The Symmetric M-Matrix and Symmetric Inverse M-Matrix Completion Problems

Leslie Hogben
Department of Mathematics
Iowa State University
Ames, IA 50011
lhogben@iastate.edu

Abstract The symmetric M-matrix and symmetric M_0-matrix completion problems are solved and results of Johnson and Smith [JS2] are extended to solve the symmetric inverse M-matrix completion problem:

1) A pattern (i.e., a list of positions in an n × n matrix) has symmetric M-completion (i.e., every partial symmetric M-matrix specifying the pattern can be completed to a symmetric M-matrix) if and only if the principal subpattern R determined by its diagonal is permutation similar to a pattern that is block diagonal with each diagonal block complete, or, in graph theoretic terms, if and only if each component of the graph of R is a clique.
2) A pattern has symmetric M_0-completion if and only if the pattern is permutation similar to a pattern that is block diagonal with each diagonal block either complete or omitting all diagonal positions, or, in graph theoretic terms, if and only if every principal subpattern corresponding to a component of the graph of the pattern either omits all diagonal positions, or includes *all* positions.
3) A pattern has symmetric inverse M-completion if and only if its graph is block-clique and no diagonal position is omitted that corresponds to a vertex in a graph-block of order > 2.

The techniques used are also applied to matrix completion problems for other classes of symmetric matrices.

1. Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion* of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A *pattern* for n × n matrices is a list of positions of an n × n matrix, that is, a subset of $\{1,...,n\} \times \{1,...,n\}$. A partial matrix *specifies a pattern* if its specified positions are exactly those listed in the pattern. Note that in this paper a pattern does not need to include all diagonal positions.

All matrices and partial matrices discussed here are real. The symbol Π will denote a class of matrices and a *Π-matrix* is a matrix in the class Π. For a particular class Π of matrices, the *Π-matrix completion problem for patterns* asks which patterns have the property that any partial Π-matrix that specifies the pattern can be completed to a Π-matrix. When a pattern has this property, we say it *has Π-completion.*

The answer to the Π-matrix completion problem obviously depends on the definition of partial Π-matrix. For many classes Π of matrices, in order for it to be possible to have a completion of a partial matrix to a Π-matrix, certain obviously necessary conditions must be satisfied. Such obviously necessary conditions are frequently taken as the definition of a partial Π-matrix, [JS1], [JS2], [H1], [H2], [H3], [H4]. Here we also take this approach of using obviously necessary conditions to define a partial Π-matrix (see below).

In this paper we concern ourselves only with matrix completion problems for patterns. Such problems have been studied for M-matrices [H2], M_0-matrices [H4], inverse M-matrices

[JS1], [H1], [H3], symmetric inverse M-matrices [JS2], and many other classes. Results on matrix completion problems and techniques are surveyed in [H4].

For α a subset of $\{1,...,n\}$, the *principal submatrix* $A[\alpha]$ is obtained from the $n \times n$ matrix A by deleting all entries a_{ij} such that $i \notin \alpha$ or $j \notin \alpha$. Similarly, the *principal subpattern* $Q[\alpha] = Q \cap (\alpha \times \alpha)$. The *principal subpattern determined by the diagonal positions* is $Q[\delta]$ where $\delta = \{i \mid (i,i) \in Q\}$.

The *characteristic matrix* of a pattern Q for $n \times n$ matrices is the $n \times n$ matrix C_Q such that $c_{ij} = 1$ if the position (i,j) is in the pattern and $c_{ij} = 0$ if (i,j) is not in the pattern. A pattern Q is *permutation similar* to a pattern R if C_Q is permutation similar to C_R. A pattern is *block diagonal* (for a particular block structure) if its characteristic matrix is block diagonal for that block structure (cf. Section 3 of [H4]).

A class Π of matrices is called a *Hereditary-Sum-Permutation-closed (HSP)* class if (1) every principal submatrix of a Π-matrix is a Π-matrix, (2) the direct sum of Π-matrices is a Π-matrix, (3) if A is a Π-matrix and P is a permutation matrix of the same size then PAP^{-1} is a Π-matrix, and (4) there is a 1×1 Π-matrix. All of the classes discussed in [H4], except those that require entries to be positive (and thus fail condition (2)), are HSP classes. For an HSP class Π we frequently define a partial matrix B to be a partial Π-matrix if any fully specified principal submatrix of B is a Π-matrix and any sign condition on the entries of a Π-matrix is respected by B (here sign condition includes nonpositive, nonnegative, sign symmetric or weakly sign symmetric, cf. [H4]). Using this definition of a partial Π-matrix is referred to as *using the HSP standard definition of a partial Π-matrix.*

A class Σ of matrices is called *symmetric* if every Σ-matrix is symmetric. A partial matrix B is *symmetric* if whenever b_{ij} is specified then so is b_{ji} and $b_{ji} = b_{ij}$. A pattern is *symmetric* (also called "positionally symmetric" and "combinatorially symmetric") if position (i,j) in the pattern implies (j,i) is also in the pattern. A class Σ of matrices is called a *Symmetric-Hereditary-Sum-Permutation-closed (SHSP)* class if Σ is both symmetric and an HSP class. For an SHSP class Σ, we frequently define a partial Σ -matrix to be a symmetric partial matrix meeting the requirements for a partial matrix of an HSP class. Using this definition of a partial Π-matrix is referred to as *using the SHSP standard definition of partial Π-matrix.*

The matrix A is called *positive stable* (respectively, *semistable*) if all the eigenvalues of A have positive (nonnegative) real part. An *M-matrix* (respectively, *M_0-matrix*) is a positive stable (semistable) matrix with nonpositive off-diagonal entries. There are many equivalent characterizations of M- and M_0-matrices [HJ]: A matrix with nonpositive off-diagonal entries is an M-matrix (M_0-matrix) if and only if every principal minor is positive (nonnegative). A matrix with nonpositive off-diagonal entries is an M-matrix if and only if it is nonsingular and its inverse is entrywise nonnegative. The notation $M_{(0)}$ will be used to mean "M (respectively, M_0)." The matrix B is an *inverse M-matrix* if B is the inverse of an M-matrix. Equivalently, an inverse M-matrix is a nonsingular, entrywise nonnegative matrix B such that B^{-1} has nonpositive off-diagonal entries. A substantial amount is known about M-matrices, M_0-matrices and inverse M-matrices [HJ], [J], [LN], including the fact that each of these classes is an HSP class.

Use the HSP standard definition of a partial Π-matrix: A *partial $M_{(0)}$-matrix* is a partial matrix such that any fully specified principal submatrix is an $M_{(0)}$-matrix and all specified off-diagonal entries are nonpositive. A *partial inverse M-matrix* is an entrywise nonnegative partial matrix such that any fully specified principal submatrix is an inverse M-matrix.

We will follow the notation of [JS2] in referring to a symmetric inverse M-matrix as a SIM matrix and we refer to a symmetric $M_{(0)}$- matrix as a $SM_{(0)}$-matrix. Note that the three classes SIM, SM and SM_0 are SHSP classes. Use the SHSP standard definition of a partial Π-matrix: A *partial SIM-matrix* is a partial inverse M-matrix that is symmetric. A *partial $SM_{(0)}$- matrix* is a partial $M_{(0)}$-matrix that is symmetric.

With the HSP and SHSP standard definitions of partial Π-matrix, HSP and SHSP classes have two basic properties that are used extensively in the study of matrix completion problems, Lemma 1.1 and Observation 1.2 below.

1.1 Lemma Let Π be an HSP (SHSP) class using the HSP (SHSP) standard definition of a partial Π-matrix. If the pattern Q has Π-completion then so does every principal subpattern $Q[\alpha]$.
Proof: Let A be a partial Π-matrix specifying $Q[\alpha]$. By (4), there is some 1×1 Π-matrix $[s]$. Extend A to a partial matrix B specifying Q by, for each (i,j) in Q but not in $Q[\alpha]$, setting $b_{ij} = s$ if $i = j$ and 0 otherwise. Then B is a partial Π-matrix because any fully specified principal submatrix $B[\beta]$ of B is permutation similar to $A[\alpha\cap\beta] \oplus [s] \oplus\ldots\oplus [s]$, which is a Π-matrix by (2). So $B[\beta]$ is a Π-matrix by (3). By hypothesis, B can be completed to a Π-matrix C. Then $C[\alpha]$, which completes A, is a Π-matrix by (1). ■

Our graph terminology follows [H4]. For a symmetric pattern Q, the *pattern-graph* of Q is the graph having $\{1,\ldots,n\}$ as its vertex set and, as its set of edges, the set of (unordered) pairs $\{i,j\}$ such that position (i,j) (and therefore also (j,i)) is in Q. If G is the pattern-graph of Q, then the pattern-graph of a principal subpattern $Q[\alpha]$ is $<\alpha>$, the subgraph induced by α. The principal subpattern $Q[\alpha]$ and the induced subgraph $<\alpha>$ are said to *correspond*; in particular, the vertex v and diagonal position (v,v) correspond. Renaming the vertices of a pattern-graph is equivalent to applying a permutation similarity to the pattern.

A *component* of a graph is a maximal connected subgraph. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects the graph; more generally, a cut-vertex is a vertex whose deletion disconnects the component containing it. A graph is *nonseparable* if it is connected and has no cut-vertices. A *block* of a graph is a subgraph that is nonseparable and is maximal with respect to this property. A (sub)graph is called a *clique* if it contains all possible edges between its vertices. A graph is *block-clique* if every block is a clique. Block-clique graphs are called "1 chordal" in [JS2].

For matrices and patterns that need not be symmetric, digraphs must be used. Let A be a (fully specified) $n \times n$ matrix. The *nonzero-digraph* of A is the digraph having as vertex set $\{1,\ldots,n\}$, and, as its set of arcs, the set of ordered pairs (i,j) such that both i and j are vertices with $i \neq j$ and $a_{ij} \neq 0$. For a pattern Q that need not be symmetric, the *pattern-digraph* of Q is the digraph having $\{1,\ldots,n\}$ as its vertex set and members (i,j) of Q with $i \neq j$ as its arcs. A digraph is *transitive* if the existence of a path from v to w implies the arc (v,w) is in the digraph. Recall that the nonzero-digraph of any inverse M-matrix is transitive [LN].

1.2 Observation Let Π is an HSP (SHSP) class. If the pattern Q is permutation similar to a block diagonal pattern in which each diagonal block has Π-completion, then Q has Π-completion by (2) and (3). Equivalently, if each principal subpattern of Q corresponding to a component of a pattern-digraph (pattern-graph) has Π-completion, then the pattern has Π-completion.

The results in 1.1 and 1.2 are already known for $SM_{(0)}$-matrices and SIM-matrices [H4].

Johnson and Smith [JS2] determined that a symmetric pattern that includes all diagonal positions has SIM completion if and only if its pattern-graph is block-clique. More general patterns that may omit some diagonal positions are classified as to SIM completion in the next section. All patterns are classified as to $SM_{(0)}$-completion in the third section.

2. Determination of patterns having SIM completion

It is well known that a graph G is block-clique if and only if for every cycle $v_1, v_2\ldots, v_k, v_1$ of G, the induced subgraph $<\{v_1, v_2\ldots, v_k\}>$ is a clique [JS1].

2.1 Theorem Let Q be a symmetric pattern and let G be its pattern-graph. If Q has SIM completion, then G is block-clique and the diagonal positions corresponding to the vertices of every cycle in G are all included in Q.

Proof: Suppose Q and G do not have the required property. Then G is contains a cycle whose induced subgraph is not a clique or Q omits the diagonal position corresponding to a vertex in a cycle. Let Γ be a shortest troublesome cycle. By renaming vertices if necessary, assume $\Gamma = 1, 2, \ldots, k, 1$ with $k > 2$, and $<\{1,\ldots,k\}>$ is not a clique or diagonal position (k,k) is not in Q.

Suppose first that $<\{1,...,k\}>$ does not contain any chord of Γ. If $k > 3$, then $Q[\{1,...,k\}]$ does not contain any complete principal subpattern of size larger than 2×2, because all chords are omitted. When $k = 3$, $(3,3)$ must be omitted from Q. Thus, in either case, $Q[\{1,...,k\}]$ does not contain any complete principal subpattern of size larger than 2×2. Define a $k \times k$ partial matrix B specifying $Q[\{1,..,k\}]$ by setting $b_{ii} = 2$ for $(i,i) \in Q[\{1,..,k\}]$, setting $b_{i\,i+1} = 1 = b_{i+1\,i}$ for $i=1,...,k-1$, and setting all other specified entries (including b_{1k} and b_{k1}) equal to 0. Since the only completely specified principal submatrices are 2×2 or smaller, and these are SIM matrices, B is a partial SIM matrix. But B cannot be completed to a SIM matrix because the nonzero-digraph of any completion of B is not transitive, since $b_{12} = \ldots = b_{k-1\,k} = 1$ and $b_{1k} = 0$. So $Q[\{1,...,k\}]$ does not have SIM completion.

Now suppose $k > 3$ and $<\{1,...,k\}>$ contains a chord of Γ. Each of the two pieces of Γ on either side of the chord, together with the chord, forms a shorter cycle. By the minimal length assumption, Q must include all diagonal positions corresponding to vertices in these two shorter cycles. Hence, Q includes all diagonal positions $(1,1), \ldots, (k,k)$. Then $<\{1,..,k\}>$ is not a clique, and thus is not block-clique. So $Q[\{1,...,k\}]$ is a symmetric pattern that includes all diagonal positions and whose graph is not block-clique, and therefore does not have SIM completion [JS2].

In either case Q does not have SIM completion because $Q[\{1,...,k\}]$ does not. ■

The properties that for any vertex v that appears in a cycle of G, $(v,v) \in Q$, and the induced subgraph of a cycle of G must be a clique are the graph analog of the pattern-digraph condition "path-clique," which is necessary for a (not necessarily symmetric) pattern to have inverse M completion [H3]. (In a digraph D, an *alternate path to a single arc* is a path $(v_1,v_2), (v_2,v_3), \ldots, (v_{k-1},v_k)$ with $k > 2$, such that (v_1,v_k) is an arc of D. A pattern-digraph is called *path-clique* if the induced subdigraph of any alternate path to a single arc is a clique and the diagonal position (v_i,v_i) is in the pattern for every vertex v_i in the path).

2.2 Theorem A symmetric pattern Q has SIM completion if and only if its pattern-graph G is block-clique and the diagonal position (v,v) is in Q for every vertex v in a block of order > 2.

Proof: (only if) By Theorem 2.1, G is block-clique. Let v be a vertex in a block H of order > 2. There are two other distinct vertices u and w in H. Since H is a clique, $\{v,u\}$, $\{u,w\}$ and $\{w,v\}$ are in H, and v occurs in a cycle. So by Theorem 2.1 (v,v) is in Q.

(if) Let H be a block of G of order > 2. By hypothesis, H is a clique and (v,v) is in Q for every vertex v in H. Thus the principal subpattern $Q[\eta]$ corresponding to H contains *all* positions, so $Q[\eta]$ trivially has SIM-completion.

Any symmetric pattern for 2×2 matrices has SIM completion [H4, remark following Lemma 4.8], so the principal subpattern of Q corresponding to any block of G of order 2 has SIM completion. Thus the principal subpattern of Q corresponding to each block of G is has SIM completion, and so Q has SIM completion by Corollary 5.6 of [H4]. ■

The following example exhibits patterns that do not include all diagonal positions, one having SIM completion and other not having SIM completion. In the diagrams of the pattern-graphs, if a diagonal position (v,v) is in the pattern, then vertex v is indicated by a solid black dot (—●—); if (v,v) is omitted, then vertex v is indicated by a hollow circle (—○—) (this follows the notation of [H3] and [H4]).

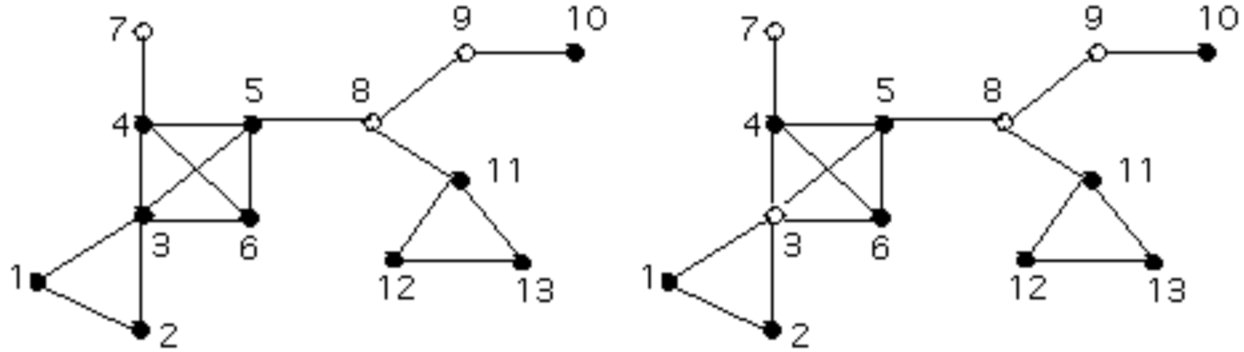

Figure 1 (a) Q_1 has SIM completion (b) Q_2 does not have SIM completion

2.3 Example The pattern Q_1 = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,3), (4,4), (4,5), (4,6), (4,7), (5,3), (5,4), (5,5), (5,6), (5,8), (6,3), (6,4), (6,5), (6,6), (7,4), (8,5), (8,9), (8,11), (9,8), (9,10), (10,9), (10,10), (11,8), (11,11), (11,12), (11,13), (12,11), (12,12), (12,13), (13,11), (13,12), (13,13)}, whose pattern-graph is shown in Figure 1(a), has SIM completion. The pattern Q_2 obtained from Q_1 by deleting the diagonal position (3,3), whose pattern-graph is shown in Figure 1(b), does not.

3. Determination of patterns having $SM_{(0)}$-completion

A partial $M_{(0)}$-matrix with all diagonal entries specified can be completed to an $M_{(0)}$-matrix if only if its zero completion (i.e., the result of setting all unspecified entries to 0) is an $M_{(0)}$-matrix, cf. [JS1], [H4]. Since a partial $SM_{(0)}$-matrix is an $M_{(0)}$-matrix, and the zero completion of a partial $SM_{(0)}$-matrix is symmetric, a partial $SM_{(0)}$-matrix with all diagonal entries specified can be completed to an $SM_{(0)}$-matrix if only if its zero completion is an $SM_{(0)}$-matrix.

3.1 Lemma If a symmetric pattern Q has $SM_{(0)}$- completion and includes positions (a,a), (b,b), (c,c), (a,b), (b,a), (b,c), and (c,b) with a<b<c, then Q also includes (a,c) and (c,a).
Proof: Suppose Q does not include (a,c) and (c,a). Then the partial matrix SM-matrix

$$A = \begin{bmatrix} 4 & -3 & ? \\ -3 & 4 & -3 \\ ? & -3 & 4 \end{bmatrix}$$

specifies Q[{a,b,c}] and cannot be completed to an SM_0-matrix because the zero-completion of A has determinant -8. Thus Q does not have $SM_{(0)}$- completion. ■

3.2 Theorem Let Q be a symmetric pattern with the property that if (v,w) is in Q then (v,v) and (w,w) are both in Q. Then the following are equivalent:
1) Q has $SM_{(0)}$-completion.
2) Q is permutation similar to a block diagonal pattern with each diagonal block containing *all* positions or consisting of a single omitted diagonal position.
3) Each component of its pattern-graph is a clique.

Proof: The equivalence of the (2) and (3) is immediate from the hypothesis about Q. If Q satisfies (3), then Q has $SM_{(0)}$-completion by Observation 1.2.

Let Q have $SM_{(0)}$-completion. Let u and v be vertices of H, a component of order > 1 of the pattern-graph of Q. Since (w,w) is in Q for every vertex visited by a path in H connecting u and v, we may apply Lemma 3.1 to eliminate one at a time from that path any vertex other than u and v. Hence {v,u} is in H and H is a clique. ■

A symmetric pattern Q has SM-completion if and only if Q[δ] does [H4, Theorem 4.6]. The principal subpattern Q[δ] satisfies the hypotheses of Theorem 3.2, so this completes the determination of patterns having SM-completion.

3.3 Corollary A symmetric pattern Q has SM-completion if and only if Q[δ] is permutation similar to a pattern that is block diagonal with all positions in each of the blocks on the diagonal in the pattern; in graph theoretic language, if and only if each component of the pattern-graph of Q[δ] is a clique.

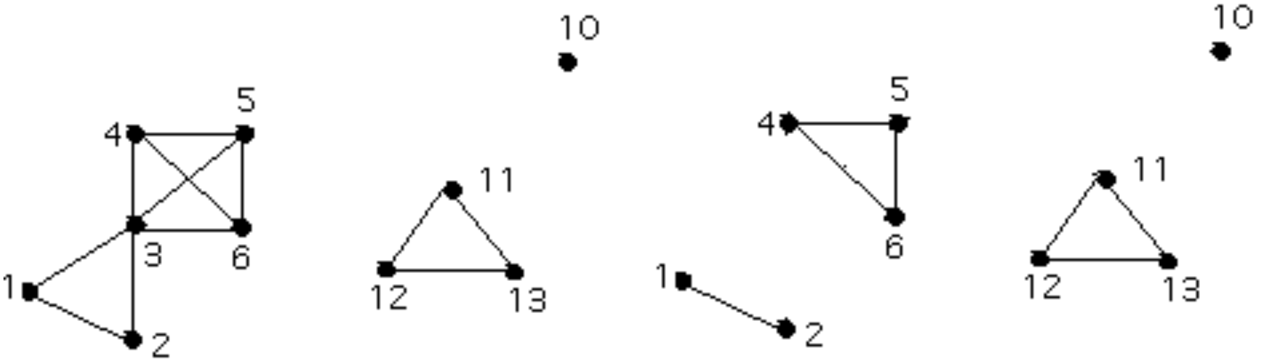

Figure 2 (a) $Q_1[\delta]$ (b) $Q_2[\delta]$

3.4 Example The pattern Q_1 in Example 2.3 does not have SM-completion because one of the components of the pattern-graph of $Q_1[\delta]$ is not a clique. The pattern Q_2 in Example 2.3 has SM-completion, because each component of $Q_2[\delta]$ is a clique. The pattern-graphs of $Q_1[\delta]$ and $Q_2[\delta]$ are shown in Figure 2(a) and (b), respectively.

3.5 Example The partial SM_0-matrix $A = \begin{bmatrix} 0 & -1 \\ -1 & ? \end{bmatrix}$ cannot be completed to an SM_0-matrix because the determinant of any completion of A equals -1.

Thus, neither Q_1 nor Q_2 from Example 2.3 has SM_0-completion, because both contain the principal subpattern R = {(4,4), (4,7), (7,4)}.

3.6 Lemma Let Π be an HSP (SHSP) class (using the HSP (SHSP) standard definition of a partial Π-matrix) such that {(a,a), (a,b), (b,a)} (with $a \neq b$) does not have Π-completion. Then if a symmetric pattern Q has Π-completion, Q is permutation similar to a block diagonal pattern in which every diagonal block either includes all diagonal positions or omits all diagonal positions, i. e., in graph theoretic terms, every principal subpattern R corresponding to a component H of the pattern-graph G of Q includes all diagonal positions or omits all diagonal positions.
Proof: Suppose R includes (v,v) and omits (w,w). Since H is a component, it is connected, and it contains a path $\{u_1,u_2\}, \{u_2,u_3\}, ..., \{u_{k-1},u_k\}$ from vertex $v = u_1$ to vertex $w = u_k$. Let t be the number such that R includes $(u_1, u_1), ... , (u_t, u_t)$ and R does not include (u_{t+1}, u_{t+1}). Then $Q[\{u_t, u_{t+1}\}] = \{(u_t, u_t),(u_t, u_{t+1}),(u_{t+1},u_t)\}$. Thus by the hypothesis, $Q[\{u_t, u_{t+1}\}]$ does not have Π-completion, and so neither does Q, by Lemma 1.1. ■

3.7 Theorem Let Q be a symmetric pattern and let G be its pattern-graph. Then Q has SM_0-completion if and only if Q is permutation similar to a pattern that is block diagonal in which each diagonal block either omits all diagonal positions or includes *all* positions, i. e., in graph theoretic language, if and only if every principal subpattern corresponding to a component of G either omits all diagonal positions, or includes *all* positions.

Proof: If Q has SM_0-completion, 3.6 shows that after permutation similarity the diagonal blocks have either no or all diagonal positions. Theorem 3.2 shows that in the latter case the block includes *all* positions.

Conversely, a pattern that omits all diagonal positions has SM_0-completion [H4, Theorem 4.7]. A pattern that includes *all* positions trivially has SM_0-completion. Since Q is permutation similar to a block-diagonal pattern in which each diagonal block has SM_0-completion, Q has SM_0-completion by 1.2. ■

3.8 Corollary Any symmetric pattern that has SM_0-completion also has SM-completion (cf. 3.3), but the converse is false (cf. 3.5). A symmetric pattern that includes all diagonal positions and has SM-completion also has SM_0-completion (cf. 3.2).

The next example shows that the assumption in Lemma 3.6 that the pattern is symmetric is necessary. A matrix is a *P_0-matrix* if every principal minor is non-negative. Use the HSP standard definition of a partial Π-matrix: A *partial P_0-matrix* is a partial matrix such that any fully specified principal submatrix is an P_0-matrix. The pattern {(a,a), (a,b), (b,a)} (with a ≠ b) does not have P_0-completion (cf. Example 3.5), so Lemma 3.6 applies to the class of P_0-matrices.

3.9 Example The pattern {(1,1), (1,2), (2,2), (2,3), (3,1)} (whose pattern-digraph is a 3-cycle) has P_0-completion, because it is asymmetric [CDHMW], and neither contains nor omits all diagonal positions.

We can also use Lemma 3.6 to complete the classification of patterns for other classes of symmetric matrices. The matrix A is *doubly nonnegative (DN)* if A is entrywise nonnegative and positive semidefinite. The matrix A is *completely positive (CP)* if A is entrywise nonnegative and $A = BB^T$ for some entrywise nonnegative $n \times m$ matrix B (the requirement that A be entrywise nonnegative is clearly redundant, but helps clarify the interpretation of using the SHSP standard definition of a partial CP-matrix). The classes DN and CP are SHSP classes [DJ]. Use the SHSP standard definitions of a partial Π-matrix: A *partial DN-matrix* is an entrywise nonnegative symmetric partial matrix such that any fully specified principal submatrix is a DN-matrix. A *partial CP-matrix* is an entrywise nonnegative symmetric partial matrix such that any fully specified principal submatrix is a CP-matrix. Drew and Johnson [DJ] established that a symmetric pattern that includes the diagonal has DN- (CP-) completion if and only if its pattern-graph is block-clique. The partial matrix $\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$ shows that {(a,a), (a,b), (b,a)} does not have DN- or CP-completion. Since any diagonally dominant nonnegative symmetric matrix is CP [K] (and thus DN), a pattern that omits all diagonal positions has CP- and DN-completion.

3.10 Corollary Let Q be a symmetric pattern and let G be its pattern-graph. Then Q has DN- (CP-) completion if and only if every principal subpattern corresponding to a component H of G either omits all diagonal positions, or includes all diagonal positions and H is block-clique.

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