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POSITIVE SEMIDEFINITE ZERO FORCING

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Abstract. The positive semidefinite zero forcing number $\mathrm{Z}_+(G)$ of a graph G was introduced in [4]. We establish a variety of properties of $\mathrm{Z}_+(G)$: Any vertex of G can be in a minimum positive semidefinite zero forcing set (this is not true for standard zero forcing). The graph parameters $\mathrm{tw}(G)$ (tree-width), $\mathrm{Z}_+(G)$, and $\mathrm{Z}(G)$ (standard zero forcing number) all satisfy the Graph Complement Conjecture (see [3]). Graphs having extreme values of the positive semidefinite zero forcing number are characterized. The effect of various graph operations on positive semidefinite zero forcing number and connections with other graph parameters are studied.

Key words. zero forcing number, maximum nullity, minimum rank, positive semidefinite, matrix, graph

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1. Introduction.

Every graph discussed is simple (no loops or multiple edges), undirected, and has a finite nonempty vertex set. In a graph G where some vertices S are colored black and the remaining vertices are colored white, the *positive semidefinite color change rule* is: If $W_1, \dots, W_k$ are the sets of vertices of the k components of $G - S$ (note that it is possible that $k = 1$), $w \in W_i$, $u \in S$, and w is the only white neighbor of u in the subgraph of G induced by $W_i \cup S$, then change the color of w to black; in this case, we say *u forces w* and write $u \to w$. Given an initial set B of black vertices, the *derived set* of B is the set of black vertices that results from applying the positive semidefinite color change rule until no more changes are possible. A *positive semidefinite zero forcing set* is an initial set B of vertices such that the derived set of B is all the vertices of G. The *positive semidefinite zero forcing number* of a graph G, denoted $\mathrm{Z}_+(G)$, is the minimum of $|B|$ over all positive semidefinite zero forcing sets $B \subseteq V(G)$. The positive semidefinite zero forcing number is a variant of the *(standard) zero forcing number* $\mathrm{Z}(G)$, which uses the same definition with a different color change rule: If u is black and w is the only white neighbor of u, then change the color of w to black. The (standard) zero forcing number was introduced in [1] as an upper bound for maximum nullity, and the positive semidefinite zero forcing number was introduced in [4] as an upper bound for positive semidefinite maximum nullity.

Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph of A*, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$.

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The *maximum positive semidefinite nullity* of G is

$$\mathrm{M}_+(G) = \max\{\operatorname{null} A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

and *minimum positive semidefinite rank* of G is

$$\mathrm{mr}_+(G) = \min\{\operatorname{rank} A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

The (standard) maximum nullity $\mathrm{M}(G)$ and (standard) minimum rank $\mathrm{mr}(G)$ use the same definitions omitting the requirement of positive semidefiniteness. It is clear that $\mathrm{mr}_+(G)+\mathrm{M}_+(G) = |G|$. In [4] it was shown that for every graph

$$\mathrm{M}_+(G) \leq \mathrm{Z}_+(G).$$

It was also shown there that

$$OS(G) + \mathrm{Z}_+(G) = |G|$$

where $OS(G)$ is a graph parameter defined in [14], and in fact shown that the complement of an OS-set is a positive semidefinite zero forcing set and the complement of a positive semidefinite zero forcing set is an OS-set. The reader is referred to [14] for the definition of OS-set and $OS(G)$.

We establish a variety of properties of $\mathrm{Z}_+(G)$. In Section 2 connections between zero forcing sets and OS-sets are applied to show that every vertex of G is in some minimum positive semidefinite zero forcing set (this is not true for standard zero forcing). It is also shown there that $\mathrm{T}(G) \leq \mathrm{Z}_+(G)$ where $\mathrm{T}(G)$ is the tree cover number of G, and cut-vertex reduction formulas for $TC(G)$ and $\mathrm{Z}_+(G)$ are established. In Section 3 it is shown that the graph parameters $\mathrm{tw}(G)$ (tree-width), $\mathrm{Z}_+(G)$, and $\mathrm{Z}(G)$ (standard zero forcing number) all satisfy the Graph Complement Conjecture (see [3]). Graphs having extreme values of the positive semidefinite zero forcing number are characterized in Section 4. The effect of various graph operations on positive semidefinite zero forcing number and connections with other graph parameters are studied in Section 5.

There are a few more graph terms that we need to define. The subgraph $G[W]$ of $G = (V, E)$ *induced* by $W \subseteq V$ is the subgraph with vertex set W and edge set $\{\{i, j\} \in E : i, j \in W\}$; $G-W$ is used to denote $G[V \setminus W]$. The graph $G - \{v\}$ is also denoted by $G - v$. The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets from V that are not in E. The *union* of $G_i = (V_i, E_i)$ is $\bigcup_{i=1}^h G_i = (\bigcup_{i=1}^h V_i, \bigcup_{i=1}^h E_i)$. The *intersection* of $G_i = (V_i, E_i)$ is $\bigcap_{i=1}^h G_i = (\bigcap_{i=1}^h V_i, \bigcap_{i=1}^h E_i)$ (provided the intersection of the vertices is nonempty). The *degree* of vertex v in graph G, $\deg_G v$, is the number of neighbors of v. A graph is *chordal* if it has no induced cycle of length 4 or more; clearly any induced subgraph of a chordal graph is chordal.

2. Tree cover number, positive semidefinite zero forcing number, and maximum positive semidefinite nullity. The *tree cover number* of a graph G, denoted $\mathrm{T}(G)$, is defined as the minimum number of vertex disjoint trees occurring as induced subgraphs of G that cover all of the vertices of G, and was introduced by Barioli, Fallat, Mitchell, and Narayan in [5]. In that paper the authors show that for any outerplanar graph G, $\mathrm{M}_+(G) = \mathrm{T}(G)$ and if G is a chordal graph, then $\mathrm{T}(G) \leq \mathrm{M}_+(G)$. It is conjectured there that $\mathrm{T}(G) \leq \mathrm{M}_+(G)$ for every graph.

2.1. Membership in a minimum positive semidefinite zero forcing set. The next theorem is an interesting consequence of the connection between OS-number and Z_+.

THEOREM 2.1. *If G is a graph and $v \in V(G)$, then there exist minimum positive semidefinite zero forcing sets B_1 and B_2 such that $v \in B_1$ and $v \notin B_2$.*

Proof. Let G be a graph and $v \in V(G)$. By Corollary 2.17 in [19], there exist OS-sets S_1 and S_2 with $|S_1| = |S_2| = OS(G)$, $v \notin S_1$ and $v \in S_2$. Then by [4, Theorem 3.6] $B_1 = \overline{S_1}$ and $B_1 = \overline{S_1}$ are minimum positive semidefinite zero forcing sets, with $v \in B_1$ and $v \notin B_2$. □

Note that the situation for positive semidefinite zero forcing as described by Theorem 2.1 is very different from (standard) zero forcing, where it is known that a graph can have a vertex that is not in any minimum zero forcing set. For example, a degree 2 vertex in a path $P_n, n \geq 3$ cannot be in a minimum zero forcing set for P_n. But we do have the extension to positive semidefinite of the property that no vertex is in every minimum zero forcing set.

COROLLARY 2.2. *If G is a connected graph of order greater than one, then*

$$\bigcap_{B \in ZFS_+(G)} B = \emptyset,$$

where $ZFS_+(G)$ is the set of all minimum positive semidefinite zero forcing sets of G.

2.2. Forcing trees. Tree cover number can be viewed as a generalization of *path cover number*, i.e., the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all of the vertices of G. It is well known that path cover number $\mathrm{P}(G)$ and maximum nullity $\mathrm{M}(G)$ are noncomparable in general, but $\mathrm{P}(G) \leq \mathrm{Z}(G)$ for every graph G. The proof uses paths of forces, and we extend this to trees of positive semidefinite forces, thus showing that $\mathrm{T}(G) \leq \mathrm{Z}_+(G)$. Let G be a graph and B a positive semidefinite zero forcing set for G. Construct the derived set, listing the forces in the order in which they were performed. This list $\mathcal{F}$ is a *chronological list of forces*. The terminology in the next definition will be justified in Theorem 2.4.

DEFINITION 2.3. Given a graph G, positive semidefinite zero forcing set B, chronological list of forces $\mathcal{F}$, and a vertex $b \in B$, define V_b to be the set of vertices w such that there is a sequence of forces $b = v_1 \to v_2 \to \cdots \to v_k = w$ in $\mathcal{F}$ (the empty sequence of forces is permitted, i.e., $b \in V_b$). The *forcing tree* T_b is the induced subgraph $T_b = G[V_b]$. The *forcing tree cover* (for the chronological list of forces $\mathcal{F}$) is $\mathcal{T} = \{T_b \,|\, b \in B\}$. An *optimal forcing tree cover* is a forcing tree cover from a chronological list of forces of a minimum positive semidefinite zero forcing set.

A graph with positive semidefinite zero forcing set with forces marked and the resulting forcing tree cover are shown in Figure 2.1.

THEOREM 2.4. *Assume G is a graph, B is a positive semidefinite zero forcing set of G, $\mathcal{F}$ is a chronological list of forces of B, and $b \in B$. Then*

1. *T_b is a tree.*
2. *The forcing tree cover $\mathcal{T} = \{T_b : b \in B\}$ is a tree cover of G.*
3. *$\mathrm{T}(G) \leq \mathrm{Z}_+(G)$.*

Proof. The sets V_b of vertices forced by distinct $b \in B$ are disjoint because each vertex of G is forced only once. If a graph H is not a tree, then $\mathrm{Z}_+(H) > 1$ (this follows from the result that H

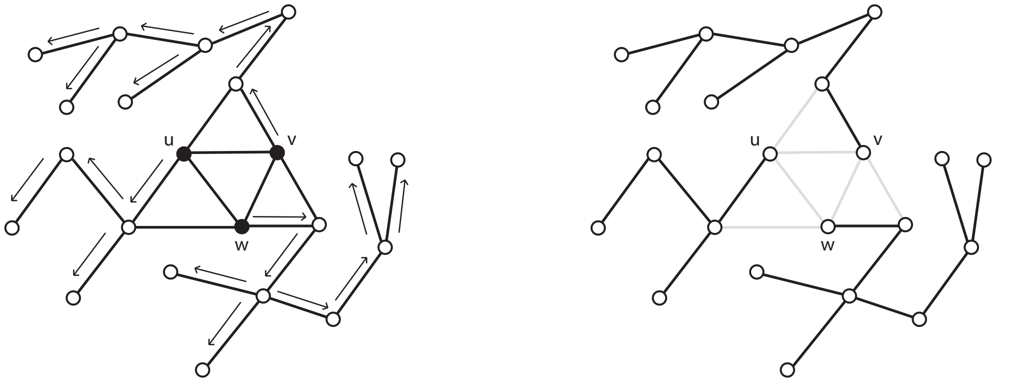


Fig. 2.1: A graph with forces marked, and the resulting forcing tree cover

not a tree implies $\mathrm{M}_+(H) > 1$ [15]). So if $T_b = G[V_b]$ is not a tree, then there must exist a vertex $v \in V_b \setminus \{b\}$ such that either $v \in B$ or v was forced through a sequence of forces from some element of B not equal to b. In either case, this contradicts the fact that the sets V_b of vertices forced by different elements of B are disjoint. Thus T_b is a tree.

Since each vertex $b \in B$ forces an induced subtree, the trees forced by distinct elements of B are disjoint, and B is a positive semidefinite zero forcing set, $\mathcal{T} = \{T_b : b \in B\}$ is a tree cover of G. Now suppose that B is a minimum positive semidefinite zero forcing set for G. Since $\mathcal{T}$ is a tree cover of G, $T(G) \leq |\mathcal{T}| = |B| = Z_+(G)$. □

2.3. Cut-vertex reduction. Cut-vertex reduction is a standard technique in the study of minimum rank. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. Suppose $G_i, i = 1, \ldots, h$ are graphs of order at least two, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^{h} G_i$ (if $h \geq 2$, then clearly v is a cut-vertex of G). Then it is established in [16] that

$$\mathrm{mr}_+(G) = \sum_{i=1}^{h} \mathrm{mr}_+(G_i).$$

Because $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$, this is equivalent to

$$\mathrm{M}_+(G) = \left(\sum_{i=1}^{h} \mathrm{M}_+(G_i)\right) - h + 1. \tag{2.1}$$

It is shown in [19] that

$$OS(G) = \sum_{i=1}^{h} OS(G_i).$$

Because $OS(G) + \mathrm{Z}_+(G) = |G|$ [4], this is equivalent to

$$\mathrm{Z}_+(G) = \left(\sum_{i=1}^{h} \mathrm{Z}_+(G_i)\right) - h + 1. \tag{2.2}$$

An analogous reduction formula is valid for tree cover number.

Proposition 2.5. *Suppose $G_i, i = 1, \ldots, h$ are graphs, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$. Then*

$$\mathrm{T}(G) = \left(\sum_{i=1}^{h} \mathrm{T}(G_i) \right) - h + 1. \tag{2.3}$$

Proof. For each G_i, let $\mathcal{T}_i$ be a tree cover of minimum cardinality. In each $\mathcal{T}_i$, there exists some T_i such that $v \in V(T_i)$. Define $T_v = \bigcup_{i=1}^h T_i$. Then $\mathcal{T} = \bigcup_{i=1}^h (\mathcal{T}_i \setminus \{T_i\}) \cup \{T_v\}$ is a tree cover for G. Therefore $\mathrm{T}(G) \leq \left(\sum_{i=1}^h \mathrm{T}(G_i) \right) - (h-1)$.

Let $\mathcal{T}$ be a minimum tree cover for G. Let T_v be the tree that includes v. For $i = 1, \ldots, h$, define $T_{v,i} = T_v \cap G_i$. For each $T \in \mathcal{T}$ such that $v \notin V(T)$, T is a subgraph of some G_i. Define $\mathcal{T}_i = \{T_{v,i}\} \cup \{T \in \mathcal{T} : T \text{ is a subgraph of } G_i\}$. Since $\mathcal{T}_i$ is a tree cover of G_i, $\mathrm{T}(G_i) \leq |\mathcal{T}_i|$. Thus

$$\sum_{i=1}^{h} \mathrm{T}(G_i) \leq \sum_{i=1}^{h} |\mathcal{T}_i| = |\mathcal{T}| + h - 1 = \mathrm{T}(G) + h - 1. \qquad \square$$

We have the following immediate consequences of the cut-vertex reduction formulas (2.1), (2.2), and (2.3).

Corollary 2.6. *Suppose $G_i, i = 1, \ldots, h$ are graphs, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^h G_i$.*

1. *If $\mathrm{M}_+(G_i) = \mathrm{Z}_+(G_i)$ for all $i = 1, \ldots, h$, then $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$.*
2. *If $\mathrm{T}(G_i) = \mathrm{Z}_+(G_i)$ for all $i = 1, \ldots, h$, then $\mathrm{T}(G) = \mathrm{Z}_+(G)$.*
3. *If $\mathrm{M}_+(G_i) = \mathrm{T}(G_i)$ for all $i = 1, \ldots, h$, then $\mathrm{M}_+(G) = \mathrm{T}(G)$.*

Corollary 2.7. *Suppose H is a graph, T is a tree, and H and T intersect in a single vertex. For $G = H \cup T$,*

1. $\mathrm{M}_+(G) = \mathrm{M}_+(H)$.
2. $\mathrm{Z}_+(G) = \mathrm{Z}_+(H)$.
3. $\mathrm{T}(G) = \mathrm{T}(H)$.

3. Graph Complement Conjecture. The *graph complement conjecture* or GCC (Conjecture 3.1 below) was stated at the 2006 American Institute of Mathematics workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns" [2].

Conjecture 3.1 (GCC). [10] *For any graph G,*

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2,$$

or equivalently,

$$\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq |G| - 2. \tag{3.1}$$

The conjecture (3.1), which is a Nordhaus-Gaddum type problem, was generalized in [3] to a variety of graph parameters related to maximum nullity, including positive semidefinite maximum nullity. For a graph parameter β related to maximum nullity, the graph compliment conjecture for β, GCC_β, is

$$\beta(G) + \beta(\overline{G}) \geq |G| - 2.$$

With this notation, GCC can be denoted $\mathrm{GCC}_{\mathrm{M}}$, and the graph compliment conjecture for positive semidefinite maximum nullity is denoted $\mathrm{GCC}_{\mathrm{M}_+}$. In this section we establish that $\mathrm{GCC}_{\mathrm{tw}}$, and hence $\mathrm{GCC}_{\mathrm{Z}_+}$ and $\mathrm{GCC}_{\mathrm{Z}}$ are true.

A *tree decomposition* of a graph G is a pair $(T, \mathcal{W})$, where T is a tree and $\mathcal{W} = \{W_t : t \in V(T)\}$ is a collection of subsets of $V(G)$ with the following properties:

1. $\bigcup_{t \in V(T)} W_t = V(G)$.
2. Every edge of G has both ends in some W_t.
3. If $t_1, t_2, t_3 \in V(T)$ and t_2 lies on a path from t_1 to t_3, then $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$.

The *bags* of the tree decomposition are the subsets W_t. The *width* of a tree decomposition is $\max\{|W_t| - 1 : t \in V(T)\}$, and the *tree-width* $\mathrm{tw}(G)$ of G is the minimum width of any tree decomposition of G. Tree-width can be characterized in terms of the clique number of chordal graphs and in terms of partial k-trees. The greatest integer r such that $K_r \subseteq G$ is the *clique number* $\omega(G)$. It follows from [11, Corollary 12.3.12] that

$$\mathrm{tw}(G) = \min\{\omega(H) - 1 : V(H) = V(G), G \subseteq H, \text{ and } H \text{ is chordal}\} \tag{3.2}$$

Note that in [11, Corollary 12.3.12], the minimum is taken over all chordal supergraphs; however, if $H \supseteq G$ is chordal, then $H[V(G)] \supseteq G$, $H[V(G)]$ is chordal, and $\omega(H[V(G)]) \leq \omega(H)$ and so we may take the minimum over only those chordal supergraphs with the same vertex set. For a positive integer k, a *k-tree* is constructed inductively by starting with a complete simple graph on $k+1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. A *partial k-tree* is a subgraph of a k-tree. Then $\mathrm{tw}(G)$ is the least positive integer k such that G is a partial k-tree [9, F12, p. 111].

A graph is *co-chordal* if its complement is chordal. A *triangulation* of a graph G is a chordal graph that is obtained from G by adding edges. A graph G is a *split graph* if there is a nonempty set $S \subset V(G)$ such that S is an independent set in G and $G - S$ is a clique. This definition of split graph differs slightly from the definition given in [17], where neither $S \neq \emptyset$ nor $S \neq V(G)$ is required. However, the two definitions are equivalent for graphs of order at least two: In case $S = V(G)$ is independent, then for any vertex $v \in V(G)$, $S' = S \setminus \{v\}$ is independent and $G - S'$ is an (order 1) clique. In case $S = \emptyset$ (so G is a clique), then for any vertex $v \in V(G)$, $S' = \{v\}$ is independent and $G - S'$ is a clique.

THEOREM 3.2. *Let $G = (V, E)$ be a graph of order at least two. Let H be a chordal supergraph of G and F be a co-chordal subgraph of G with $V(G) = V(H) = V(F)$. Then for some clique of H and some clique of $\overline{F}$, the union of their vertex sets is all of V.*

Proof. Since $F \subseteq G \subseteq H$ and H is chordal, H is a triangulation of F. Let $\Gamma \subseteq H$ be a minimal triangulation of F. Since F is co-chordal, it is $2K_2$ free (see, for example [17, Fact 2]), so by [17, Corollary 7], Γ is a split graph. Let S be an independent set of vertices such that $\Gamma - S$ is a clique. Since S is independent, $\overline{\Gamma[S]} = \overline{\Gamma}[S]$ is also a clique. Since $\Gamma \subseteq H$, $\Gamma - S \subseteq H$ and since $F \subseteq \Gamma$ with the same vertex set, $\overline{\Gamma} \subseteq \overline{F}$ and so $\overline{\Gamma}[S] \subseteq \overline{F}$. Finally, it is obvious that $(V \setminus S) \cup S = V$. □

THEOREM 3.3. *GCC_{tw} is true, i.e.,* $\text{tw}(G) + \text{tw}(\overline{G}) \geq |G| - 2$.

Proof. Let G be a graph. By (3.2), we can choose chordal graphs $H \supseteq G$ and $H' \supseteq \overline{G}$ such that $\omega(H) = \text{tw}(G) + 1$, $\omega(H') = \text{tw}(\overline{G}) + 1$, and $V(G) = V(H) = V(H')$. Observe that Theorem 3.2 can be applied with H as H and $\overline{H'}$ as F in the theorem. So there exist cliques $K_r \subseteq H$ and $K_{r'} \subseteq H'$ such that $V(G) = V(K_r) \cup V(K_{r'})$. Therefore,

$$|G| = |V(K_r) \cup V(K_{r'})| \leq |K_r| + |K_{r'}| \leq \omega(H) + \omega(H') = \text{tw}(G) + \text{tw}(\overline{G}) + 2. \qquad \square$$

Since for every graph G, $\text{tw}(G) \leq \text{Z}_+(G) \leq \text{Z}(G)$, we have the following corollary.

COROLLARY 3.4. *GCC_{Z_+} and GCC_{Z} are true, i.e.,*

$$\text{Z}_+(G) + \text{Z}_+(\overline{G}) \geq |G| - 2 \qquad \textit{and} \qquad \text{Z}(G) + \text{Z}(\overline{G}) \geq |G| - 2.$$

Note that GCC_{Z_+} also follows from [18, Proposition 9].

4. Graphs with extreme positive semidefinite zero forcing number. In this section we show that for graphs having very low or very high maximum positive semidefinite nullity or positive semidefinite zero forcing number, these two parameters are equal. Since characterizations of graphs having very low or very high maximum positive semidefinite nullity are known, these extend to graphs having very low or very high positive semidefinite zero forcing number.

It is well known that $\text{M}_+(G) = 1$ if and only if G is a tree if and only if $\text{Z}_+(G) = 1$ (the first equivalence is established in [15], and the latter follows from $\text{M}_+(G) \leq \text{Z}_+(G)$ and the fact that any one vertex is a positive semidefinite zero forcing set for a tree). Graphs that have $\text{M}_+(G) = 2$ are characterized in [15] (note that here a graph is required to be simple whereas in [15] multigraphs are considered).

A connected graph is *nonseparable* if it does not have a cut-vertex. A *block* of a graph is a maximal nonseparable subgraph.

THEOREM 4.1. *Let G be a graph. The following are equivalent.*

1. $\text{Z}_+(G) = 2$,
2. $\text{M}_+(G) = 2$,
3. *Either*
   (a) *G is the disjoint union of two trees, or*
   (b) *G is connected, exactly one block of G has a cycle, and G does not have a K_4 or T_3 minor.*

Proof. (2) $\Leftrightarrow$ (3): This follows from Theorems 4.3 and 2.2 in [15] and the fact that $\text{M}_+(G) = 1$ if and only if G is a tree.
(1) $\Rightarrow$ (2) because $\text{M}_+(G) \leq \text{Z}_+(G)$ and $\text{M}_+(G) = 1 \Leftrightarrow \text{Z}_+(G) = 1$.

(3) ⇒ (1): By hypothesis, G has no K_4 minor, so $\mathrm{tw}(G) \leq 2$ (see [9, F31, p. 112]). It is shown in [12] that if $\mathrm{tw}(G) \leq 2$, then $\mathrm{Z}_+(G) = \mathrm{M}_+(G)$. (Note that [12] defines tree-width in terms of partial k-trees, but as noted in Section 3, that definition is equivalent to the standard definition used here.) □

Corollary 4.2. *If* $\mathrm{Z}_+(G) \leq 3$, *then* $\mathrm{Z}_+(G) = \mathrm{M}_+(G)$.

Proof. If $\mathrm{Z}_+(G) = 3$, then $\mathrm{M}_+(G) \leq 3$, but $\mathrm{M}_+(G) \leq 2$ would imply $\mathrm{Z}_+(G) \leq 2$ by Theorem 4.1 and the fact that $\mathrm{M}_+(G) = 1 \Leftrightarrow \mathrm{Z}_+(G) = 1$. □

Observe that $\mathrm{Z}_+(V_8) = 4$ but $\mathrm{M}_+(V_8) = 3$, so for $\mathrm{Z}_+(G) \geq 4$ there is no result analogous to Corollary 4.2.

Theorem 4.4 below, which characterizes high positive semidefinite zero forcing number, follows from the characterization of graphs having $\mathrm{mr}_+(G) \leq 2$ in [7], using the parameter mz_+ and the next proposition. Define $\mathrm{mz}_+(G) = |G| - \mathrm{Z}_+(G)$. Since $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$, $\mathrm{mz}_+(G) \leq \mathrm{mr}_+(G)$. The proof of Proposition 4.3 below is the same as the proof of Proposition 4.4 in [1].

Proposition 4.3. *If H is an induced subgraph of G, then* $\mathrm{mz}_+(H) \leq \mathrm{mz}_+(G)$.

Theorem 4.4. *Let G be a graph. The following are equivalent.*

1. $\mathrm{Z}_+(G) \geq |G| - 2$,
2. $\mathrm{M}_+(G) \geq |G| - 2$,
3. *G has no induced* $P_4, K_{1,3}, P_3\dot{\cup}K_2, 3K_2$

Proof. (1) ⇒ (3) by Proposition 4.3, because $\mathrm{mz}_+(H) = 3$ for $H = P_4, K_{1,3}, P_3\dot{\cup}K_2$, or $3K_2$. (3) ⇒ (2) by Theorem 8 in [7]. (2) ⇒ (1) since $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$. □

It is clear that $\mathrm{M}_+(G) = |G|$ if and only if G has no edges, and the same is true for $\mathrm{Z}_+(G)$. Similarly, $\mathrm{M}_+(G) = |G| - 1 \Leftrightarrow G = K_r \cup sK_1 \Leftrightarrow \mathrm{Z}_+(G) = |G| - 1$. The next corollary is analogous to Corollary 4.2.

Corollary 4.5. *If* $\mathrm{M}_+(G) \geq |G| - 3$, *then* $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$.

5. Effects of graph operations on Z_+.

We examine the effect of various graph operations, including vertex deletion, edge deletion, edge subdivision, and edge contraction on positive semidefinite zero forcing number.

5.1. Vertex deletion.

The effect of vertex deletion (and edge deletion) on the (standard) zero forcing number was established in [13], where this was described using the language of spreads, i.e., the difference between the parameter evaluated on G and on G with a vertex or edge deleted. In this section we examine the effect of vertex deletion on positive semidefinite zero forcing number.

Definition 5.1. *Let G be a graph and v be a vertex in G.*

1. *The positive semidefinite rank spread of v is* $\mathrm{r}_v^+(G) = \mathrm{mr}_+(G) - \mathrm{mr}_+(G - v)$.
2. *The positive semidefinite null spread of v is* $\mathrm{n}_v^+(G) = \mathrm{M}_+(G) - \mathrm{M}_+(G - v)$.
3. *The positive semidefinite zero spread of v is* $\mathrm{z}_v^+(G) = \mathrm{Z}_+(G) - \mathrm{Z}_+(G - v)$.

Observation 5.2. *For any graph G and vertex v,*

1. $0 \le \mathrm{r}_v^+(G)$.
2. $\mathrm{n}_v^+(G) \le 1$.
3. $\mathrm{r}_v^+(G) + \mathrm{n}_v^+(G) = 1$.

The proof of the next proposition is the same as part of the proof of Theorem 2.3 in [13].

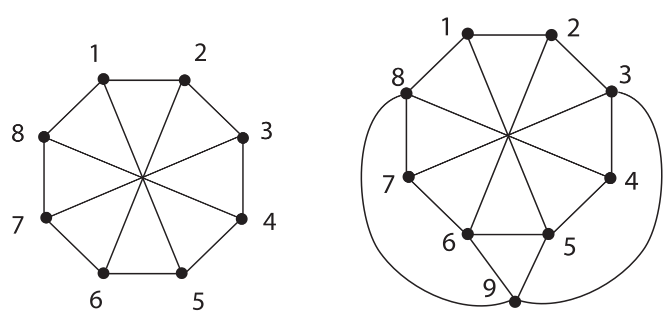
Proposition 5.3. *Let G be a graph and v be a vertex in G. Then $\mathrm{Z}_+(G - v) \ge \mathrm{Z}_+(G) - 1$, so $\mathrm{z}_v^+(G) \le 1$.*

However, there is no upper bound for $\mathrm{r}_v^+(G)$ and no lower bound for $\mathrm{n}_v^+(G)$ and $\mathrm{z}_v^+(G)$ as exhibited in the following example.

Example 5.4. *The complete bipartite graph $K_{1,s}$ with $s \ge 2$ has $\mathrm{mr}_+(K_{1,s}) = s$ and $\mathrm{M}_+(K_{1,s}) = 1 = \mathrm{Z}_+(K_{1,s})$. However if v is the cut-vertex, then $K_{1,s} - v$ has no edges and thus $\mathrm{mr}_+(K_{1,s} - v) = 0$ and $\mathrm{M}_+(K_{1,s} - v) = s = \mathrm{Z}_+(K_{1,s} - v)$. Thus $\mathrm{r}_v^+(K_{1,s}) = s$ and $\mathrm{n}_v^+(K_{1,s}) = 1 - s = \mathrm{z}_v^+(K_{1,s})$.*

As is the case with (standard) zero forcing number and maximum nullity [13], the parameters $\mathrm{n}_v^+(G)$ and $\mathrm{z}_v^+(G)$ are not comparable.

Example 5.5. The graph V_8 (also known as the Möbius ladder on 8 vertices) shown in Figure 5.1a has $\mathrm{M}_+(G) = 3$ and $\mathrm{Z}_+(G) = 4$ [19, 4]. Since $\{1, 2, 3\}$ is a positive semidefinite zero forcing set for $V_8 - 8$, $\mathrm{Z}_+(V_8 - 8) \le 3$. Then by Corollary 4.2, $\mathrm{M}_+(V_8 - 8) = \mathrm{Z}_+(V_8 - 8)$, so $\mathrm{n}_8^+(V_8) < \mathrm{z}_8^+(V_8)$. (It is not difficult to find a matrix $A \in \mathcal{S}_+(V_8 - 8)$ with $\operatorname{rank} A = 4$, so $\mathrm{M}_+(V_8 - 8) \ge 3$, $\mathrm{M}_+(V_8 - 8) = \mathrm{Z}_+(V_8 - 8) = 3$, and $\mathrm{n}_8^+(V_8) = 0$ and $\mathrm{z}_8^+(V_8) = 1$.)



(a) V_8 (b) G_9

Fig. 5.1: The graphs V_8 and G_9

Example 5.6. The graph G_9 in Figure 5.1b has a positive semidefinite zero forcing set

$\{3,4,7,8\}$ so $\mathrm{Z}_+(G_9) \le 4$. Since

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 3 & -5 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & -4 & 1 & 0 & -1 \end{bmatrix}$$

is an orthogonal representation of G_9 in $\mathbb{R}^5$ (i.e., $B^TB \in \mathcal{S}_+(G_9)$), $\mathrm{M}_+(G_9) \ge 4$. Thus $\mathrm{Z}_+(G_9) = \mathrm{M}_+(G_9) = 4$. Since $G_9 - 9 = V_8$, $\mathrm{z}_9^+(G_9) < \mathrm{n}_9^+(G_9)$ (in fact, $\mathrm{z}_9^+(G_9) = 0$ and $\mathrm{n}_9^+(G_9) = 1$).

As in [13], we have the following observation.

Observation 5.7. *Let G be a graph such that $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ and let v be a vertex of G.*

1. $\mathrm{n}_v^+(G) \ge \mathrm{z}_v^+(G)$.
2. *If $\mathrm{z}_v^+(G) = 1$, then $\mathrm{n}_v^+(G) = 1$.*

In the case of standard maximum nullity and zero forcing number, $\mathrm{M}(G) = \mathrm{Z}(G)$ and $\mathrm{n}_v(G) = -1$ imply $\mathrm{z}_v(G) = -1$. However, since there are no lower bounds on $\mathrm{z}_v^+(G)$ and $\mathrm{n}_v^+(G)$, we do not have any bound based on $\mathrm{n}_v^+(G) = -1$, as the next example shows.

Example 5.8. Let H be the graph obtained from G_9 in Example 5.6 by appending two leaves to vertex 9. Then by cut-vertex reduction (2.1) and (2.2), $\mathrm{M}_+(H) = 4 + 1 + 1 - 3 + 1 = \mathrm{Z}_+(G)$. Since $H - 9 = V_8 \dot\cup 2K_1$, $\mathrm{M}_+(G) = 5$ and $\mathrm{Z}_+(G) = 6$. Thus $\mathrm{n}_9^+(H) = -1$ and $\mathrm{z}_9^+(H) = -2$.

A tree cover $\mathcal{T}$ of G contains a vertex v as a *singleton* if $\{v\}$ (with no other vertices and no edges) is one of the trees in $\mathcal{T}$. The proof of the next proposition is the same as the proof of Theorem 2.7 in [13].

Proposition 5.9. *Let G be a graph and $v \in V(G)$. Then there exists an optimal forcing tree cover of G that contains v as a singleton if and only if $\mathrm{z}_v^+(G) = 1$.*

Remark 5.10. For the (standard) zero forcing number Z, we know that if G is a graph, $v \in V(G)$, B is a minimum zero forcing set, and $v \in B$, then $\mathrm{z}_v(G) \ge 0$. However, this is not the case for $\mathrm{z}_v^+(G)$, because for any vertex v, there is a minimum positive semidefinite zero forcing set containing v by Theorem 2.1, yet there are vertices that have negative spread (such as in Example 5.4).

For a graph G, the *neighborhood* of $v \in V(G)$ is $N_G(v) = \{w \in V(G) : w \text{ is adjacent to } v\}$. Vertices v and w of G are called *duplicate vertices* if $N_G(v) \cup \{v\} = N_G(w) \cup \{w\}$. Observe that duplicate vertices are necessarily adjacent. It was shown in [8] that if v is a duplicate vertex in a connected graph G of order at least three, then $\mathrm{mr}_+(G-v) = \mathrm{mr}_+(G)$, so $\mathrm{M}_+(G-v) = \mathrm{M}_+(G)-1$.

Proposition 5.11. *If v and w are duplicate vertices in a connected graph G with $|G| \ge 3$, then $\mathrm{Z}_+(G-v) = \mathrm{Z}_+(G) - 1$, or equivalently, $\mathrm{z}_v^+(G) = 1$.*

Proof. Choose a minimum positive semidefinite zero forcing set B that contains v. We show that $B - \{v\}$ is a positive semidefinite zero forcing set for $G - v$. Proposition 5.3 then implies that $B - \{v\}$ is a minimum positive semidefinite zero forcing set for $G - v$.

Observe that in G, unless v forces w, v cannot perform a force until w is black. If v does not force w in G, then either $w \in B$ or there is a $u \in N_G(w)$ such that $u \to w$. In the latter case, u also forces w in $G - v$ starting with black vertices $B - \{v\}$. Then in $G - v$, w can perform any forces that v had performed in G. So if v does not force w in G, then $B - \{v\}$ is a positive semidefinite zero forcing set for $G - v$.

So assume v forces w, then at the stage at which $v \to w$, all vertices in $N_G(v) - \{w\}$ are black. So in $G - v$, $B - \{v\}$ can still force all the vertices in $N_{G-v}(w)$. Since $|G| \geq 3$ and G is connected, $N_{G-v}(w) \neq \emptyset$, and any $u \in N_{G-v}(w)$ can force w (since w is an isolated vertex after all the currently black vertices are deleted from $G - v$). As before, all remaining forces can then be performed. Therefore $B - \{v\}$ is a positive semidefinite zero forcing set. □

5.2. Edge deletion. If e is an edge of G, then $G - e$ is the graph obtained from G by deleting e. In this section we examine the effect of edge deletion on positive semidefinite zero forcing number, using spread terminology.

DEFINITION 5.12. *Let G be a graph and e be an edge in G.*

1. *The positive semidefinite rank edge spread of e is $\mathrm{r}_e^+ = \mathrm{mr}_+(G) - \mathrm{mr}_+(G - e)$.*
2. *The positive semidefinite null edge spread of e is $\mathrm{n}_e^+(G) = \mathrm{M}_+(G) - \mathrm{M}_+(G - e)$.*
3. *The positive semidefinite zero edge spread of e is $\mathrm{z}_e^+(G) = \mathrm{Z}_+(G) - \mathrm{Z}_+(G - e)$.*

OBSERVATION 5.13. *For any graph G and edge e of G, $\mathrm{r}_e^+(G) + \mathrm{n}_e^+(G) = 0$.*

PROPOSITION 5.14. *Let G be a graph and $e = \{i, j\}$ be an edge in G. Then*

1. $-1 \leq \mathrm{r}_e^+(G) \leq 1$,
2. $-1 \leq \mathrm{n}_e^+(G) \leq 1$,
3. $-1 \leq \mathrm{z}_e^+(G) \leq 1$.

Proof. Nylen [20] established that the (standard) rank edge spread is between -1 and 1, and the same proof establishes that $\mathrm{r}_e^+(G) \leq 1$. For the other inequality in part (1), choose a matrix $A \in \mathcal{S}_+(G)$ having $\operatorname{rank} A = \mathrm{mr}_+(G)$, and let $\mathbf{e}_k$ denote the kth standard basis vector in $\mathbb{R}^n$. Define $A' = A + (\mathbf{e}_i - a_{ij}\mathbf{e}_j)(\mathbf{e}_i - a_{ij}\mathbf{e}_j)^T$. Then $A' \in \mathcal{S}_+(G - e)$ and $\operatorname{rank} A' \leq \operatorname{rank} A + 1 = \mathrm{mr}_+(G) + 1$, so $\mathrm{r}_e^+(G) \geq -1$. Part (2) follows from part (1) and Observation 5.13. Part (3) can be proven by the same method used to prove Theorem 2.17 in [13] (although Theorem 2.1 could be used to simplify the proof). □

As is the case with (standard) zero forcing number and maximum nullity [13], the parameters $\mathrm{n}_e^+(G)$ and $\mathrm{z}_e^+(G)$ are not comparable.

EXAMPLE 5.15. The graph V_8 has $\mathrm{M}_+(G) = 3$ and $\mathrm{Z}_+(G) = 4$ [19, 4]. Consider the edge $e = \{1, 8\}$. Since $\{1, 2, 3\}$ is a positive semidefinite zero forcing set for $V_8 - e$, $\mathrm{Z}_+(V_8 - e) \leq 3$. Then by Corollary 4.2, $\mathrm{M}_+(V_8 - 8) = \mathrm{Z}_+(V_8 - 8)$, so $\mathrm{n}_8^+(V_8) < \mathrm{z}_8^+(V_8)$.

EXAMPLE 5.16. In Example 5.6 it was shown that the graph G_9 has $\mathrm{Z}_+(G_9) = \mathrm{M}_+(G_9) = 4$. Let $e_1 = \{3, 9\}$, $e_2 = \{5, 9\}$, $e_3 = \{6, 9\}$, $e_4 = \{8, 9\}$. Define $H_0 = G_9$ and $H_k = G_9 - \{e_1, \ldots, e_k\}$

for $k = 1, \ldots, 4$. Note that $H_4 = V_8 \dot\cup K_1$, so $\mathrm{Z}_+(H_4) = 5$ and $\mathrm{M}_+(H_4) = 4$. Since

$$-1 = \mathrm{Z}_+(H_0) - \mathrm{Z}_+(H_4) = \mathrm{z}^+_{e_1}(H_0) + \mathrm{z}^+_{e_2}(H_1) + \mathrm{z}^+_{e_3}(H_2) + \mathrm{z}^+_{e_4}(H_3), \text{ and}$$
$$0 = \mathrm{M}_+(H_0) - \mathrm{M}_+(H_4) = \mathrm{n}^+_{e_1}(H_0) + \mathrm{n}^+_{e_2}(H_1) + \mathrm{n}^+_{e_3}(H_2) + \mathrm{n}^+_{e_4}(H_3),$$

necessarily there exists a $k \in \{1, 2, 3, 4\}$ such that $\mathrm{z}^+_{e_k}(H_{k-1}) < \mathrm{n}^+_{e_k}(H_{k-1})$.

Observation 5.17. *Let G be a graph such that $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ and let e be an edge of G.*

1. $\mathrm{n}^+_e(G) \geq \mathrm{z}^+_e(G)$.
2. *If $\mathrm{z}^+_e(G) = 1$, then $\mathrm{n}^+_e(G) = 1$.*
3. *If $\mathrm{n}^+_e(G) = -1$, then $\mathrm{z}^+_e(G) = -1$.*

The proof of the next proposition is the same as the proof of Theorem 2.21 in [13].

Proposition 5.18. *Let G be a graph and $e \in E(G)$. If $\mathrm{z}^+_e(G) = -1$, then for every optimal forcing tree cover of G, e is an edge in some forcing tree. Equivalently, if there is an optimal forcing tree cover of G such that e is not an edge in any tree, then $\mathrm{z}^+_e(G) \geq 0$.*

Question 5.19. *Is the converse of Proposition 5.18 true? That is, if G is a graph, e is an edge of G, and $\mathrm{z}^+_e(G) \geq 0$, must there exist an optimal forcing tree cover $\mathcal{T}$ of G such that e is not an edge of any tree in $\mathcal{T}$?*

Proposition 5.20. *Let G be a graph and $e = \{v, w\}$ be an edge of G. If $\mathrm{z}^+_e(G) = 1$, then there exists an optimal forcing tree cover $\mathcal{T}$, such that e is not an edge of any tree in $\mathcal{T}$.*

Proof. Let G be a graph and $e = \{v, w\}$ be an edge of G with $\mathrm{z}^+_e(G) = 1$. Since $\mathrm{z}^+_e(G) = 1$ we know that $Z_+(G) = Z_+(G - e) + 1$. Let B be a minimum positive semidefinite zero forcing set for $G - e$ such that $v \in B$. Note that B is not a positive semidefinite zero forcing set for G since $|B| < Z_+(G)$. Furthermore, $w \notin B$ because if it were, then adding the edge e back into our graph would not change what v and w could force, implying that B would force G. Now we let $B' = B \cup \{w\}$. Then B' forces G and $|B'| = Z_+(G)$, so B' is a minimum positive semidefinite zero forcing set for G and e is not in the forcing tree cover of any chronological forces of B'. □

The converse of Proposition 5.20 is false.

Example 5.21. For the edge e of the graph G shown in Figure 5.2, $\mathrm{Z}_+(G) = \mathrm{Z}_+(G - e) = 2$, so $\mathrm{z}^+_e(G) = 0$, but e is not in any tree in the forcing tree cover of the chronological list of forces shown in Figure 5.2.

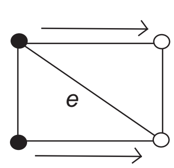


Fig. 5.2: A chronological list of forces in the graph G that does not contain edge e

5.3. Edge subdivision and edge contraction. The effect of edge contraction and edge subdivision on the (standard) zero forcing number was established in [21]. The *contraction* of edge

$e = \{u, v\}$ of G, denoted G/e, is obtained from G by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. In [21] it is shown that $\mathrm{Z}(G) - 1 \le \mathrm{Z}(G/e) \le \mathrm{Z}(G) + 1$. The first inequality remains true but the second does not.

Proposition 5.22. *Let G be a graph and $e = \{u, v\} \in E(G)$. Then $\mathrm{Z}_+(G) - 1 \le \mathrm{Z}_+(G/e)$.*

Proof. Let w be the vertex of G/e obtained by identifying u and v. Choose a minimum positive semidefinite zero forcing set B' of G/e that contains w (this is possible by Theorem 2.1). Then $B = B' \setminus \{w\} \cup \{u, v\}$ is a positive semidefinite zero forcing set for G, so $\mathrm{Z}_+(G) \le \mathrm{Z}_+(G/e) + 1$. □

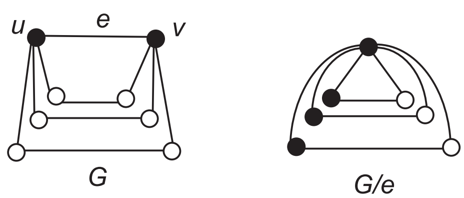


Fig. 5.3: A graph G with $\mathrm{Z}_+(G/e) = \mathrm{Z}_+(G) + 2$.

Example 5.23. Let G be the graph obtained from k copies of C_4 by identifying a common edge $e = \{u, v\}$ as shown shown on the left in Figure 5.3 for $k = 3$; G/e as shown on the right in Figure 5.3, and the black vertices are minimum positive semidefinite zero forcing sets for each of the graphs G and G/e. Then $\mathrm{Z}_+(G) = 2$ and $\mathrm{Z}_+(G/e) = k + 1$, so $\mathrm{Z}_+(G/e) = \mathrm{Z}_+(G) + (k - 1)$.

The *subdivision* of edge $e = \{u, v\}$ of G, denoted G_e, is the graph from G obtained by deleting e and inserting a new vertex w adjacent exactly to u and v. In the case of contraction, the result for positive semidefinite zero forcing was the same as for (standard) zero forcing. It was shown in [21] that $\mathrm{Z}(G) \le \mathrm{Z}(G_e) \le \mathrm{Z}(G) + 1$, and each of the inequalities can be equality, but the case of positive semidefinite zero forcing is simpler.

Theorem 5.24. *Let G be a graph and $e = \{u, v\} \in E(G)$. Then $\mathrm{Z}_+(G_e) = \mathrm{Z}_+(G)$ and any positive semidefinite zero forcing set for G is a positive semidefinite zero forcing set for G_e.*

Proof. In G_e, denote the vertex added to G in the subdivision by w. Let B be a positive semidefinite zero forcing set for G and $\mathcal{F}$ a chronological list of forces. Without loss of generality, either $u \to v$ or neither forces the other in $\mathcal{F}$. In G_e, color the vertices in B black. If $u \to v$ in $\mathcal{F}$, replace this by $u \to w \to v$ and otherwise perform the same forces as in $\mathcal{F}$. If neither u nor v forces the other in $\mathcal{F}$, then $u \to w$ after all the forces in $\mathcal{F}$ have been performed in G_e. In either case, if $u \to x \ne v$ when v is white, then x and v are in different components of $G - S$ (where S is the set of black vertices at this stage). Then x and w are in different components of $G_e - S$, and the forcing can continue as before. A similar argument holds for $v \to x \ne u$ when u is white. Thus B is a positive semidefinite zero forcing set for G_e. By choosing B so that $|B| = \mathrm{Z}_+(G)$, $\mathrm{Z}_+(G_e) \le \mathrm{Z}_+(G)$.

Now let B be a minimum positive semidefinite zero forcing set for G_e with $u \in B$. If $w \in B$, then set $B' = B \setminus \{w\} \cup \{v\}$; otherwise set $B' = B$. Then B' is a positive semidefinite zero forcing set for G. Since $|B'| = |B|$, $\mathrm{Z}_+(G) \le \mathrm{Z}_+(G_e)$. □

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