A variant
on the graph parameters of
Colin de Verdière:
Implications to the minimum
rank of graphs

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$S_n$ denotes the set of $n \times n$ real symmetric matrices. All matrices are real and symmetric.

All graphs are simple.

The graph of $n \times n$ matrix $A$:
$\mathcal{G}(A) = (V, E)$
$V = \{1, ..., n\}$,
$E = \{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$.

The family of symmetric matrices associated with a graph $G$ of order $n$:

$$S(G) = \{A \in S_n \mid \mathcal{G}(A) = G\}.$$

For matrix $A$, $S_A = S(\mathcal{G}(A))$. 
The minimum rank of graph $G$:

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank } A.$$ 

The maximum eigenvalue multiplicity:

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$ 

$$M(G) + \text{mr}(G) = |V(G)|.$$ 

**Problem** Determine the minimum rank (or maximum eigenvalue multiplicity) of a graph $G$.

- If $G$ is the disjoint union of graphs $G_i$ then

$$M(G) = \sum M(G_i)$$

$$\text{mr}(G) = \sum \text{mr}(G_i)$$
• If $G$ is connected, 
  $mr(G) = 0$ iff $G$ is single vertex.
  $mr(G) = 1$ iff $G = K_n$, $n \geq 2$.

• If $H$ is an induced subgraph of $G$ then 
  $mr(H) \leq mr(G)$.

• $M(G) = 1$ if and only if $G$ is a path. 
  [Fiedler 1969]

• For a connected graph $G$, $mr(G) \leq 2$ iff $G$ 
  does not contain as an induced subgraph 
  any of $P_4, K_{3,3,3}$ or 

  ![Graphs](attachment:attachment.png)

  [Barrett, van der Holst, and Loewy 2004]
For a tree $T$,

$$M(T) = P(T) = \Delta(T)$$

where

$P(T)$ is the path cover number

$$\Delta(T) = \max\{p - |Q| : T - Q \text{ is } p \text{ paths}\}.$$  

[Johnson and Leal-Duarte 1999]

There are good algorithms for computing $\Delta(T), P(T)$.

For a unicyclic graph $G$,

$$P(G) \geq M(G)$$

[Barioli, Fallat, Hogben 2004]

Method to compute minimum rank of a vertex sum from minimum rank of the summands

[Barioli, Fallat, Hogben 2004]
Colin de Verdière (1990) defined a new graph parameter $\mu$ that bounds $M$ from below and has nice properties.

**Definition** $X$ *fully annihilates* $A$ if
1. $AX = 0$;
2. $A \circ X = 0$;
3. $I_n \circ X = 0$.

**Definition** The matrix $A$ has the *Strong Arnold Property* (SAP) if the zero matrix is the only symmetric matrix that fully annihilates $A$.

$\text{corank } A = \text{nullity } A = \text{multiplicity of eigenvalue } 0$, so $M(G) = \max_{A \in S(G)} \text{corank } A$.

**Definition**

$$\mu(G) = \max \{\text{corank } L\} \quad \text{such that}$$

1. $L \in S(G)$, and is a generalized Laplacian matrix (off-diagonal entries nonpositive).
2. $L$ has exactly one negative eigenvalue (with multiplicity one).
3. $L$ has SAP.
Colin de Verdière showed

- $\mu(G) \leq 1$ iff $G$ is a disjoint union of paths.
- $\mu(G) \leq 2$ iff $G$ is outerplanar.
- $\mu(G) \leq 3$ iff $G$ is planar
- $\mu$ is minor monotone, i.e., if $H$ is obtained from $G$ by a sequence of edge deletions, isolated vertex deletions and edge contractions (in any order), then $\mu(H) \leq \mu(G)$. 
To study minimum rank, generalized Laplacians and number of negative eigenvalues are not relevant.

**Example**

$$K_{2,2,2}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -3 & -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & 2 & 1 \end{bmatrix} = 2.$$  

No generalized Laplacian of $K_{2,2,2}$ has rank 2.
But minor monotonicity is useful. Our new parameter:

$$\xi(G) = \max\{\text{corank } A : A \in S(G), A \text{ has SAP}\}.$$ 

- $$\mu(G) \leq \xi(G) \leq M(G)$$ for any $$G$$
- $$\xi(G) \geq 1$$ for any $$G$$ (because any corank 1 matrix has SAP)

**Example** $$K_{2,2,2}$$ continued.

$$\xi(K_{2,2,2}) = 4$$ because $$A$$ has SAP.

$$\mu(K_{2,2,2}) = 3$$ since $$K_{2,2,2}$$ is planar.

**Example** $$\xi(K_p) = M(K_p) = p - 1.$$
**Theorem** If $G$ is the disjoint union of graphs $G_i, i = 1, \ldots, h$ then

$$\xi(G) = \max_{i=1,\ldots,h} \xi(G_i)$$

**Corollary** $\xi(G) = 1$ iff $G$ is a disjoint union of paths.
Example Here is why you can’t sum the $\xi(G_i)$:

Let $x_1 = [-2, 1, 1]^T$, $x_2 = [-1, 1]^T$, so $A_i x_i = 0$

$\hat{x}_1 = [-2, 1, 1, 0, 0]^T$, $\hat{x}_2 = [0, 0, 0, -1, 1]^T$

$X = \hat{x}_1 \hat{x}_2^T + \hat{x}_2 \hat{x}_1^T =$

\[
\begin{bmatrix}
0 & 0 & 0 & 2 & -2 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 1 \\
2 & -1 & -1 & 0 & 0 \\
-2 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

fully annihilates $A$, so $A$ doesn’t not have SAP.
In our work on $\xi$ we have followed the treatment of $\mu$ given by van der Holst, Lovász, and Shrijver (1999).

- SAP comes from manifold theory.
- $\mathcal{R}_A = \{B : \text{rank } B = \text{rank } A\}$.
- $S_A = S(\mathcal{G}(A))$.
- $A$ has SAP iff manifolds $\mathcal{R}_A$ and $S_A$ intersect transversally at $A$.
- Transversal intersection allows perturbation.
**Theorem** If \( H \) is a subgraph of \( G \) then 
\[ \xi(H) \leq \xi(G). \]

Proof: \( H \) can be obtained from \( G \) by a series of deletions of edges and isolated vertices. Deletion of isolated vertices cannot increase \( \xi \) by disjoint union theorem.

Obtain \( G' \) from \( G \) by deleting edge \( uv \).

Choose \( A' \) \( \xi \)-optimal for \( G' \) so \( A' \) has SAP and \( R_{A'} \) and \( S_{A'} \) intersect transversally at \( A' \).

\( S(t) \) is the manifold of matrices obtained from matrices \( B \) in \( S_{A'} \) by replacing the \( u, v \)- and \( v, u \)-entries of \( B \) by \( t \).

\[ R(t) = R_{A'}. \]

For sufficiently small positive \( t_0 \), \( R(t_0) \) and \( S(t_0) \) intersect transversally at some \( A \). So \( A \) has SAP, \( G(A) = G \).

\[ \xi(G') = \text{corank } A' = \text{corank } A \leq \xi(G'). \]
**Corollary** If $G$ has $q$ independent vertices then $\xi(G) \leq |V(G)| - q + 1$.

Proof: Add edges between independent vertices to obtain $\tilde{G}$ having path $P_q$ as induced subgraph.

\[ q - 1 = \text{mr}(P_q) \leq \text{mr}(\tilde{G}). \]

\[ |V(G)| - (q - 1) \geq |V(\tilde{G})| - \text{mr}(\tilde{G}) = M(\tilde{G}) \geq \xi(\tilde{G}) \geq \xi(G). \]

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**Corollary** $\xi(K_{p,q}) = p + 1$ $(1 \leq p \leq q, 3 \leq q)$.

Proof: $p + 1 = \mu(K_{p,q}) \leq \xi(K_{p,q}) \leq p + q - (q - 1)$
Lemma Let $A = \begin{bmatrix} \alpha & b_1^T & b_2^T & \cdots & b_k^T \\ b_1 & A_1 & 0 & \cdots & 0 \\ b_2 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_k & 0 & 0 & \cdots & A_k \end{bmatrix}$,

$\text{corank } A_i \geq \text{corank } A_{i+1}$ for $i = 1, \ldots, k-1$. If $A$ has SAP, then

1. $\text{corank } A_2 \leq 1$.

2. $\text{corank } A_3 \leq 1$, and, if $\text{corank } A_3 = 1$, then $\text{corank } A_1 = \text{corank } A_2 = 1$.

3. $\text{corank } A_i = 0$ for $i \geq 4$.

Proof uses same ideas as disjoint union theorem proof, using nonzero vectors in $\ker \begin{bmatrix} b_i^T \\ A_i \end{bmatrix}$.
**Theorem** If $T$ is a tree that is not a path, then $\xi(T) = 2$.

Proof: Choose $A$ $\xi$-optimal for $T$.

$\text{corank } A \geq 2$.

By Parter-Wiener Theorem, there is a vertex $v$ such that $\text{corank } A(v) = \text{corank } A + 1$ and 0 is an eigenvalue of at least 3 components of $A(v)$.

So by Lemma, 0 must be a simple eigenvalue of exactly 3 components of $A(v)$.

$\xi(A) = \text{corank } A = \text{corank } A(v) - 1 = 2$. 
**Theorem** If $T$ is a tree and $\xi(T) < M(T)$, then we can add an edge to $T$ to obtain graph $G$ such that $M(G) < M(T)$.

**Example**

\[ M(T) = P(T) = 8 \quad M(G) \leq P(G) = 7 \]
**Theorem** Let $G$ be vertex-sum at $v$ of graphs $G_1, \ldots, G_k$. Then

$$
\max_{i=1}^{k} \xi(G_i) \leq \xi(G) \leq \max_{i=1}^{k} \xi(G_i) + 1.
$$

Note: For a vertex sum,

$$
\mu(G) = \max_{i=1}^{k} \mu(G_i).
$$

**Example**

![Graphs with vertices and edges]

$\xi(G_1) = 2$  $\xi(G_2) = 1$  $\xi(G) = 3$

because $\text{mr}(G_1) \geq 4, \text{mr}(G) \geq 4$ and there is a matrix with SAP having corank 3.
**Theorem** If $H$ is obtained from $G$ by contraction of an edge of $G$, then $\xi(H) \leq \xi(G)$.

Proof is basically same as that given for $\mu$ by van der Holst, Lovász, and Shrijver (1999).

**Corollary** $\xi$ is minor monotone.

**Future directions**

1. Finish extension of some properties of $\mu$ to $\xi$, such as maximum decrease in $\xi$ when deleting a vertex.

2. Exploit contraction monotonicity.