Matrix Completion Problems for Pairs of Related Classes of Matrices

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Abstract For a class X of real matrices, a list of positions in an $n \times n$ matrix (a pattern) is said to have X-completion if every partial X-matrix that specifies exactly these positions can be completed to an X-matrix.
If X and X_0 are classes that satisfy the conditions

(1) any partial X-matrix is a partial X_0-matrix,

(2) for any X_0-matrix A and $\varepsilon > 0$, $A + \varepsilon I$ is a X-matrix, and

(3) for any partial X-matrix A, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is a partial X-matrix (where $\tilde{I}$ is the partial identity matrix specifying the same pattern as A)

then any pattern that has X_0-completion must also have X-completion.
However, there are usually patterns that have X-completion that fail to have X_0-completion.

This result applies to many pairs of subclasses of P- and P_0-matrices defined by the same restriction on entries, including the classes P/P_0-matrices, (weakly) sign-symmetric P/P_0-matrices, and non-negative P/P_0-matrices. It also applies to other related pairs of subclasses of P_0-matrices, such as the pairs classes of P-/$P_{0,1}$-matrices, (weakly) sign-symmetric $P/P_{0,1}$-matrices and non-negative $P/P_{0,1}$-matrices.

Furthermore, any pattern that has (weakly sign-symmetric, sign-symmetric, non-negative) P_0-completion must also have (weakly sign-symmetric, sign-symmetric, non-negative) $P_{0,1}$-completion, although these pairs of classes do not satisfy condition (3).

Similarly, the class of inverse M-matrices and its topological closure do not satisfy condition (3), but the conclusion remains true, and the matrix completion problem for the topological closure of the class of inverse M-matrices is solved for patterns containing the diagonal.

1. Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not (both no entries specified and all entries specified are also allowed). A *completion* of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1,...,n\} \times \{1,...,n\}$. A partial matrix *specifies the pattern* if its specified entries are exactly those listed in the pattern. Note that in this paper a pattern does not need to include all diagonal positions. For a class X of real matrices, we say a pattern *has X-completion* if every partial X-matrix specifying the pattern can be completed to an X-matrix. The matrix completion problem (for patterns) for the class of X-matrices is to determine which patterns have X-completion.

Applications of matrix completion problems arise in situations where some data are known but other data are not available, and it is known that the full data matrix must have a certain property. Examples include geophysical problems such as seismic reconstruction problems and electrical and computer engineering problems including data transmission, coding, and image enhancement. Matrix completion problems also arise in optimization and the study of Euclidean distance matrices.

Matrix completion problems have been studied for many classes of matrices, including positive definite matrices [GJSW], P-matrices [JK], [DH], P_0-matrices [CDHMW], M-matrices [Ho2], M_0-matrices [Ho4], inverse M-matrices [JS1], [Ho1], [Ho3] and many other subclasses of P- and P_0-matrices [FJTU], [Ho4]. A variety of techniques have been developed that apply to matrix completion problems for many classes. In this paper, we examine the specific relationship between the solutions to the matrix completion problems for certain pairs of classes.

The answer to the X-matrix completion problem obviously depends on the definition of a partial X-matrix. For many classes X of matrices, in order for it to be possible to have a completion of a partial matrix to an X-matrix, certain obviously necessary conditions must be satisfied. Such obviously necessary conditions are frequently taken as the definition of a partial X-matrix, as we do here.

For α a subset of $\{1,...,n\}$, the *principal submatrix* $A(\alpha)$ is obtained from the $n \times n$ matrix A by deleting all rows and columns not in α. For all of the classes X of matrices discussed in this paper, membership in the class is inherited by principal submatrices. Thus in order for a partial X-matrix to have a completion to an X-matrix, it is certainly necessary that every fully specified principal submatrix be an X-matrix. For some classes this is sufficient to define a partial X-matrix. Other classes have entry sign patterns (e.g., all entries are non-negative), so any specified entries must also satisfy the sign pattern. Explicit definitions of a partial matrix for the classes discussed are given in Table 1 and Section 3.

A *principal minor* of A is the determinant of a principal submatrix of A. The matrix $A \in \mathbf{R}^{nxn}$ is a P- (respectively, P_0-, $P_{0,1}$-) matrix if every principal minor is positive (non-negative, non-negative and all diagonal elements of A are positive). We examine the relationship between the solutions to the matrix completion problems for pairs of related subclasses of P_0-matrices. If X and Y are classes of matrices with $X \subseteq Y$, in general it is not possible to conclude either that a pattern that has Y completion must have X completion (because the completion to a Y-matrix may not be an X-matrix) or that a pattern that has X completion must have Y completion (because there may be a partial Y-matrix that is not a partial X-matrix). However, in cases where there is a natural relationship between the classes X and Y, it is sometimes possible to conclude that any pattern that has Y-completion has X-completion. The pair of related classes may be defined by the same entry restriction (see Table 1 below) on the classes of P- and P_0-matrices, on the classes P- and $P_{0,1}$–matrices, or on the classes $P_{0,1}$- and P_0-matrices. These pairs of classes are studied in Section 2.

Alternatively, the pair may be a class and its topological closure. For a class X of matrices, the matrix A is in the *topological closure* of X if A is the limit of a sequence A_n of matrices in X. The determinant is a continuous function of the entries, so any matrix in the topological closure of the class of P-matrices is a P_0-matrix. If A is a P_0-matrix and $\varepsilon > 0$, then $A + \varepsilon I$ is a P-matrix [HJ2], so any P_0-matrix is the limit of P-matrices. Thus the class of P_0-matrices is the topological closure of the class of P-matrices. The matrix completion problem for the topological closure of the inverse M-matrices is solved in Section 3 for patterns that include all diagonal positions.

In all these cases it is established that if a pattern has completion for the larger class in the pair then it has completion for smaller class in the pair, and that there is a pattern that has completion for the smaller class in the pair that does not have completion for the larger class in the pair.

2. Pairs of Π/Π_0-classes

2.1 Definition The classes of matrices X and X_0 are referred to as a *pair of Π/Π_0-classes* if

1. Any partial X-matrix is a partial X_0-matrix.
2. For any X_0-matrix A and $\varepsilon > 0$, $A + \varepsilon I$ is a X-matrix.

3. For any partial X-matrix A, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is a partial X-matrix (where $\tilde{I}$ is the partial identity matrix specifying the same pattern as A).

For any partial P-matrix A, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is a partial P-matrix (where $\tilde{I}$ is the partial identity matrix specifying the same pattern as A), because the determinant is a continuous function of the entries of the matrix. Hence the classes P- matrices and P_0-matrices are a pair of Π/Π_0-classes.

Table 1 provides definitions of various pairs of subclasses of P- and P_0-matrices and partial matrices for these classes (cf. [Ho4]). The pairs X/X_0 of classes listed in Table 1 are "natural" in the sense that the class of X-matrices is a subclass of P-matrices and the class of X_0-matrices is the analogous subclass of P_0-matrices. These pairs are all pairs of Π/Π_0-classes (statement 1 is obvious and statements 2 and 3 follow from statements 2 and 3 for P- and P_0- matrices).

Table 1: Pairs of Π / Π_0 -Classes

Class X/X_0	Definition of a X/X_0-matrix A: A is a P/P_0 - matrix and	Definition of a partial X/X_0-matrix A: Every fully specified principal submatrix of A is a X/X_0-matrix and whenever the listed entries are specified,
P/P_0-matrices		
weakly sign-symmetric P/P_0-matrices	$a_{ij} a_{ji} \geq 0$ for each i,j	$a_{ij} a_{ji} \geq 0$
sign-symmetric P/P_0-matrices	$a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each i,j	$a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$
non-negative P/P_0-matrices	$a_{ij} \geq 0$ for all i,j	$a_{ij} \geq 0$
M/M_0-matrices	$a_{ij} \leq 0$ for all $i \neq j$	$a_{ij} \leq 0$ if $i \neq j$
symmetric M/M_0-matrices	symmetric and $a_{ij} \leq 0$ for all $i \neq j$	$a_{ji} = a_{ij}$ and $a_{ij} \leq 0$ if $i \neq j$
positive definite/semidefinite	symmetric	$a_{ji} = a_{ij}$

2.2 Theorem For a pair of Π/Π_0-classes, if a pattern has Π_0-completion then it must also have Π-completion.

Proof: Let Q be a pattern that has Π_0-completion, and let A be a partial Π-matrix specifying Q. Let $\tilde{I}$ be the partial identity matrix specifying the pattern Q. There is a $\delta > 0$ such that $B = A - \delta \tilde{I}$ is a partial Π-matrix, and hence a partial Π_0-matrix. Since Q has Π_0-completion, B can be completed to a Π_0-matrix $\hat{B}$. Then $\hat{A} = \hat{B} + \delta I$ is a Π-matrix that completes A. Thus Q has Π-completion. ■

2.3 Corollary

- Any pattern that has P_0-completion has P-completion.
- Any pattern that has weakly sign-symmetric P_0-completion has weakly sign-symmetric P-completion.
- Any pattern that has sign-symmetric P_0-completion has sign-symmetric P-completion.
- Any pattern that has non-negative P_0-completion has non-negative P-completion.
- Any pattern that has M_0-completion has M-completion.
- Any pattern that has symmetric M_0-completion has symmetric M-completion.
- Any pattern that has positive semidefinite-completion has positive definite-completion.

For Π/Π_0 the classes of P/P_0-matrices, weakly sign-symmetric P/P_0-matrices, and non-negative P/P_0-matrices, Corollary 2.3 provides new information. For Π/Π_0 the class of positive definite/semidefinite matrices and (symmetric) M/M_0-matrices, Corollary 2.3 does not provide any new information, because these completion problems have already been solved [GJSW], [JS2], [Ho2], [Ho4], [Ho5]. For Π/Π_0 the classes of sign-symmetric P/P_0-matrices, Corollary 2.3 again does not provide any new information, because the sign-symmetric P_0-completion problem has been solved and every pattern that has sign-symmetric P_0-completion is already known to have sign-symmetric P-completion [FJTU], [Ho4].

It is interesting to note that for all the classes in Corollary 2.3, the conclusion is false if the roles of Π and Π_0 are reversed (assuming patterns are allowed to omit some diagonal positions), as the following examples show.

2.4 Example Let $Q_1 = \{(1,1), (1,2), (2,1)\}$. The pattern Q_1 has (non-negative, weakly sign-symmetric, sign-symmetric) P-completion, (symmetric) M-completion and positive definite completion [GJSW], [JK], [Ho2], [Ho4]. The partial matrices $A_{1+} = \begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$ and $A_{1-} = \begin{bmatrix} 0 & -1 \\ -1 & ? \end{bmatrix}$ specify Q_1. A_{1+} is a partial (weakly sign-symmetric, sign-symmetric, non-negative) P_0-matrix and a partial positive semi-definite matrix that cannot be completed to a P_0-matrix (and hence cannot be completed to any of the subclasses) because det $\hat{A}_{1+}$ = -1 for any completion $\hat{A}_{1+}$ of A_{1+}. A_{1-} is a partial (symmetric) M_0-matrix that cannot be completed to a M_0-matrix.

Note that pattern Q_1 omits a diagonal position. For Π/Π_0 the pair of classes positive definite/semidefinite, M-/M_0-, or symmetric M-/M_0- matrices, the only patterns that have Π-completion that do not have Π_0-completion are patterns that omit diagonal positions (such as Q_1) [GJSW], [FJTU], [Ho2], [Ho4], [Ho5]. However, Examples 2.5, 2.6 and 2.7 and Lemma 2.8 below show that this is not the case for pairs P-/P_0-, sign-symmetric P-/P_0-, weakly sign-symmetric P-/P_0-, and non-negative P-/P_0-matrices.

2.5 Example Let $Q_2 = \{(1,1), (2,1), (2,2)\}$. The pattern Q_2 has sign-symmetric P-completion [Ho4, Theorem 3.3]. The partial sign-symmetric P_0-matrix $A_2 = \begin{bmatrix} 0 & ? \\ 1 & 1 \end{bmatrix}$ specifies Q_2 and cannot be completed to a sign-symmetric P_0-matrix, because for any sign-symmetric completion $\hat{A}_2$ of A_2, det $\hat{A}_2 < 0$.

2.6 Example The pattern $Q_3 = \{(1,1), (1,2), (2,1)\ (2,2), (2,3), (3,1), (3,2)\ (3,3)\}$ has P-completion [JK]. The partial P_0-matrix $A_3 = \begin{bmatrix} 0 & -1 & ? \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ specifies Q_3 and cannot be completed to a P_0-matrix because det $\hat{A}_3$ = -1 for any completion $\hat{A}_3$ of A_3.

2.7 Example Let $Q_4 = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,1), (4,4)\}$. The pattern Q_4 has non-negative P-completion [Ho4, Corollary 8.5]. The partial non-negative and weakly sign symmetric

P_0-matrix $A_4 = \begin{bmatrix} 0 & 1 & ? & ? \\ ? & 0 & 1 & ? \\ ? & ? & 0 & 1 \\ 1 & ? & ? & 0 \end{bmatrix}$ specifies Q_4 and cannot be completed to a non-negative P_0-matrix nor to a weakly sign symmetric P_0-matrix [Ho4, Example 9.8].

2.8 Lemma The pattern Q_4 has weakly sign symmetric P- and weakly sign symmetric $P_{0,1}$-completion.

Proof: Because the class of weakly sign symmetric P- ($P_{0,1}$-) matrices is closed under multiplication by positive diagonal matrices, we may assume that all diagonal entries in a partial weakly sign symmetric P- ($P_{0,1}$-) matrix are 1. The partial weakly sign symmetric $P_{0,1}$-matrix $A_4 = \begin{bmatrix} 1 & a_{12} & ? & ? \\ ? & 1 & a_{23} & ? \\ ? & ? & 1 & a_{34} \\ a_{41} & ? & ? & 1 \end{bmatrix}$ specifies Q_4. If any of a_{12}, a_{23}, a_{34}, a_{41} is 0, then A can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing all unspecified entries to be 0. If all of a_{12}, a_{23}, a_{34}, a_{41} are nonzero, then without loss of generality (by use of a diagonal similarity) we may assume that $a_{12} = a_{23} = a_{34} =1$. If $a_{41} < 0$ then A can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing all unspecified entries to be 0. If $a_{41} > 0$ then A can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing the 1,3 and 2,4 entries to be 1 and all other unspecified entries to be 0. ■

It is also possible to apply Theorem 2.2 to other pairs of classes. In particular, it applies to the pair of classes P-matrices and $P_{0,1}$-matrices, as well as to pairs of analogously defined subclasses of these classes. To see that these are Π/Π_0-pairs, note that property (1) is clear, (2) follows from property (2) for P_0-matrices, and (3) follows property (3) of P-matrices.

2.9 Corollary

- Any pattern that has $P_{0,1}$-completion also has P-completion.
- Any pattern that has weakly sign-symmetric $P_{0,1}$-completion also has weakly sign-symmetric P-completion.
- Any pattern that has sign-symmetric $P_{0,1}$-completion also has sign-symmetric P-completion.
- Any pattern that has non-negative $P_{0,1}$-completion also has non-negative P-completion.

Again, for all the classes in Corollary 2.9, the statement of Corollary 2.9 is false if the roles of Π and Π_0 are reversed (assuming patterns are allowed to omit some diagonal positions), as the following example shows.

2.10 Example Let $Q_5 = \{(1,1), (1,2), (1,3), (2,1) (2,2), (2,3), (3,1), (3,2)\}$. The pattern Q_5 has (weakly sign-symmetric, sign-symmetric, non-negative) P-completion [JK], [Ho4, Theorem 4.6].

The partial (weakly sign-symmetric, sign-symmetric, non-negative) $P_{0,1}$-matrix $A_5 = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & ? \end{bmatrix}$ specifies Q_5 and cannot be completed to a $P_{0,1}$-matrix (and hence cannot be completed to any of the subclasses) because det $\hat{A}_5 = -1$ for any completion $\hat{A}_5$ of A_5.

The remark about omitting diagonal positions is not necessary for the pair of classes sign-symmetric P-/$P_{0,1}$-matrices, as the following example shows.

2.11 Example Let Q_6 = {(1,1), (1,2), (2,1) (2,2), (3,1), (3,2), (3,3)}. The pattern Q_6 has sign-symmetric P-completion [Ho4, Lemma 3.3]. The partial sign-symmetric $P_{0,1}$-matrix $A_6 = \begin{bmatrix} 4 & 2 & ? \\ 2 & 1 & ? \\ 4 & -1 & 1 \end{bmatrix}$ specifies Q_6 and cannot be completed to a sign -symmetric $P_{0,1}$-matrix [Ho4, Example 3.4].

Although Theorem 2.2 does not apply to the pair of classes $P_{0,1}$-matrices and P_0-matrices (because condition (3) of the definition of pair of Π/Π_0-classes is not true, as the $P_{0,1}$-matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ shows), the conclusion remains true:

2.12 Theorem

- Any pattern that has P_0-completion also has $P_{0,1}$-completion.
- Any pattern that has weakly sign-symmetric P_0-completion also has weakly sign-symmetric $P_{0,1}$-completion.
- Any pattern that has sign-symmetric P_0-completion also has sign-symmetric $P_{0,1}$-completion.
- Any pattern that has non-negative P_0-completion also has non-negative $P_{0,1}$-completion.

Proof: Let Q be a pattern that has P_0-completion, and let A be a partial $P_{0,1}$-matrix specifying Q. Clearly A is a partial P_0-matrix. Since Q has P_0-completion, A can be completed to a P_0-matrix $\hat{A}$. If $\hat{A}$ is not a $P_{0,1}$-matrix, then one or more diagonal entries are 0 (this is the only distinction between a $P_{0,1}$-matrix and a P_0-matrix). Since A was a partial $P_{0,1}$-matrix, any diagonal entry specified in A was positive. Let $D = [d_{ij}]$ be defined by $d_{ii} = 0$ if $(i,i) \in Q$, $d_{ii} = 1$ if $(i,i) \notin Q$, and $d_{ij} = 0$ if $i \neq j$. Then $\hat{A}+D$ completes A to a $P_{0,1}$-matrix, because it has positive diagonal and, as the sum of a P_0-matrix and a non-negative diagonal matrix, $\hat{A}+D$ is a P_0-matrix. Thus Q has $P_{0,1}$-completion. The same argument works for the weakly sign-symmetric, sign-symmetric and non-negative pairs of subclasses. ■

Again the statements in Theorem 2.12 are false if the roles of $P_{0,1}$ and P_0 are reversed. The Venn diagrams shown in Figure 1 summarize the results established in Corollary 2.9, Theorem 2.12, and the examples. All regions shown in the diagrams are non-empty. The pattern Q_2 has sign-symmetric $P_{0,1}$-completion and does not have sign-symmetric P_0-completion ([Ho4, Lemma 4.8] and Example 2.5); Q_3 has $P_{0,1}$-completion and does not have P_0-completion ([Ho4, Lemma 8.1] and Example 2.6); and Q_4 has non-negative $P_{0,1}$-completion [Ho4, Corollary 8.5] and weakly sign-symmetric $P_{0,1}$-completion (Lemma 2.8) and does not have either non-negative P_0-completion or weakly sign-symmetric P_0-completion [Ho4, Example 9.8].

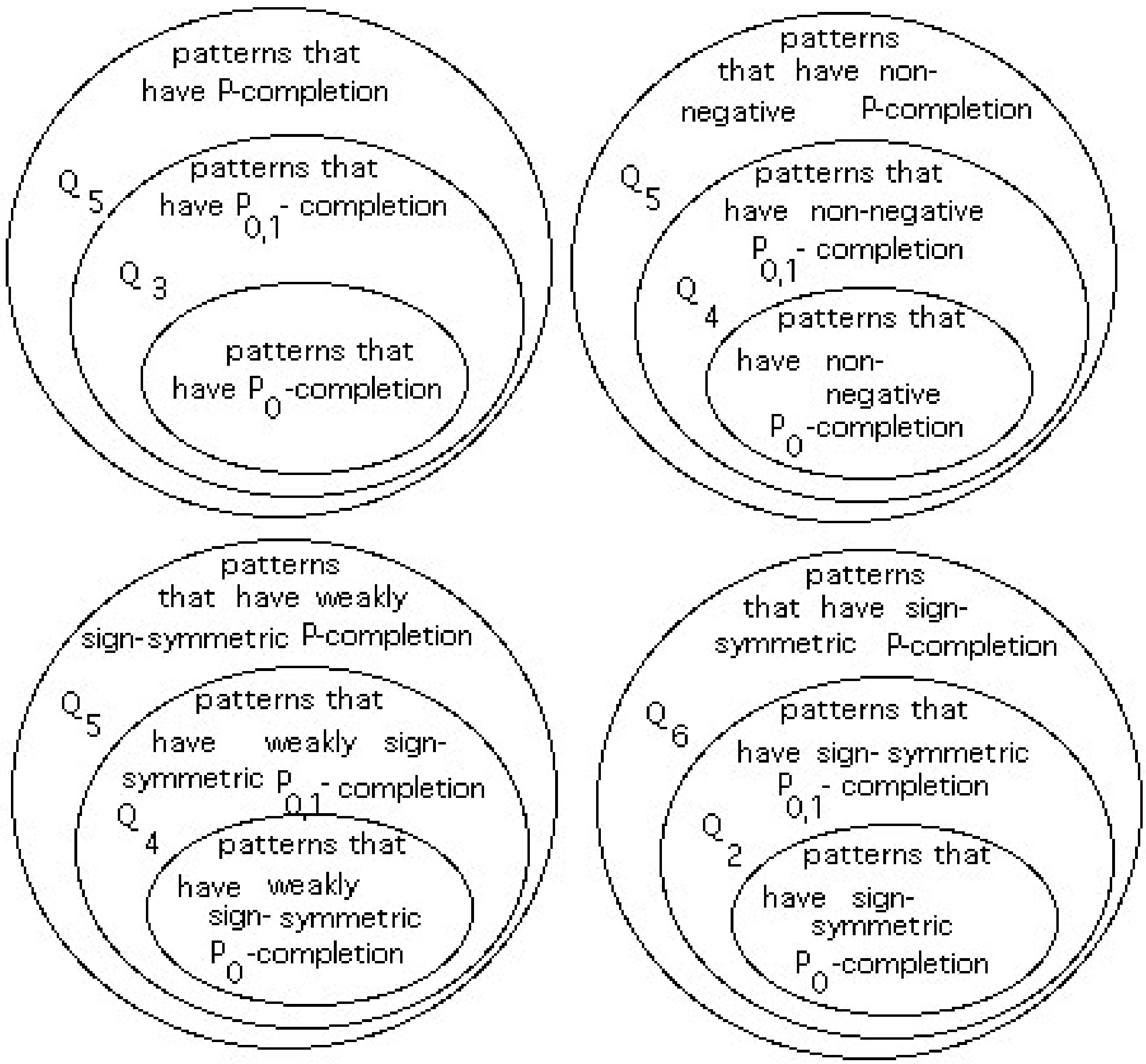

Figure 1: Relationships among the sets of patterns having completion for various classes of matrices.

All the results presented so far inferring X-completion of a pattern from Y-completion of a pattern involved pairs of classes $X \subseteq Y$. For certain special patterns it is also possible to infer completion of a pattern for the larger class from completion for the smaller class.

A pattern Q is *asymmetric* if $(i,j) \in Q$ implies $(j,i) \notin Q$.

2.13 Theorem

- Any asymmetric pattern that has P-completion also has $P_{0,1}$-completion.
- Any asymmetric pattern that has weakly sign-symmetric P-completion also has weakly sign-symmetric $P_{0,1}$-completion.
- Any asymmetric pattern that has sign-symmetric P-completion also has sign-symmetric $P_{0,1}$-completion.
- Any asymmetric pattern that has non-negative P-completion also has non-negative $P_{0,1}$-completion.

Proof: Let Q be an asymmetric pattern that has P-completion, and let A be a partial $P_{0,1}$-matrix specifying Q. Since the pattern is asymmetric, there are no fully specified principal submatrices of size large than 1, and the size 1 matrices are P-matrices. Thus A is a partial P-matrix, and can be completed to a P-matrix $\hat{A}$. Clearly $\hat{A}$ is a $P_{0,1}$-matrix that completes A. The same argument works for the weakly sign-symmetric, sign-symmetric and non-negative pairs of subclasses. ■

Since it is known [CDHMW] that every asymmetric pattern has both P-completion and P_0-completion, either Theorem 2.12 or Theorem 2.13 may be used to conclude that every asymmetric pattern has $P_{0,1}$-completion.

3. The Topological Closure of the Class of Inverse M-matrices

In this section patterns are assumed to contain all diagonal positions.

A matrix is defined to be an *inverse M-matrix* if it is nonsingular and its inverse is an M-matrix, i.e., a P-matrix with non-positive off-diagonal entries. Any inverse M-matrix is non-negative [HJ2]. Every principal submatrix of an inverse M-matrix is an inverse M-matrix [J1]. A partial matrix is a *partial inverse M-matrix* if every fully specified principal submatrix is an inverse M-matrix and all specified entries are non-negative. In [JS2], a matrix is defined to be a *singular inverse M-matrix* if it is singular and in the topological closure of the class of inverse M-matrices. We say the matrix A is *TCIM-matrix* if it is in the topological closure of the class of inverse M-matrices, that is, if A is the limit of a sequence of inverse M-matrices. Since the entries of A^{-1} are continuous functions of the entries of A (as long as A remains nonsingular) and a nonsingular limit of M-matrices is an M-matrix, any nonsingular TCIM-matrix is an inverse M-matrix. Any TCIM-matrix is non-negative. A matrix is a *partial TCIM-matrix* if every fully specified principal submatrix is a TCIM-matrix and all specified entries are non-negative.

We examine the relationship between the inverse M-matrix completion problem and the TCIM-matrix completion problem. Because the class of P_0-matrices is the topological closure of the class of P-matrices, one might hope to apply Theorem 2.2 to the pair of classes inverse M-/TCIM-matrices; however, this pair of classes fails to satisfy the third required property, as the next example shows:

3.1 Example Let $B = \begin{bmatrix} 8 & 3 & 1 \\ 4 & 6 & 2 \\ 4 & 3 & 5 \end{bmatrix}$. Then

$$(B - \delta I)^{-1} = \frac{1}{\det(B - \delta I)} \begin{bmatrix} 24 - 11\delta + \delta^2 & -12 + 3\delta & \delta \\ -12 + 4\delta & 36 - 13\delta + \delta^2 & -12 + 2\delta \\ -12 + 4\delta & -12 + 3\delta & 36 - 14\delta + \delta^2 \end{bmatrix}.$$ By choosing $\delta = 0$, it is clear that B is an inverse M-matrix. However, 1,3-entry of $(B - \delta I)^{-1}$ will be positive for $\delta > 0$, so there does not exist $\delta > 0$ such that $B - \delta I$ is an inverse M-matrix.

However, the other two properties in Theorem 2.2 remain true (the first is obvious):

3.2 Lemma [J2] For any TCIM-matrix A and $\varepsilon > 0$, $A + \varepsilon I$ is an inverse M-matrix.
Proof: Suppose A is a TCIM-matrix. Since A is in the closure of the class of inverse M-matrices, there is a sequence A_n of inverse M-matrices with $\lim_{n \to \infty} A_n = A$. Then for $\varepsilon > 0$, $\lim_{n \to \infty} (A_n + \varepsilon I) = A + \varepsilon I$. Since A_n is an inverse M-matrix, so is $A_n + \varepsilon I$, because the sum of an inverse M-matrix and a non-negative diagonal matrix is an inverse M-matrix [J1]. So $A + \varepsilon I$ is in the closure of the inverse M-matrices. But A is a P_0-matrix, so $A + \varepsilon I$ is a P-matrix, and

hence is nonsingular. Therefore $A + \varepsilon I$ is an inverse M-matrix. ■

In addition to the property that a principal submatrix of an inverse-M matrix is an inverse M-matrix, the class of inverse M-matrices (like the classes discussed in section 2) possesses two other properties that are important in the study of matrix completions: it is closed under permutation similarity and direct sums. Hence the class of TCIM-matrices is closed under permutation similarity and direct sums, i.e., if A and B are TCIM matrices and P is a permutation matrix of the same size as A, then $P^{-1}AP$ and $A \oplus B$ are TCIM-matrices. (Thus the classes inverse M-matrices and TCIM-matrices are HSP classes as defined in [Ho5]).

In recent years graphs and digraphs have been used very effectively to study matrix completion problems. Here we shall use digraphs to study the matrix completion problem for TCIM-matrices.

A *digraph* $G = (V_G,E_G)$ is a finite set of positive integers V_G, whose members are called *vertices*, and a set E_G of ordered pairs (v,u) of distinct vertices, called *arcs*. The *order* of a digraph is the number of vertices.

The digraph $G = (V_G,E_G)$ is *isomorphic* to the digraph $H = (V_H,E_H)$ by isomorphism ϕ if ϕ is a one-to-one map from V_G onto V_H and $(v,w) \in E_G$ if and only if $(\phi(v), \phi(w)) \in E_H$.

A *subdigraph* of the digraph $G = (V_G,E_G)$ is a digraph $H = (V_H,E_H)$, where $V_H \subseteq V_G$ and $E_H \subseteq E_G$ (note that $(v,u) \in E_H$ requires $v,u \in V_H$ since H is a digraph). If $W \subseteq V_G$, the *subdigraph of G induced by W, <W>*, is the digraph (W,E_W) with $E_W = E_G \cap (W \times W)$. A subdigraph induced by a subset of vertices is also called an *induced subdigraph*.

A *path* (respectively, *semipath*) in a digraph G=(V,E) is sequence of vertices $v_1, v_2, ..., v_{k-1},v_k$ in V such that for i=1,…,k-1 the arc $(v_i,v_{i+1}) \in E$ (respectively, $(v_i,v_{i+1}) \in E$ or $(v_{i+1},v_i) \in E$) and all vertices are distinct except possibly $v_1 = v_k$. Clearly, a path is a semipath, although the converse is false. A (semi)path is *open* if the first and last vertices are distinct. The length of the (semi)path $v_1, v_2, ..., v_{k-1},v_k$ is k-1.

A digraph is *connected* if there is a semipath from any vertex to any other vertex (a digraph of order 1 is connected); otherwise it is *disconnected*. A *component* of a digraph is a maximal connected subdigraph. A digraph is *strongly connected* if there is a path from any vertex to any other vertex. Clearly, a strongly connected digraph is connected, although the converse is false.

A digraph G=(V,E) is a *source/sink cut bipartite digraph* if V can be partitioned into disjoint sets S (*sources*) and T (*sinks*), such that for any $(u,v) \in E$, $u \in S$ and $v \in T$.

Let A be a (fully specified) $n \times n$ matrix. The *nonzero-digraph* of A is the digraph having vertex set {1,...,n} and, as arcs, the ordered pairs (i,j) where $i \neq j$ and $a_{ij} \neq 0$. The *characteristic matrix* of a pattern Q for $n \times n$ matrices is the $n \times n$ matrix C such that $c_{ij} = 1$ if the position $(i,j) \in Q$ and $c_{ij} = 0$ if $(i,j) \notin Q$. For a pattern Q that contains all diagonal positions, the *digraph of Q* is the nonzero-digraph of the characteristic matrix of Q.

A digraph G = (V,E) with V = {1,…,n} is called a *pattern-digraph*. Clearly such a digraph is the digraph of the pattern for $n \times n$ matrices $Q_G = E \cup \{(v,v): 1 \leq v \leq n\}$. A partial matrix that specifies Q is also referred to as specifying the digraph of Q and a pattern-digraph G is referred to as having TCIM-completion if Q_G does.

A pattern Q is *permutation similar* to a pattern R if there is a permutation π of {1,…,n} such that $R = \{((\pi(i), \pi(j)): (i,j) \in Q\}$. Equivalently, C_Q is permutation similar to C_R. Relabeling the vertices of a digraph diagram, which performs a digraph isomorphism, corresponds to performing a permutation similarity on the pattern. Since the class of TCIM-matrices is closed under permutation similarity, we are free to relabel digraphs as desired.

Note that when using digraphs, patterns are assumed to include all diagonal positions.

3.3 Lemma Let Q be a pattern that has TCIM-completion, and let G be the digraph of Q. Then every pattern-digraph isomorphic to an induced subdigraph of G has TCIM-completion.

Proof: Suppose the pattern-digraph K is isomorphic to the induced subdigraph H of G by isomorphism ϕ. Let A be a partial TCIM-matrix specifying K. Define a partial matrix B specifying Q by defining b_{ij} for $(i,j) \in Q$ as follows: If $i, j \in V_H = \phi(V_K)$, $b_{ij} = a_{\phi^{-1}(i)\,\phi^{-1}(j)}$. If $i \notin V_H$, $b_{ii} = 1$. If $i \neq j$ and $i \notin V_H$ or $j \notin V_H$, $b_{ij} = 0$. Any fully specified submatrix of B is permutation similar to a fully specified submatrix of A or to the direct sum of an identity matrix with such a matrix. Since the class of TCIM-matrices is closed under direct sums and permutation similarity, B is a partial TCIM-matrix. Since Q has TCIM-completion, we can complete B to a TCIM-matrix $\hat{B}$. The principal submatrix $\hat{B}(V_H)$ then is used to define a completion $\hat{A}$ of A by $\hat{a}_{ij} = \hat{b}_{\phi(i)\phi(j)}$. ■

3.4 Lemma Let G be a pattern-digraph. If every component of G is isomorphic to a pattern-digraph that has TCIM-completion, then G has TCIM-completion.
Proof: Relabel the vertices of G to obtain an isomorphic digraph G' in which the vertices of each component are consecutive numbers. Every component of G' is isomorphic to a component of G and hence to a pattern-digraph that has TCIM-completion. Any partial TCIM-matrix A specifying G' is a block diagonal partial matrix, with each diagonal block corresponding to a component of G'. Complete each of these blocks to a TCIM-matrix via the isomorphism to a pattern-digraph that has TCIM-completion, resulting in a partial matrix B. Since the class of TCIM-matrices is closed under direct sums, B can be completed to a TCIM-matrix by setting all entries outside the diagonal blocks to 0. Since the class of TCIM-matrices is closed under permutation similarity, any partial TCIM-matrix specifying G can be transformed into a partial TCIM-matrix specifying G', completed to a TCIM-matrix, and transformed back. Thus G has TCIM-completion. ■

3.5 Lemma The pattern-digraphs shown in Figure 2 do not have TCIM- completion.
Proof: Note that each digraph shown in Figure 2 contains an open path of length 2.

The pattern-digraphs q=3 n=3, q=4, n=2, 3, 4 and q=5 all include the (1,2), (2,3) and (1,3) positions, as well as the diagonal (1,1), (2,2) and (3,3), and all omit at least one off-diagonal position. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ a & 1 & 1 \\ b & c & 1 \end{bmatrix}$, where each of a, b, c is either 0 or unspecified with at least one unspecified, so A is a partial TCIM-matrix. When values are chosen for any of a, b, c to obtain a completion $\hat{A}$, $(\hat{A}+\varepsilon I)^{-1} = \dfrac{1}{\det(\hat{A}+\varepsilon I)} \begin{bmatrix} 1-c+2\varepsilon+\varepsilon^2 & -1-\varepsilon & 1 \\ -a+b-a\varepsilon & 1+2\varepsilon+\varepsilon^2 & -1-\varepsilon \\ -b+ac-b\varepsilon & b-c-c\varepsilon & 1-a+2\varepsilon+\varepsilon^2 \end{bmatrix}$. Since $(\hat{A}+\varepsilon I)^{-1}{}_{13} = \dfrac{1}{\det(\hat{A}+\varepsilon I)} > 0$, $(\hat{A}+\varepsilon I)^{-1}$ is not an M-matrix, $(\hat{A}+\varepsilon I)$ is not an inverse M-matrix and $\hat{A}$ is not a TCIM matrix. Thus the pattern-digraphs q=3 n=3, q=4, n=2, q=4, n=3, q=4 n=4 and q=5 do not have TCIM-completion.

The pattern-digraphs q=2 n=2, q=3 n=1, 2, 4, and q=4 n=2 all include the (1,2) and (2,3) positions, as well as the diagonal (1,1), (2,2) and (3,3), and all omit (1,3). Let $B = \begin{bmatrix} 1 & 1 & ? \\ a & 0 & 1 \\ b & c & 1 \end{bmatrix}$, where ? is unspecified and each of a, b, c is either 0 or unspecified, so B is a partial TCIM-matrix. If *a* is unspecified, any completion $\hat{B}$ of B to a TCIM-matrix must set *a* to 0, because *a* must be chosen non-negative, and det $\hat{B}(\{1,2\})=-a \geq 0$, since $\hat{B}$ must be a P_0-matrix. Similarly, *c* must be 0.

Thus any completion of B to a TCIM-matrix must be of the form $\hat{B}_y = \begin{bmatrix} 1 & 1 & y \\ 0 & 0 & 1 \\ b & 0 & 1 \end{bmatrix}$, where y (and b if necessary) is/are specific choice(s) for entry(ies) 1,3 (and 3,1). In order for $\hat{B}_y$ to be a TCIM-matrix, $\hat{B}_y + \varepsilon I$ must be an inverse M-matrix for every $\varepsilon > 0$. But

$$(\hat{B}_y + \varepsilon I)^{-1} = \frac{1}{\det(\hat{B}_y + \varepsilon I)} \begin{bmatrix} \varepsilon + \varepsilon^2 & -1-\varepsilon & 1-\varepsilon y \\ b & 1+2\varepsilon+\varepsilon^2 - by & -1-\varepsilon \\ -b\varepsilon & b & \varepsilon + \varepsilon^2 \end{bmatrix}.$$

For any y there is an $\varepsilon_y > 0$ such that $0 < 1-\varepsilon_y y$, and so $(\hat{B}_y + \varepsilon I)_{13} = \frac{1-\varepsilon_y y}{(\hat{B}_y + \varepsilon I)} > 0$. Thus $(\hat{B}_y + \varepsilon_y I)^{-1}$ is not an M-matrix and $\hat{B}_y + \varepsilon_y I$ is not an inverse M-matrix. Thus $\hat{B}_y$ is not a TCIM-matrix and B cannot be completed to a TCIM-matrix. Thus the pattern-digraphs q=2 n=2, q=3 n=1, 2, 4, and q=4 n=2 do not have TCIM-completion. ■

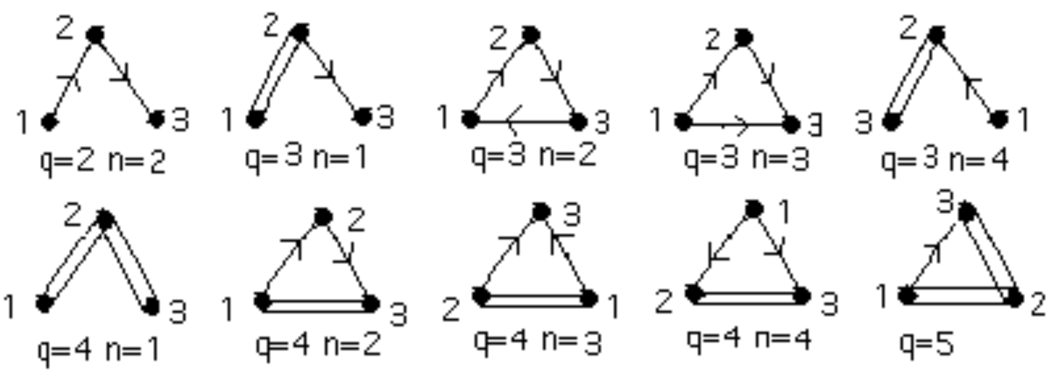

Figure 2 Pattern-digraphs that do not have TCIM-completion (notation from [Ha]).

3.6 Lemma A digraph G is a source/sink cut bipartite digraph if and only if G does not contain any path of length 2.
Proof: If G = (V,E) is a source/sink cut bipartite digraph (with sources S and sinks T) and (u,v) ∈ E, then u ∈ S and v ∈ T, so G does not contain any arcs (u,v) and (v,w) and hence does not contain a path u, v, w of length 2. If G does not contain any path of length 2, let S = {u: (u,v) ∈ E} and T = E – S. If (u,v) ∈ E, then there does not exist (v,w) in E because then u, v, w would be a path of length 2. Thus v ∉ S so v ∈ T and G is a source/sink cut bipartite digraph. ■

3.7 Lemma If a connected digraph G does not contain an induced subdigraph isomorphic any to the digraphs in Figure 2, then G is a clique or a source/sink cut bipartite digraph.
Proof: Suppose G is not a source/sink cut bipartite digraph. Then G contains a path Γ of length 2.

For any path u, v, w of length 2 in G, we show that <u,v,w> is a clique: If w = u then <u,v,w> = <u,v> is a clique. If w ≠ u, an examination of the table of digraphs of order 3 in [Ha] shows that the only digraphs of order 3 that contain an open path of length 2 are those in Figure 2 and the clique on 3 vertices, so <u,v,w> is a clique.

Let H be maximal among cliques in G that contain the clique induced by the path Γ of length 2. If H ≠ G, then since G is connected, there is a vertex z not in H and a vertex x in H such that (z,x) or (x,z) is in H. The order of H is greater than or equal to 2, so let y be any other vertex in H. Since H is a clique, (x,y) and (y,x) are in H. So either G contains both (z,x) and (x,y) or G contains both (y,x) and (x,z), and z, x, y or y, x, z is a path of length 2. So as before, <x,y,z> is a clique, so (z,x),(x,z),(y,z),(z,y) are in G for vertex x and any other vertex y of H. Then <z,V(H)> is a clique and H is not maximal, contradicting the choice of H. Thus G is a clique. ■

The *zero-completion* of a partial matrix is the matrix obtained by setting all unspecified entries to zero.

3.8 Theorem A pattern Q that includes all diagonal positions has TCIM-completion if and only if each component of its digraph G is a source/sink cut bipartite digraph or a clique. For such a pattern Q, the zero-completion of a partial TCIM-matrix specifying Q is a TCIM-matrix.
Proof: Consider first a pattern-digraph H that is a source/sink cut bipartite digraph. By relabeling the vertices, we may assume that S={1,…,k} and T = {k+1,…,n}. Let B be any non-negative partial matrix specifying H and let B_0 be the zero completion of B. For any ε>0, $B_0+\varepsilon I$ can be partitioned as $\begin{bmatrix} D_1 & B_{12} \\ 0 & D_2 \end{bmatrix}$ with D_1 and D_2 positive diagonal matrices and B_{12} non-negative. From the formula for the inverse of a partitioned matrix [HJ1], $(B_0+\varepsilon I)^{-1} = \begin{bmatrix} D_1^{-1} & -D_1^{-1}B_{12}D_2^{-1} \\ 0 & D_2^{-1} \end{bmatrix}$. Thus $(B_0+\varepsilon I)^{-1}$ is an M-matrix, $B_0+\varepsilon I$ is an inverse M-matrix and B_0 is a TCIM-matrix. So H has TCIM-completion.

Thus if each component of the digraph G of Q is a source/sink cut bipartite digraph or a clique then each component is isomorphic to a pattern-digraph that has TCIM-completion, so by Lemma 3.4, G (and Q) have TCIM-completion. Note that all the completions used involve setting all unspecified entries to 0, i.e., the zero-completion.

For the converse, suppose Q has TCIM-completion and let G be its digraph. Let H be a component of G. Any induced subdigraph of H is an induced subdigraph of G, so by Lemmas 3.3 and 3.5, H cannot contain an induced subdigraph isomorphic to any of the digraphs in Fig. 2. So by Lemma 3.7, H is a clique or a source/sink cut bipartite digraph. ■

Even though Theorem 2.2 does not apply, the conclusion of Theorem 2.2 remains true, because a pattern that includes all diagonal positions has inverse M-completion if and only if every strongly connected induced subdigraph is path/cycle-clique [Ho1]. Any clique or source/sink cut bipartite digraph is path/cycle-clique.

3.9 Corollary If a pattern that includes all diagonal positions has TCIM-completion then it has inverse M-completion.

Like some of the classes studied in Section 2, there are patterns that include all diagonal positions and have inverse M-completion but do not have TCIM-completion. For example, the pattern whose digraph is q=2 n=2 has inverse M-completion [Ho1], but does not have TCIM completion.

4. Conclusion

It should be noted that the conclusion of Theorem 2.2 is false for some pairs consisting of a class X and its topological closure. For example, the topological closure of the class of sign-symmetric P_0-matrices is the class of weakly sign-symmetric P_0-matrices. And the pattern

Q_2 = {(1,1), (2,1), (2,2)} does have weakly sign-symmetric P_0-completion (set the unspecified entry to 0) but does not have sign-symmetric P-completion, as the partial sign-symmetric P_0-matrix $A_2 = \begin{bmatrix} 0 & ? \\ 1 & 1 \end{bmatrix}$ shows.

An important distinction between the pair of classes sign-symmetric P_0-/weakly sign-symmetric P_0-matrices and the pair of classes inverse M-/TCIM-matrices is that in the latter case the first and second conditions of Definition 2.1 hold, but the second condition does not hold in the former. Note also that these two properties were sufficient in the discussion of pairs of subclasses of $P_{0,1}$- and P_0-matrices in Section 2. Properties (1) and (2) of Definition 2.1 appear to be the key ingredients in deducing completion results about one class from another.

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