The Minimal Rank Problem for Finite Fields

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References

[BvdHL1] Barrett, van der Holst, Loewy, Graphs whose Minimal Rank is Two, Electronic Journal of Linear Algebra, volume 11 (2004) , pp. 258-280.

[BvdHL2] Barrett, van der Holst, Loewy, Graphs whose Minimal Rank is Two: The Finite Fields Case, Electronic Journal of Linear Algebra, volume 14 (2005), pp. 32-42.

[BM] Barrett, March, The Minimal Rank Problem over a Finite Field with a Prime Number of Elements, in preparation

Setup

$$F \text{ - a field}$$

$$G = (V, E) \text{ - a graph}$$

$$V = \{1, 2, \ldots, n\}$$

$S(F, G)$

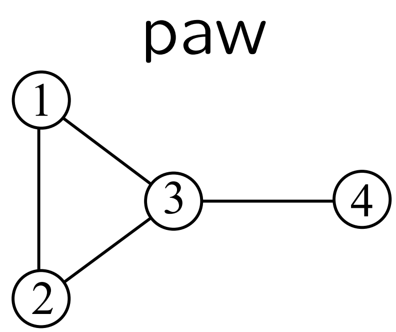
- set of all symmetric $n \times n$ matrices A with graph G.

This means: For $i \neq j$

$$a_{ij} \neq 0 \iff ij \in E$$

no condition on the diagonal entries

Example:



$$S(F, \text{paw}) = \left\{ \begin{bmatrix} a & w & x & 0 \\ w & b & y & 0 \\ x & y & c & z \\ 0 & 0 & z & d \end{bmatrix} | a, b, c, d, w, x, y \in F, wxyz \neq 0 \right\}$$

The zeros correspond to the missing edges 14, 24.

Problem

Given a field F and a graph G, find

$$\text{mr}(F,G) = \min\{\text{rank}\, A | A \in S(F,G)\}$$

Idea of the Question: How much can you tell about a matrix if you only know where the zeros are?

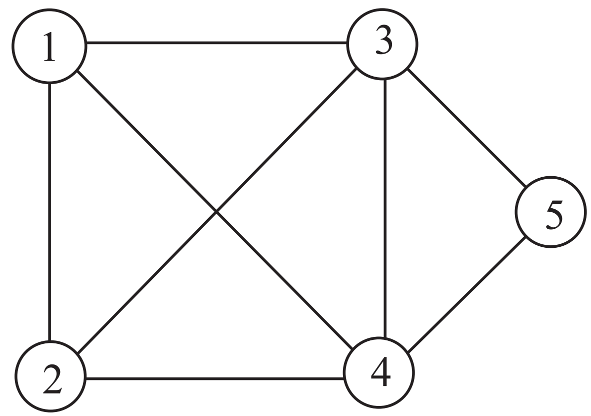
Related to other questions:

maximum multiplicity of eigenvalues

degeneracies in chemical bonding theory

The answer can depend on the field:

Let G be the graph



clique sum of K_4 and K_3 on K_2

two missing edges: 15, 25

$F = \mathbb{R}$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$A \in S(\mathbb{R}, G)$ rank $A = 2$

But if char $F = 2$, the $3, 4$ element is $1 + 1 = 0$, not 2 and this $A \notin S(F, G)$

$F = F_2$:

Any $A \in S(F_2, G)$ has the form $\begin{bmatrix} d_1 & 1 & 1 & 1 & 0 \\ 1 & d_2 & 1 & 1 & 0 \\ 1 & 1 & d_3 & 1 & 1 \\ 1 & 1 & 1 & d_4 & 1 \\ 0 & 0 & 1 & 1 & d_5 \end{bmatrix}$

Know all off-diagonal entries now.

$A[145|235] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d_5 \end{bmatrix}$ has determinant 1 so rank $A \geq 3$,

and $\text{mr}(F_2, G) \geq 3$.

Theorems (Barrett, van der Holst & Loewy)

char$F \neq 2$

F infinite: $\mathrm{mr}(F,G) \leq 2 \iff$

$$G^c = (K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

F finite with p^t elements: $\mathrm{mr}(F,G) \leq 2 \iff$ Either

$$G^c = (K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \quad k \leq (p^t - 1)/2,$$

or

$$G^c = (K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \quad k \leq (p^t + 1)/2$$

Theorems $\quad \mathrm{char} F = 2$

F infinite: $\quad \mathrm{mr}(F,G) \le 2 \iff$ Either

$$G^c = (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r$$

or

$$G^c = (K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

F finite with 2^t elements: $\quad \mathrm{mr}(F,G) \le 2 \iff$ Either

$$G^c = (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r, \qquad k \le 2^t + 1,$$

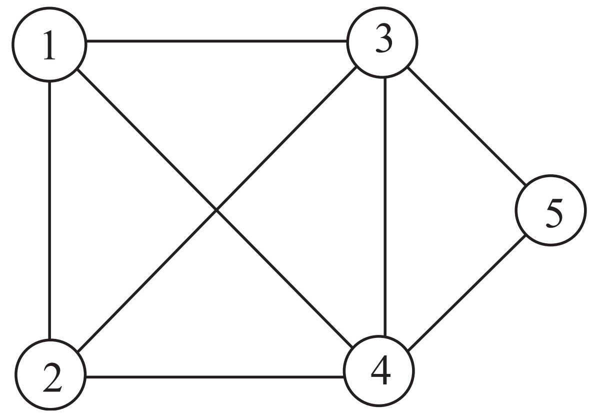
or

$$G^c = (K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \quad k \le 2^{t-1}$$

Corollary: $\text{mr}(F_2, G) \leq 2 \iff$ Either

$G^c = (K_p \cup K_q \cup K_r) \vee K_t$ or $G^c = (K_r \cup K_{p,q}) \vee K_t$

Recall the graph



It's complement is $P_3 \cup 2K_1$

Is it one of the forms above?

No dominating vertex in $P_3 \cup 2K_1$

Is $P_3 \cup 2K_1$ of the form $K_p \cup K_q \cup K_r$ or $K_r \cup K_{p,q}$?

Not $K_p \cup K_q \cup K_r$ because of P_3.

Not $K_r \cup K_{p,q}$ because $P_3 \cup K_1$ is not induced in $K_{p,q}$

But $P_3 \cup 2K_1$ is induced in the union of two bipartite graphs or in the union of two complete graphs and a bipartite graph $\Longrightarrow$

its complement is a rank 2 graph for any other field.

The methods in the papers [BvdHL1], [BvdH2] do not extend in any straightforward way to the problem of characterizing graphs with $\text{mr}(F,G) \leq k$ for $k \geq 3$.

However, it is possible to obtain results of this sort for finite fields using a quite different method which makes explicit use of the finiteness of F.

The method is straightforward but the complexity grows exponentially with the rank k.

We had

Corollary: $\mathrm{mr}(F_2, G) \le 2 \iff$ Either

$$G^c = (K_p \cup K_q \cup K_r) \vee K_t \quad \text{or} \quad G^c = (K_r \cup K_{p,q}) \vee K_t$$

Taking complements

$\mathrm{mr}(F_2, G) \le 2 \iff$ Either

$$G = K_{p,q,r} \cup K_t^c \quad \text{or} \quad G = [(K_p \cup K_q) \vee K_r^c] \cup K_t^c$$

Assume that $A \in S_n(F_2)$ with $\operatorname{rank} A \leq 2$.

$\operatorname{rank} A < 2 \implies A$ is the zero matrix, or is permutation similar to $J_k \oplus O_{n-k}$

So assume $\operatorname{rank} A = 2$.

Then $A = U^t B U$,

$B - 2 \times 2$, symmetric, invertible $\quad U - 2 \times n$

Either B is congruent to I_2 or $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So WLOG either $A = U^t U$ or $A = U^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U$

U has at most 4 distinct columns: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We may assume that identical columns are contiguous

Write $U = \begin{bmatrix} E_1 & E_2 & J & O \end{bmatrix}$

Either $A = \begin{bmatrix} E_1^t \\ E_2^t \\ J^t \\ O^t \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & 0 \end{bmatrix} = \begin{bmatrix} J_p & 0 & J_{p,r} & 0 \\ 0 & J_q & J_{q,r} & 0 \\ J_{r,p} & J_{r,q} & 0_r & 0 \\ 0 & 0 & 0 & 0_t \end{bmatrix}$

or $A = \begin{bmatrix} E_1^t \\ E_2^t \\ J^t \\ O^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & 0 \end{bmatrix} = \begin{bmatrix} 0_p & J_{p,q} & J_{p,r} & 0 \\ J_{q,p} & 0_q & J_{q,r} & 0 \\ J_{r,p} & J_{r,q} & 0_r & 0 \\ 0 & 0 & 0 & 0_t \end{bmatrix}$

The graph of the first matrix is $[(K_p \cup K_q) \vee K_r^c] \cup K_t^c$

The graph of the second is $K_{p,q,r} \cup K_t^c$

Every block in the above matrices is either a 0 matrix or a J matrix. So it would suffice to use $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, and compute

$$A = U^t U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

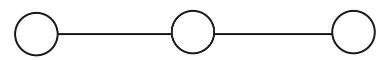
$$A = U^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

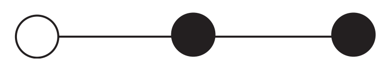
Then if we interpret each diagonal 1 as a clique and each diagonal 0 as an independent set, we again obtain the graphs in Corollary 1.

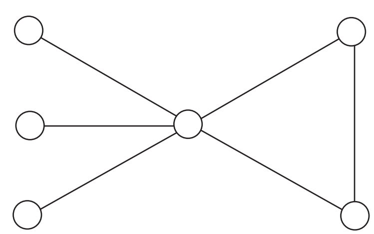
True also when rank $A > 2$.

Gives a straightforward process for constructing the graphs of all rank 3, rank 4, etc. matrices.

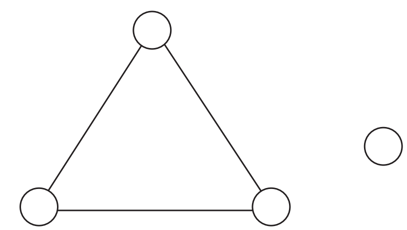
Complexity grows exponentially with rank.

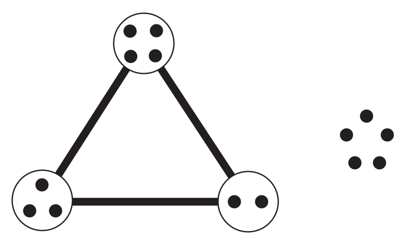
Definition Graph X: 

Marked graph X_B: 

Substitution graph: 

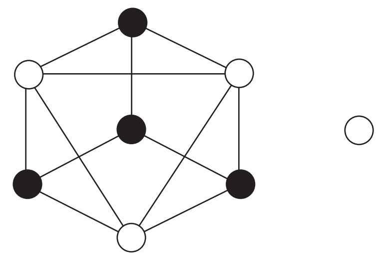
Example

What graphs are substitution graphs of  ?

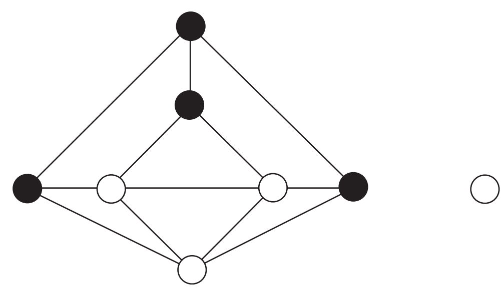
Example: 

Answer $K_{p,q,r} \cup K_t^c$

Let G_8 be the marked graph



G_8 is a planar graph as can be seen from the alternative representation



Theorem:

$\text{mr}(F_2, G) \leq 3 \iff G$ is a substitution graph of $G8$.

Proof: $A = U^t B U$, $B - 3 \times 3$, symmetric, invertible,

$$U = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

Consider possible forms for B. Interpret answer as a substitution graph. □

For each prime number p and each positive integer k, there is a family of marked graphs $\mathcal{F}_{p,k}$, such that

$\text{mr}(F_p, G) \leq k \iff G$ is a substitution graph of a graph in $\mathcal{F}_{p,k}$.

Open Questions:

1. How to recognize a substitution graph of G_8 or $X_{p,k}$

2. What are the forbidden subgraphs for G_8 or $X_{p,k}$?

3. Does every substitution graph of G_8 have minimum rank 3 over $\mathbb{R}$?

4. Find a graph G that has minimum rank 3 over $\mathbb{R}$ that is not a substitution graph of G_8.

5. Let G be any graph, let F_{fin} be any finite field, and F_∞ be any infinite field such char $F_{\mathsf{fin}} = \text{char}\, F_\infty$

   If $\text{mr}(F_{\mathsf{fin}}, G) \leq k$, is $\text{mr}(F_\infty, G) \leq k$?

6. Have substitution graphs occurred in any other settings in combinatorial matrix theory?