A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs

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S_n denotes the set of $n \times n$ real symmetric matrices. All matrices are real and symmetric.

All graphs are simple.

The *graph* of $n \times n$ matrix A:
$\mathcal{G}(A) = (V, E)$
$V = \{1, ..., n\}$,
$E = \{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$.

The family of symmetric matrices associated with a graph G of order n:

$$\mathcal{S}(G) = \{A \in S_n \mid \mathcal{G}(A) = G\}.$$

For matrix A, $\mathcal{S}_A = \mathcal{S}(\mathcal{G}(A))$.

The minimum rank of graph G:

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank}\, A.$$

The maximum eigenvalue multiplicity:

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$

$$M(G) + \text{mr}(G) = |V(G)|.$$

Problem Determine the minimum rank (or maximum eigenvalue multiplicity) of a graph G.

- If G is the disjoint union of graphs G_i then

$$M(G) = \sum M(G_i)$$

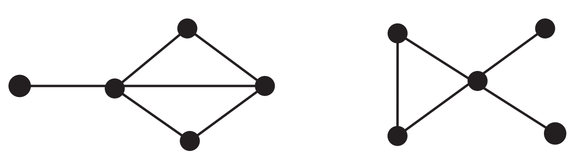
$$\text{mr}(G) = \sum \text{mr}(G_i)$$

- If G is connected,
  mr$(G) = 0$ iff G is single vertex.
  mr$(G) = 1$ iff $G = K_n,\ n \geq 2$.

- If H is an induced subgraph of G then mr$(H) \leq$ mr(G).

- $M(G) = 1$ if and only if G is a path.
  [Fiedler 1969]

- For a connected graph G, mr$(G) \leq 2$ iff G does not contain as an induced subgraph any of $P_4, K_{3,3,3}$ or



[Barrett, van der Holst, and Loewy 2004]

- For a tree T,

$$M(T) = P(T) = \Delta(T)$$

  where
  $P(T)$ is the path cover number

$$\Delta(T) = \max\{p - |Q| : T - Q \text{ is } p \text{ paths}\}.$$

  [Johnson and Leal-Duarte 1999]

  There are good algorithms for computing $\Delta(T), P(T)$.

- For a unicyclic graph G,
  $P(G) \geq M(G)$
  [Barioli, Fallat, Hogben 2004]

- Method to compute minimum rank of a vertex sum from minimum rank of the summands
  [Barioli, Fallat, Hogben 2004]

Colin de Verdière (1990) defined a new graph parameter μ that bounds M from below and has nice properties.

Definition X *fully annihilates* A if
1. $AX = 0$;
2. $A \circ X = 0$;
3. $I_n \circ X = 0$.

Definition The matrix A has the *Strong Arnold Property* (SAP) if the zero matrix is the only symmetric matrix that fully annihilates A.

$\operatorname{corank} A = \operatorname{nullity} A =$ multiplicity of eigenvalue 0, so $M(G) = \max\limits_{A \in \mathcal{S}(G)} \operatorname{corank} A$.

Definition

$$\mu(G) = \max\{\operatorname{corank} L\} \qquad \text{such that}$$

1. $L \in \mathcal{S}(G)$, and is a generalized Laplacian matrix (off-diagonal entries nonpositive).
2. L has exactly one negative eigenvalue (with multiplicity one).
3. L has SAP.

Colin de Verdière showed

- $\mu(G) \leq 1$ iff G is a disjoint union of paths.

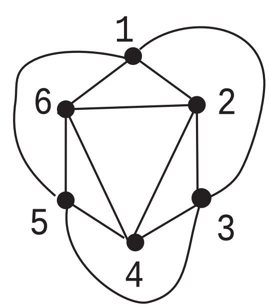
- $\mu(G) \leq 2$ iff G is outerplanar.

- $\mu(G) \leq 3$ iff G is planar

- μ is *minor* monotone, i.e., if H is obtained from G by a sequence of edge deletions, isolated vertex deletions and edge contractions (in any order), then $\mu(H) \leq \mu(G)$.

To study minimum rank, generalized Laplacians and number of negative eigenvalues are not relevant.

Example



$K_{2,2,2}$

$$\operatorname{rank} A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -3 & -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & 2 & 1 \end{bmatrix} = 2.$$

No generalized Laplacian of $K_{2,2,2}$ has rank 2.

But minor monotonicity is useful. Our new parameter:

$\xi(G) = \max\{\text{corank}\, A : A \in \mathcal{S}(G), A \text{ has SAP}\}$.

- $\mu(G) \leq \xi(G) \leq M(G)$ for any G

- $\xi(G) \geq 1$ for any G
  (because any corank 1 matrix has SAP)

Example $K_{2,2,2}$ continued.
$\xi(K_{2,2,2}) = 4$ because A has SAP.
$\mu(K_{2,2,2}) = 3$ since $K_{2,2,2}$ is planar.

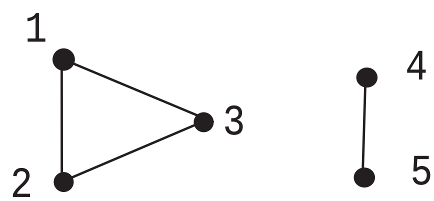
Example $\xi(K_p) = M(K_p) = p - 1$.

Theorem If G is the disjoint union of graphs $G_i, i = 1, \ldots, h$ then

$$\xi(G) = \max_{i=1,\ldots,h} \xi(G_i)$$

Corollary $\xi(G) = 1$ iff G is a disjoint union of paths.

Example Here is why you can't sum the $\xi(G_i)$:



$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = A_1 \oplus A_2.$$

Let $x_1 = [-2, 1, 1]^T$, $x_2 = [-1, 1]^T$, so $A_i x_i = 0$

$\hat{x}_1 = [-2, 1, 1, 0, 0]^T$, $\hat{x}_2 = [0, 0, 0, -1, 1]^T$

$$X = \hat{x}_1 \hat{x}_2^T + \hat{x}_2 \hat{x}_1^T = \begin{bmatrix} 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 2 & -1 & -1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \end{bmatrix}$$

fully annihilates A, so A doesn't not have SAP.

In our work on ξ we have followed the treatment of μ given by van der Holst, Lovász, and Shrijver (1999).

- SAP comes from manifold theory.

- $\mathcal{R}_A = \{B : \operatorname{rank} B = \operatorname{rank} A\}$.

- $\mathcal{S}_A = \mathcal{S}(\mathcal{G}(A))$.

- A has SAP iff manifolds $\mathcal{R}_A$ and $\mathcal{S}_A$ intersect transversally at A.

- Transversal intersection allows perturbation.

Theorem If H is a subgraph of G then $\xi(H) \leq \xi(G)$.

Proof: H can be obtained from G by a series of deletions of edges and isolated vertices. Deletion of isolated vertices cannot increase ξ by disjoint union theorem.

Obtain G' from G by deleting edge uv.

Choose A' ξ-optimal for G' so A' has SAP and $\mathcal{R}_{A'}$ and $\mathcal{S}_{A'}$ intersect transversally at A'.

$\mathcal{S}(t)$ is the manifold of matrices obtained from matrices B in $\mathcal{S}_{A'}$ by replacing the u,v- and v,u-entries of B by t.

$\mathcal{R}(t) = \mathcal{R}_{A'}$.

For sufficiently small positive t_0, $\mathcal{R}(t_0)$ and $\mathcal{S}(t_0)$ intersect transversally at some A. So A has SAP, $\mathcal{G}(A) = G$.

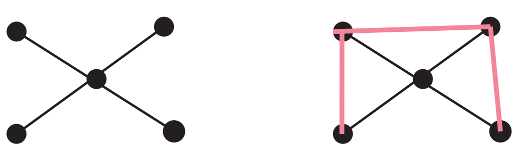
$\xi(G') = \operatorname{corank} A' = \operatorname{corank} A \leq \xi(G)$.

Corollary If G has q independent vertices then $\xi(G) \leq |V(G)| - q + 1$.

Proof: Add edges between independent vertices to obtain $\widetilde{G}$ having path P_q as induced subgraph.

$q - 1 = \text{mr}(P_q) \leq \text{mr}(\widetilde{G})$.

$$\begin{aligned}|V(G)| - (q-1) &\geq |V(\widetilde{G})| - \text{mr}(\widetilde{G}) = M(\widetilde{G}) \\ &\geq \xi(\widetilde{G}) \geq \xi(G).\end{aligned}$$



Corollary $\xi(K_{p,q}) = p + 1$ $(1 \leq p \leq q, 3 \leq q)$.

Proof: $p+1 = \mu(K_{p,q}) \leq \xi(K_{p,q}) \leq p+q-(q-1)$

Lemma Let $A = \begin{bmatrix} \alpha & \mathbf{b}_1^T & \mathbf{b}_2^T & \dots & \mathbf{b}_k^T \\ \mathbf{b}_1 & A_1 & 0 & \dots & 0 \\ \mathbf{b}_2 & 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_k & 0 & 0 & \dots & A_k \end{bmatrix}$, corank $A_i \geq$ corank A_{i+1} for $i = 1, \dots, k-1$. If A has SAP, then

1. corank $A_2 \leq 1$.

2. corank $A_3 \leq 1$, and, if corank $A_3 = 1$, then corank $A_1 =$ corank $A_2 = 1$.

3. corank $A_i = 0$ for $i \geq 4$.

Proof uses same ideas as disjoint union theorem proof, using nonzero vectors in ker $\begin{bmatrix} \mathbf{b}_i^T \\ A_i \end{bmatrix}$.

Theorem If T is a tree that is not a path, then $\xi(T) = 2$.

Proof: Choose A ξ-optimal for T.

$\operatorname{corank} A \geq 2$.

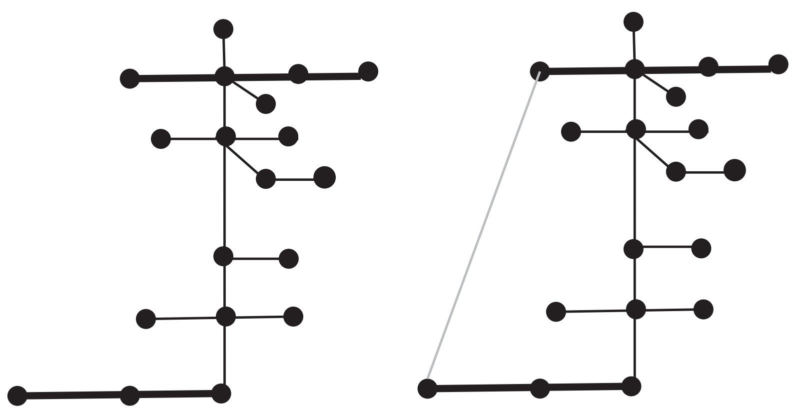
By Parter-Wiener Theorem, there is a vertex v such that $\operatorname{corank} A(v) = \operatorname{corank} A + 1$ and 0 is an eigenvalue of at least 3 components of $A(v)$.

So by Lemma, 0 must be a simple eigenvalue of exactly 3 components of $A(v)$.

$\xi(A) = \operatorname{corank} A = \operatorname{corank} A(v) - 1 = 2$.

Theorem If T is a tree and $\xi(T) < M(T)$, then we can add an edge to T to obtain graph G such that $M(G) < M(T)$.

Example



$M(T) = P(T) = 8$ $\qquad$ $M(G) \leq P(G) = 7$

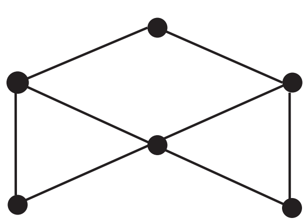
Theorem Let G be vertex-sum at v of graphs $G_1, \ldots, G_k$. Then

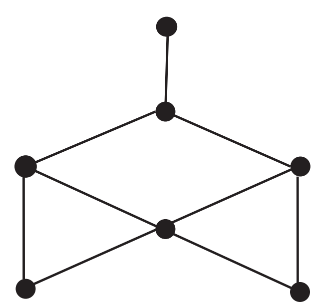
$$\max_{i=1}^{k} \xi(G_i) \leq \xi(G) \leq \max_{i=1}^{k} \xi(G_i) + 1.$$

Note: For a vertex sum,

$$\mu(G) = \max_{i=1}^{k} \mu(G_i).$$

Example





$\xi(G_1) = 2$ $\quad$ $\xi(G_2) = 1$ $\quad$ $\xi(G) = 3$

because $\text{mr}(G_1) \geq 4, \text{mr}(G) \geq 4$ and there is a matrix with SAP having corank 3.

Theorem If H is obtained from G by contraction of an edge of G, then $\xi(H) \leq \xi(G)$.

Proof is basically same as that given for μ by van der Holst, Lovász, and Shrijver (1999).

Corollary ξ is minor monotone.

Future directions

1. Finish extension of some properties of μ to ξ, such as maximum decrease in ξ when deleting a vertex.

2. Exploit contraction monotonicity.