathrm{M}(\overleftrightarrow{H_5} - \{v, w\}) = \mathrm{Z}(\overleftrightarrow{H_5} - \{v, w\}) = 2$, so $r_v(\overleftrightarrow{H_5} - w) = 1$. Clearly $\mathrm{mr}(\overleftrightarrow{H_5}[\{v, w\}]) = 1$ and $\mathrm{mr}(\overleftrightarrow{H_5}[\{v, w\}] - v) = 0$, so $r_v(\overleftrightarrow{H_5}[\{v, w\}]) = 1$. The *type* of the cut-vertex v of a digraph Γ, $\mathrm{type}_v(\Gamma)$, is a subset of $\{\mathrm{C}, \mathrm{R}\}$, where $\mathrm{C} \in \mathrm{type}_v(\Gamma)$ if there exists a matrix $A' \in \mathrm{M}(\Gamma - v)$ with rank $A' = \mathrm{mr}(\Gamma - v)$ and a vector $\mathbf{z}$ in range A' that has the in-pattern of v, and similarly for rows. Thus, $\mathrm{type}_v(\overleftrightarrow{H_5}[\{v, w\}]) = \emptyset$, so by [5, Theorem 2.8], $r_v(\overleftrightarrow{H_5}) = 2$. So $\mathrm{mr}(\overleftrightarrow{H_5}) = \mathrm{mr}(\overleftrightarrow{H_5} - \{v, w\}) + \mathrm{mr}(\overleftrightarrow{H_5}[\{w\}]) + 2 = 6 + 0 + 2 = 8$ and $\mathrm{M}(\overleftrightarrow{H_5}) = 2$. Since $\mathrm{P}(\overleftrightarrow{H_5}) = \mathrm{P}(H_5) = 3$, $\mathrm{P}(\overleftrightarrow{H_5}) > \mathrm{M}(\overleftrightarrow{H_5})$. It is easy to find an example of a digraph Γ with $\mathrm{P}(\Gamma) < \mathrm{M}(\Gamma)$, e.g. Example 2.13, so M and P are noncomparable.

Proposition 3.8. *Suppose that both G and $\overleftrightarrow{G}$ have field independent minimum rank. Then* $\mathrm{mr}(G) = \mathrm{mr}(\overleftrightarrow{G})$ *and* $\mathrm{M}(G) = \mathrm{M}(\overleftrightarrow{G})$.

Proof. Since both G and $\overleftrightarrow{G}$ have field independent minimum rank $\mathrm{mr}(G) = \mathrm{mr}^{\mathbb{Z}_2}(G)$ and $\mathrm{mr}^{\mathbb{Z}_2}(\overleftrightarrow{G}) = \mathrm{mr}(\overleftrightarrow{G})$. Furthermore, $\mathcal{S}^{\mathbb{Z}_2}(G) = \mathcal{M}^{\mathbb{Z}_2}(\overleftrightarrow{G})$, so

$$\mathrm{mr}(G) = \mathrm{mr}^{\mathbb{Z}_2}(G) = \mathrm{mr}^{\mathbb{Z}_2}(\overleftrightarrow{G}) = \mathrm{mr}(\overleftrightarrow{G}).$$

□

The converse of Proposition 3.8 is not true, however.

Example 3.9. Let G be the full house graph, shown in Figure 3.2. It is well known that $\mathrm{mr}^{\mathbb{Z}_2}(G) = 3$, yet $\mathrm{mr}(G) = 2 = \mathrm{mr}(\overleftrightarrow{G})$.

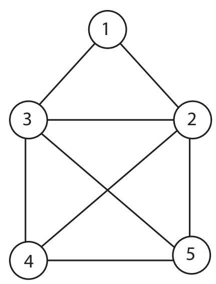

Fig. 3.2. *The full house graph*

4. Digraphs in general. In this section, we present some minimum rank, maximum nullity, and zero forcing results for digraphs in general, where any pair of vertices may or may not have an arc in either direction. We begin with two (undirected) graph properties that do not extend to digraphs.

Sinkovic [11] has shown that for any outerplanar graph G, $\mathrm{M}(G) \le \mathrm{P}(G)$. This is not true for digraphs, because it was was shown in Example 2.13 that the outerplanar digraph $\vec{G}_R$ has $\mathrm{M}(\vec{G}_R) = \mathrm{Z}(\vec{G}_R) = 3 > 2 = \mathrm{P}(\vec{G}_R)$, and $\vec{G}_R$ is outerplanar (although Figure 2.2 is not drawn that way).

The *complement* of a graph $G = (V, E)$ (or digraph $\Gamma = (V, E)$) is the graph $\overline{G} = (V, \overline{E})$ (or digraph $\overline{\Gamma} = (V, \overline{E})$), where $\overline{E}$ consists of all two element sets of vertices (or all ordered pairs of distinct vertices) that are not in E. The Graph Complement Conjecture (GCC) is equivalent to the statement that for any graph G, $\mathrm{M}(G) + \mathrm{M}(\overline{G}) \ge |G| - 2$. This statement is generalized in [3]: For a graph parameter β related to maximum nullity, the Graph Compliment Conjecture for β, GCC_β, is $\beta(G) + \beta(\overline{G}) \ge |G| - 2$. With this notation, GCC can be denoted $\mathrm{GCC}_{\mathrm{M}}$. The Graph Compliment Conjecture for zero forcing number, $\mathrm{Z}(G) + \mathrm{Z}(\overline{G}) \ge |G| - 2$, denoted $\mathrm{GCC}_{\mathrm{Z}}$, is actually the Graph Complement Theorem for zero forcing [7]. However, as the following example shows, $\mathrm{GCC}_{\mathrm{Z}}$ does not hold for digraphs, and since for any digraph $\mathrm{M}(\Gamma) \leq \mathrm{Z}(\Gamma)$, $\mathrm{GCC_M}$ does not hold for digraphs. A tournament provides a counterexample.

Example 4.1. For the Hessenberg tournament of order n, denoted $\vec{K}_n^{(H)}$, $\mathrm{Z}(\vec{K}_n^{(H)}) = 1$ because $\vec{H}_n$ is a Hessenberg path. Because $\vec{K}_n^{(H)}$ is self-complementary,

$$\mathrm{Z}(\vec{K}_n^{(H)}) + \mathrm{Z}(\overline{\vec{K}_n^{(H)}}) = 2,$$

but for $n \geq 5$, $n - 2 = |\vec{K}_n^{(H)}| - 2 \geq 3$.

Some properties of minimum rank for graphs do remain true for digraphs. For a graph G, it is well-known that if K_r is a subgraph of G then $\mathrm{M}(G) \geq r - 1$ (see, for example, [2] and the references therein). An analogous result holds true for digraphs, although the proof is different than those usually given for graphs.

Theorem 4.2. *Suppose* $\overleftrightarrow{K_r}$ *is a subgraph of a digraph* Γ. *Then* $\mathrm{M}(\Gamma) \geq r - 1$.

Proof. First, we order the vertices of Γ so that the subdigraph induced on the vertices $1, 2, \ldots, r$ is $\overleftrightarrow{K_r}$. We will construct $L \in \mathcal{M}(\Gamma)$ with $\operatorname{rank} L \leq n - r + 1$, where L is partitioned as $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $A \in \mathcal{M}(K_r)$. We may choose $D \in \mathcal{M}(\Gamma[\{r+1, \ldots, n\}])$ so that $\operatorname{rank} D = n - r$. We now choose C to be any matrix with the correct zero-nonzero pattern. Denote the ith column of C by $\mathbf{c}_i$ and the jth column of D by $\mathbf{d}_j$. Since D has full rank, there exist coefficients $d_{i,1}, \ldots, d_{i,n-r}$ such that $\mathbf{c}_i = d_{i,1}\mathbf{d}_1 + \cdots + d_{i,n-r}\mathbf{d}_{n-r}$ for $1 \leq i \leq r$.

Now, we choose B to be any matrix with the correct zero-nonzero pattern and denote the jth column of B by $\mathbf{b}_j$. Then define E to be the $r \times r$ matrix whose ith column is equal to $d_{i,1}\mathbf{b}_1 + \cdots + d_{i,n-r}\mathbf{b}_{n-r}$. Therefore, the matrix $L' = \begin{bmatrix} E & B \\ C & D \end{bmatrix}$ has $\operatorname{rank} L' = n - r$. Let p be a real number greater than the absolute value of every entry of E. Define $A := E + pJ_r$, where J_r is the $r \times r$ matrix all of whose entries are 1, so $A \in \mathcal{M}(K_r)$, $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{M}(\Gamma)$ and $\operatorname{rank} L \leq n - r + 1$. □

In [6], the maximum number of edges in a graph G of order n with a prescribed zero forcing number k is shown to be $kn - \binom{k+1}{2}$. We similarly seek to bound the number of arcs that a digraph Γ of order n may possess given $\mathrm{Z}(\Gamma) = k$.

Theorem 4.3. *Suppose* Γ *is a digraph of order* n *with* $\mathrm{Z}(\Gamma) = k$. *Then,*

$$|E(\Gamma)| \leq \binom{n}{2} - \binom{k}{2} + k(n-1). \tag{4.1}$$

Proof. We prove that (4.1) holds for a digraph Γ of order $n \geq k + 1$ whenever $\mathrm{Z}(\Gamma) \leq k$ (since for a graph G of order n, $\mathrm{Z}(G) \leq n - 1$). The proof is by induction on n, for a fixed positive integer k. The base case is $n = k + 1$, or $k = n - 1$, and the inequality (4.1) reduces to

$$|E(\Gamma)| \leq \binom{n}{2} - \binom{n-1}{2} + (n-1)^2 = \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} + (n-1)^2 = n(n-1).$$

Any digraph Γ of order n has at most $2\binom{n}{2} = n^2 - n$ arcs and thus the claim holds.

Now assume (4.1) holds true for any digraph of order $n-1$ that has a zero forcing set of cardinality k. Let Γ be a digraph of order n with $\mathrm{Z}(\Gamma) \leq k$. Let B be a zero forcing set of Γ where $|B| = k$. Suppose $\mathcal{F}$ is a chronological list of forces for B and that the first force of $\mathcal{F}$ occurs on the arc (v, w). Then, $(B \setminus \{v\}) \cup \{w\}$ is a zero forcing set of cardinality k for $\Gamma - v$ and the induction hypothesis applies to $E(\Gamma - v)$. In order to determine an upper bound on $|E(\Gamma)|$ we determine the maximum number of arcs incident with v in Γ. Since v forces w first in $\mathcal{F}$, $(v, x) \in E(\Gamma)$ implies $x \in (B \setminus \{v\}) \cup \{w\}$. Furthermore, $E(\Gamma)$ contains at most $n-1$ arcs of the form (x, v), one for each vertex $x \neq v$. Therefore,

$$\begin{aligned} |E(\Gamma)| \leq |E(\Gamma - v)| + k + (n-1) &\leq \binom{n-1}{2} - \binom{k}{2} + k(n-2) + k + (n-1) \\ &= \binom{n}{2} - \binom{k}{2} + k(n-1). \end{aligned}$$

□

In the paper [6], the edge bound is used to show that the zero forcing number must be at least half the average degree. However, a Hessenberg tournament (cf. Example 2.15) has half of all possible arcs and $\mathrm{Z}(\vec{H}_n) = 1$, so the analogous result is not true for digraphs, and any correct result of this type for digraphs is not likely to be useful.

For a digraph Γ where $\mathrm{Z}(\Gamma) = k$, Theorem 4.3 gives an upper bound for the number of arcs Γ may possess. However, the proof also suggests that equality is achievable in (4.1) when $n > k$. The following provides a construction of a class of digraphs for which (4.1) is sharp.

THEOREM 4.4. *Let k be a fixed positive integer. Then for each integer $n > k$ and each partition $\pi = (n_1, \ldots, n_k)$ of n, there exists a digraph $\Gamma_{n,k,\pi}$ of order n for which $\mathrm{Z}(\Gamma_{n,k,\pi}) = k$, the forcing chains of $\Gamma_{n,k,\pi}$ have lengths $n_1, n_2, \ldots, n_k$ respectively, and*

$$|E(\Gamma_{n,k,\pi})| = \binom{n}{2} - \binom{k}{2} + k(n-1).$$

Proof. For $1 \leq i \leq k$, let Γ_i be a full Hessenberg path on n_i vertices. Among all the Γ_i, there are a total of $\sum_{i=1}^{k} \left[(n_i - 1) + \binom{n_i}{2}\right]$ arcs. Within each Γ_i we denote the initial vertex of the Hessenberg path by b_i and the terminal vertex of the Hessenberg path by t_i. We define $B = \{b_i : 1 \le i \le k\}$ and $T = \{t_i : 1 \le i \le k\}$, and note that B and T will intersect if $n_i = 1$ for some i. To create $\Gamma_{n,k,\pi}$ we start with $\bigcup_{i=1}^{k} \Gamma_i$ and add arcs between the Γ_i in the following manner:

(1) For $1 \le j < i \le k$, we add all arcs from vertices in Γ_i to vertices in Γ_j. This adds a total of $\sum_{i<j} n_i n_j$ arcs.

(2) Add all arcs from vertex t_i to vertices in other Γ_j. For each i, this adds $\sum_{j \ne i} n_j = n - n_i$ arcs. Over all i, this adds a total of $kn - \sum_{i=1}^{k} n_i = (k-1)n$ arcs.

Some arcs have been double-counted, which must be reflected in the overall total. In particular, arcs from t_i to all vertices in Γ_j (for $j < i$) have been double-counted. For an arc from t_i to a vertex v of Γ_j where $v \ne t_j$, we replace the double-counted arc by an arc from v to b_i. Therefore, we need only remove from the total count the number of arcs from t_j to t_i for $1 \le i < j \le k$. There are a total of $\binom{k}{2}$ such arcs. Thus, we have

$$\begin{aligned}
|E| &= \sum_{i=1}^{k} \left[(n_i - 1) + \binom{n_i}{2} \right] + \sum_{1 \le i < j \le k} n_i n_j + (k-1)n - \binom{k}{2} \\
&= n - k + \left(\sum_{i=1}^{k} \binom{n_i}{2} + \sum_{1 \le i < j \le k} n_i n_j \right) + kn - n - \binom{k}{2} \\
&= \binom{n}{2} - \binom{k}{2} + k(n-1).
\end{aligned}$$

By Theorem 4.3, we know that $Z(\Gamma_{n,k,\pi}) \ge k$. We claim that B is a zero forcing set for $\Gamma_{n,k,\pi}$ and that a chronological list of forces exists for which the forcing chains have lengths $n_1, \ldots, n_k$ respectively. Assume that each vertex of B is blue. Γ_1 is a Hessenberg path and the only arcs coming from vertices of Γ_1 point to vertices in B with the exception of arcs coming from t_1. Thus, forcing may occur along Γ_1, where t_1 is the last vertex forced. We then proceed to Γ_2 and so on through all Γ_i. When we get to Γ_i, the only arcs coming from vertices of Γ_i point to vertices in B or to the already blue vertices of Γ_h where $h < i$, with the exception of arcs coming from t_i (which is not used to perform a force). So, forcing may occur along Γ_i until all vertices are blue. Therefore B is a zero forcing set for $\Gamma_{n,k,\pi}$ and $Z(\Gamma_{n,k,\pi}) = k$. □

Although the digraph $Z(\Gamma_{n,k,\pi})$ achieve equality in the bound (4.3), these are not the only digraphs that do so.

Example 4.5. Let Γ be the digraph of order $n = 8$ in Figure 4.1. Note that $\{1, 5\}$ is a zero forcing set of Γ. Since $\deg^+(v) \ge 2$ for all vertices v, $k = Z(\Gamma) = 2$. Also note that Γ has 41 arcs, the same number of arcs as each digraph $\Gamma_{8,2,\pi}$.

In all of the digraphs $\Gamma_{n,k,\pi}$ constructed in Theorem 4.4, the forcing process may be completed one forcing chain at a time, and we show that this is not true for Γ.

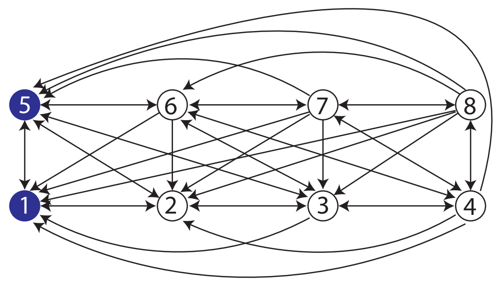

Fig. 4.1. *A digraph with the maximum number of arcs for its zero forcing number.*

Since $\deg^+(v) \geq 3$ for all vertices v other than vertex 1, vertex 1 must be contained in any minimum zero forcing set of Γ along with one of the two out-neighbors of vertex 1. Therefore, the only minimum zero forcing sets are $B_1 = \{1,5\}$ and $B_2 = \{1,2\}$. If we color the vertices of B_1 blue, then the first three forces must occur along the arcs $(1,2),(2,3)$, and $(5,6)$, in that order. If we color the vertices of B_2 blue, then the first three forces must occur along the arcs $(1,5),(2,3)$, and $(5,6)$, in that order. In either case, neither of the two forcing chains is completely blue before the forcing process must begin on the other. Therefore, Γ is not equal to any of the $\Gamma_{n,2,\pi}$ constructed in Theorem 4.4.

Although $\mathrm{M}(\Gamma)$ does not necessarily equal $\mathrm{Z}(\Gamma)$ for all digraphs Γ, we get equality for all $\Gamma_{n,k,\pi}$ constructed in the proof of Theorem 4.4.

Proposition 4.6. *If k and n are positive integers where $k < n$ and $\Gamma_{n,k,\pi}$ is one of the digraphs constructed in the proof of Theorem 4.4, then* $\mathrm{M}(\Gamma_{n,k,\pi}) = \mathrm{Z}(\Gamma_{n,k,\pi})$.

Proof. We adopt the notation and definitions used in the proof of Theorem 4.4. By construction, each of the k vertices in T has an arc to every other vertex of $\Gamma_{n,k,\pi}$. So for all vertices $v \notin T$, $\deg^-(v) \geq k$. Now we consider $t_i \in T$. If $t_i \neq b_i$, then there is an arc to t_i from another vertex of Γ_i. There are also arcs to t_i from all other vertices of T and therefore $\deg^-(t_i) \geq k$. We now consider the case where $t_i = b_i$. By construction, the subgraph induced on B is $\overleftrightarrow{K_k}$ and thus there is an arc to t_i from each of the other $k-1$ vertices of B. Furthermore, since $n > k$ there is a vertex $t_j \in T$ for which $t_j \neq b_j$. By construction, there is also an arc from t_j to t_i and therefore $\deg^-(t_i) \geq k$. This implies that $\delta^-(\Gamma_{n,k,\pi}) \geq k$. Since $\mathrm{M}(\Gamma_{n,k,\pi}) \geq \max\{\delta^-(\Gamma_{n,k,\pi}), \delta^+(\Gamma_{n,k,\pi})\}$ [5], $k \leq \mathrm{M}(\Gamma_{n,k,\pi}) \leq Z(\Gamma_{n,k,\pi}) \leq k$, and so $\mathrm{M}(\Gamma_{n,k,\pi}) = \mathrm{Z}(\Gamma_{n,k,\pi})$. $\square$

For $k = n-1$, $\Gamma_{n,k,\pi}$ is the digraph $\overleftrightarrow{K}_n$, so for $n \geq 4$, $\mathrm{P}(\Gamma_{n,k,\pi}) = \lceil \frac{n}{2} \rceil < n-1 = \mathrm{Z}(\Gamma_{n,k,\pi})$.

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