The Minimal Rank Problem for Finite Fields

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References


[BM] Barrett, March, The Minimal Rank Problem over a Finite Field with a Prime Number of Elements, in preparation
Setup

\[ F - \text{a field} \]
\[ G = (V, E) - \text{a graph} \]
\[ V = \{1, 2, \ldots, n\} \]

\[ S(F, G) \]
- set of all symmetric \( n \times n \) matrices \( A \) with graph \( G \).

This means: For \( i \neq j \)

\[ a_{ij} \neq 0 \iff ij \in E \]

no condition on the diagonal entries
Example:

\[ S(F, \text{paw}) = \left\{ \begin{bmatrix} a & w & x & 0 \\ w & b & y & 0 \\ x & y & c & z \\ 0 & 0 & z & d \end{bmatrix} \mid a, b, c, d, w, x, y \in F, wxyz \neq 0 \right\} \]

The zeros correspond to the missing edges 14, 24.
Problem

Given a field $F$ and a graph $G$, find

$$mr(F, G) = \min \{ \text{rank } A \mid A \in S(F, G) \}$$

Idea of the Question: How much can you tell about a matrix if you only know where the zeros are?

Related to other questions:

- maximum multiplicity of eigenvalues
- degeneracies in chemical bonding theory
The answer can depend on the field:

Let $G$ be the graph

clique sum of $K_4$ and $K_3$ on $K_2$

two missing edges: 15, 25
\( F = \mathbb{R} \):

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 2 & 2 & 1 \\
1 & 1 & 2 & 2 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

\( A \subseteq S(\mathbb{R}, G) \quad \text{rank} \ A = 2 \)

But if char \( F = 2 \), the 3, 4 element is \( 1 + 1 = 0 \), not 2
and this \( A \notin S(F, G) \)
\[ F = F_2: \]

Any \( A \in S(F_2, G) \) has the form

\[
\begin{bmatrix}
 d_1 & 1 & 1 & 1 & 0 \\
 1 & d_2 & 1 & 1 & 0 \\
 1 & 1 & d_3 & 1 & 1 \\
 1 & 1 & 1 & d_4 & 1 \\
 0 & 0 & 1 & 1 & d_5
\end{bmatrix}
\]

Know all off-diagonal entries now.

\( A[145|235] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & d_5 \end{bmatrix} \)

has determinant 1 so rank \( A \geq 3 \),

and \( \text{mr}(F_2, G) \geq 3 \).
Theorems (Barrett, van der Holst & Loewy)

\[ \text{char} F \neq 2 \]

\[
F \text{ infinite: } \quad \text{mr}(F, G) \leq 2 \iff G^c = (K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r
\]

\[
F \text{ finite with } p^t \text{ elements: } \quad \text{mr}(F, G) \leq 2 \iff \text{Either}
\]
\[
G^c = (K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \quad k \leq (p^t-1)/2,
\]

or
\[
G^c = (K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \quad k \leq (p^t + 1)/2
\]
Theorems \( \text{char} F = 2 \)

\( F \) infinite: \( \text{mr}(F, G) \leq 2 \iff \) Either

\[
G^c = (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \lor K_r
\]
or

\[
G^c = (K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \lor K_r
\]

\( F \) finite with \( 2^t \) elements: \( \text{mr}(F, G) \leq 2 \iff \) Either

\[
G^c = (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \lor K_r, \quad k \leq 2^t + 1,
\]
or

\[
G^c = (K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \lor K_r, \quad k \leq 2^{t-1}
\]
Corollary: $\text{mr}(F_2, G) \leq 2 \iff$ Either

\[ G^c = (K_p \cup K_q \cup K_r) \lor K_t \quad \text{or} \quad G^c = (K_r \cup K_{p,q}) \lor K_t \]

Recall the graph

![Graph Image]

It's complement is $P_3 \cup 2K_1$

Is it one of the forms above?
No dominating vertex in $P_3 \cup 2K_1$

Is $P_3 \cup 2K_1$ of the form $K_p \cup K_q \cup K_r$ or $K_r \cup K_{p,q}$?

Not $K_p \cup K_q \cup K_r$ because of $P_3$.

Not $K_r \cup K_{p,q}$ because $P_3 \cup K_1$ is not induced in $K_{p,q}$

But $P_3 \cup 2K_1$ is induced in the union of two bipartite graphs or in the union of two complete graphs and a bipartite graph $\implies$

its complement is a rank 2 graph for any other field.
The methods in the papers [BvdHL1], [BvdH2] do not extend in any straightforward way to the problem of characterizing graphs with \( \text{mr}(F, G) \leq k \) for \( k \geq 3 \).

However, it is possible to obtain results of this sort for finite fields using a quite different method which makes explicit use of the finiteness of \( F \).

The method is straightforward but the complexity grows exponentially with the rank \( k \).
We had

Corollary: $\text{mr}(F_2, G) \leq 2 \iff$ Either

$G^c = (K_p \cup K_q \cup K_r) \lor K_t$ \quad or \quad $G^c = (K_r \cup K_{p,q}) \lor K_t$

Taking complements

$\text{mr}(F_2, G) \leq 2 \iff$ Either

$G = K_{p,q,r} \cup K_t^c$ \quad or \quad $G = [(K_p \cup K_q) \lor K_r^c] \cup K_t^c$
Assume that $A \in S_n(F_2)$ with rank $A \leq 2$.

$\text{rank } A < 2 \iff A \text{ is the zero matrix, or is permutation similar to } J_k \oplus O_{n-k}$

So assume rank $A = 2$.

Then $A = U^t B U$,

$B - 2 \times 2$, symmetric, invertible $U - 2 \times n$
Either $B$ is congruent to $I_2$ or $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So WLOG either $A = U^t U$ or $A = U^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U$

$U$ has at most 4 distinct columns: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We may assume that identical columns are contiguous

Write $U = \begin{bmatrix} E_1 & E_2 & J & O \end{bmatrix}$
Either \( A = \begin{bmatrix} E_1^t \\ E_2^t \\ J^t \\ O^t \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & 0 \end{bmatrix} = \begin{bmatrix} J_p & 0 & J_{p,r} & 0 \\ 0 & J_q & J_{q,r} & 0 \\ J_{r,p} & J_{r,q} & 0 & 0 \\ 0 & 0 & 0 & 0_t \end{bmatrix} \)

or \( A = \begin{bmatrix} E_1^t \\ E_2^t \\ J^t \\ O^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & 0 \end{bmatrix} = \begin{bmatrix} 0_p & J_{p,q} & J_{p,r} & 0 \\ J_{q,p} & 0 & J_{q,r} & 0 \\ J_{r,p} & J_{r,q} & 0 & 0 \\ 0 & 0 & 0 & 0_t \end{bmatrix} \)

The graph of the first matrix is \([(K_p \cup K_q) \lor K_r^c] \cup K_t^c \)

The graph of the second is \(K_{p,q,r} \cup K_t^c \)
Every block in the above matrices is either a 0 matrix or a $J$ matrix. So it would suffice to use $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, and compute

$$A = U^t U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$A = U^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then if we interpret each diagonal 1 as a clique and each diagonal 0 as an independent set, we again obtain the graphs in Corollary 1.
True also when rank $A > 2$.

Gives a straightforward process for constructing the graphs of all rank 3, rank 4, etc. matrices.

Complexity grows exponentially with rank.

**Definition**  
Graph $X$: \[\text{Diagram of graph X}\]

Marked graph $X_B$: \[\text{Diagram of marked graph X_B}\]

Substitution graph: \[\text{Diagram of substitution graph}\]
Example

What graphs are substitution graphs of $K_{p,q,r} \cup K^c_t$?

Example:

Answer $K_{p,q,r} \cup K^c_t$
Let $G_8$ be the marked graph

$G_8$ is a planar graph as can be seen from the alternative representation

Theorem:

$\text{mr}(F_2, G) \leq 3 \iff G$ is a substitution graph of $G_8$. 
Proof: $A = U^t B U$, $B \in \mathbb{R}^{3 \times 3}$, symmetric, invertible,

$$U = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

Consider possible forms for $B$. Interpret answer as a substitution graph.

For each prime number $p$ and each positive integer $k$, there is a family of marked graphs $\mathcal{F}_{p,k}$, such that

$$\text{mr}(\mathcal{F}_p, G) \leq k \iff G \text{ is a substitution graph of a graph in } \mathcal{F}_{p,k}.$$
Open Questions:

1. How to recognize a substitution graph of $G_8$ or $X_{p,k}$

2. What are the forbidden subgraphs for $G_8$ or $X_{p,k}$?

3. Does every substitution graph of $G_8$ have minimum rank 3 over $\mathbb{R}$?

4. Find a graph $G$ that has minimum rank 3 over $\mathbb{R}$ that is not a substitution graph of $G_8$. 
5. Let $G$ be any graph, let $F_{\text{fin}}$ be any finite field, and $F_{\infty}$ be any infinite field such that $\text{char } F_{\text{fin}} = \text{char } F_{\infty}$.

If $\text{mr}(F_{\text{fin}}, G) \leq k$, is $\text{mr}(F_{\infty}, G) \leq k$?

6. Have substitution graphs occurred in any other settings in combinatorial matrix theory?