THE Q-MATRIX COMPLETION PROBLEM*

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Abstract. A real $n \times n$ matrix is a Q-matrix if for every $k = 1, 2, \ldots, n$ the sum of all $k \times k$ principal minors is positive. A digraph D is said to have Q-completion if every partial Q-matrix specifying D can be completed to a Q-matrix. In this paper we study the Q-completion problem, give sufficient conditions for a digraph to have Q-completion, give necessary conditions for a digraph to have Q-completion, and characterize those digraphs of order at most four that have Q-completion.

Key words. Partial matrix, matrix completion, Q-matrix, Q-completion, digraph.

AMS subject classifications. 15A48, 05C50

1. Introduction. A real $n \times n$ matrix A is a *P-matrix (P_0-matrix)* if for every $k = 1, 2, \ldots, n$, all the $k \times k$ principal minors are positive (nonnegative). A real $n \times n$ matrix A is a *P_0^+-matrix* if for every $k = 1, 2, \ldots, n$, all the $k \times k$ principal minors are nonnegative and at least one principal minor of each order is positive. A real $n \times n$ matrix A is a *Q-matrix* if for every $k = 1, 2, \ldots, n$, $S_k(A) > 0$, where $S_k(A)$ denotes the sum of all $k \times k$ principal minors. Clearly the set of P_0^+-matrices is the intersection of the set of P_0-matrices with the set of Q-matrices, and every P-matrix is a P_0^+ matrix. Spectral properties of Q-matrices and other related classes are discussed in [3], [5] and [6].

A *partial matrix* is a square array in which some entries are specified, while others are free to be chosen. A partial matrix B is a *partial P-matrix* if every fully specified principal submatrix of B has positive determinant. In many cases, the pattern of specified entries is sufficient to guarantee that any partial P-matrix with this pattern of specified entries can be completed to a P-matrix, and the (combinatorial) P-matrix completion problem involves the study of such patterns. Patterns of entries are usually described by digraphs (see Subsection 1.1). The P-matrix completion problem has been studied (e.g., [9], [4]); for a discussion and bibliography of completion problems of P-matrix classes see [7] and [8].

We can define a partial Q-matrix in a similar way: A partial matrix B is a *partial Q-matrix* if $S_k(B) > 0$ for every $k = 1, 2, ..., n$ for which all $k \times k$ principal submatrices are fully specified. A more useful characterization of a partial Q-matrix is given in Section 2.

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1.1. Digraphs. Most of the following graph-theoretic terms can be found in any standard reference, such as [1], [11], or [12] (note that graph terminology varies with the author: what we call a digraph is called a simple digraph in [12]). A *digraph* $D = (V_D, A_D)$ consists of a non-empty finite set V_D of *vertices* and a set A_D of ordered pairs of vertices, called *arcs*. The *order* of D, denoted $|D|$, is the number of vertices of D. A digraph $H = (V_H, A_H)$ is a *subdigraph* of a digraph D if $V_H \subseteq V_D$ and $A_H \subseteq A_D$. A subdigraph $H = (V_H, A_H)$ of $D = (V_D, A_D)$ is an *induced subdigraph* if $A_H = (V_H \times V_H) \cap A_D$; in this case we also denote H by $D[V_H]$. A subdigraph H of digraph D is a *spanning* subdigraph if $V_H = V_D$. The *complement* of a digraph $D = (V_D, A_D)$ is the graph $\overline{D} = (V_D, A_{\overline{D}})$, where $(v, w) \in A_{\overline{D}}$ if and only if $(v, w) \notin A_D$. A digraph D is *symmetric* if $(v, w) \in A_D$ implies $(w, v) \in A_D$. A *complete digraph* on n vertices, denoted K_n, is a digraph having all possible arcs (including all loops).

A *path* P in a digraph D is a subdigraph of D whose distinct vertices and arcs can be written in an alternating sequence

$$v_1\,(v_1, v_2)\,v_2\,(v_2, v_3)\,v_3, \ldots, v_{k-1}\,(v_{k-1}, v_k)\,v_k.$$

This path can also be written as $v_1 \to v_2 \to \cdots \to v_k$ and can be denoted $D\langle v_1, v_k\rangle$. If $D\langle v_1, v_k\rangle$ is a path in D and (v_k, v_1) is also an arc in D, then $D\langle v_1, v_k\rangle$ together with (v_k, v_1) is a *cycle*. The *length* of a path P (respectively, cycle C), denoted $|P|$ (respectively, $|C|$), is the number of its arcs. A *k-cycle* is a cycle of length k; a 1-cycle (i.e., an arc $(v, v), v \in V_D$) is generally called a *loop*.

It is useful to associate a partial matrix with a digraph that describes the positions of the specified entries in the partial matrix. Let $\langle n\rangle = \{1, 2, \ldots, n\}$. We say that an $n \times n$ partial matrix B *specifies* a digraph D if $D = (\langle n\rangle, A_D)$, and for $1 \le i, j \le n$, $(i, j) \in A_D$ if and only if the entry b_{ij} of B is specified.

A digraph D has *P-completion* if every partial P-matrix specifying D can be completed to a P-matrix. The study of the P-matrix completion problem was initiated in [9], where it was shown that any symmetric digraph has P-completion, and an example was given of a digraph that does not have P-completion. Similarly, we say a digraph D has *Q-completion* if every partial Q-matrix specifying D can be completed to a Q-matrix.

There is another way to associate a digraph to a (complete) matrix that describes the positions of the nonzero entries of the matrix; this association is particularly useful when studying Q-completions of families of matrices with certain specified entries. The *digraph* of an $n \times n$ matrix A is $\mathcal{D}(A) = (\langle n\rangle, A_{\mathcal{D}(A)})$ where
$A_{\mathcal{D}(A)} = \{(i, j) : a_{ij} \neq 0\}$.

Let π be a permutation of V. A *permutation digraph* is a digraph of the form $D_\pi = (V, A_\pi)$ where $A_\pi = \{(v, \pi(v)) : v \in V\}$. Clearly each component of a permutation digraph is a cycle.

Observation 1.1. ([2, p. 292]). *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \sum (\mathrm{sgn}\ \pi) a_{1\pi(1)} \cdots a_{n\pi(n)}$$

where the sum is taken over all permutations π of $\langle n\rangle$ such that D_π is a spanning subdigraph of $\mathcal{D}(A)$, and the sum over the empty set is zero.

A *permutation subdigraph* of a digraph D is a permutation digraph that is a subdigraph of D. A digraph D is *stratified* if D has a permutation subdigraph of order k for every $k = 2, 3, ..., |D|$.

A *signing* of a digraph is an assignment of a sign ($+$ or $-$) to each arc of the digraph; the sign of arc (v, w) is denoted $\mathrm{sgn}(v, w)$. The result of a signing is called a *signed digraph.* The *sign of a path* $P = v_1 \to v_2 \to \cdots \to v_k$ in a signed digraph is the product of the signs of the arcs of the path. The *sign of a cycle* $C = v_1 \to v_2 \to \cdots \to v_k \to v_1$ in a signed digraph is $\mathrm{sgn}(C) = (-)^{k+1}\mathrm{sgn}(v_1, v_2) \dots \mathrm{sgn}(v_{k-1}, v_k)\mathrm{sgn}(v_k, v_1)$.

1.2. Overview. In this paper we study the following (combinatorial) Q-matrix completion problem: Determine which digraphs have the property that every partial Q-matrix specifying the digraph can be completed to a Q-matrix. That is, determine which digraphs have Q-completion. Our main results are presented in Section 2. Specifically, we establish some sufficient conditions for a digraph to have Q-completion (Subsection 2.1) and some necessary conditions for a digraph to have Q-completion (Subsection 2.2). There remains a gap between these criteria, and in Subsection 2.4 we present techniques that are useful for completing partial Q-matrices specifying digraphs not classified by either the sufficient or necessary conditions. In Subsection 2.3 we classify digraphs of order at most four as to Q-completion.

The property of being a Q-matrix is preserved under similarity and transposition, but it is not inherited by principal submatrices as can easily be verified. Thus the Q-matrix completion problem is quite different from completion problems involving P-matrix classes, where principal submatrices inherit the properties of the class under consideration. In Subsection 2.5 we discuss the relationship between the (combinatorial) Q-matrix completion problem and the (combinatorial) P-matrix completion problem.

2. Partial Q-matrices and the Q-matrix completion problem.

Recall that a partial matrix B is a partial Q-matrix if $S_k(B) > 0$ for every $k = 1, 2, ..., n$ for which all $k \times k$ principal submatrices are fully specified. Note that if all 1×1 principal submatrices of B are fully specified (i.e., all diagonal elements are specified), this implies $\mathrm{Tr}(B) > 0$, and if for some $k \geq 2$, all $k \times k$ principal submatrices are fully specified, then B is fully specified (and therefore B is a Q-matrix).

PROPOSITION 2.1. *A partial matrix B is a partial Q-matrix if and only if exactly one of the following holds:*

(i) at least one diagonal entry of B is not specified,

(ii) all diagonal entries of B are specified, $\mathrm{Tr}(B) > 0$*, and at least one off-diagonal entry of B is not specified,*

(iii) all entries of B are specified and B is a Q-matrix.

2.1. Sufficient conditions for Q-completion. Notice that if B specifies the digraph D, then the entries in B in positions described by $\overline{D}$ are free to be chosen to complete B. In the next example we use the idea of signing cycles positively in $\overline{D}$ to obtain a completion to a Q-matrix.

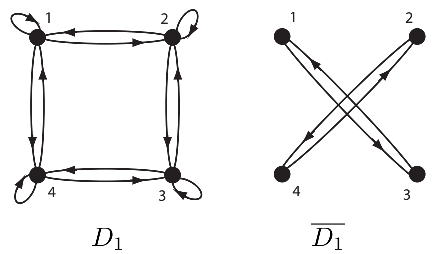

FIG. 2.1. *The digraph D_1 has Q-completion*

EXAMPLE 2.2. Let D_1 be the digraph in Figure 2.1. We sign the arcs of $\overline{D_1}$ as follows: $\text{sgn}(1,3) = \text{sgn}(2,4) = +$ and $\text{sgn}(3,1) = \text{sgn}(4,2) = -$. Then the sign of each of the two 2-cycles in $\overline{D_1}$ is $+$. For a partial Q-matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & x_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & x_{24} \\ x_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & x_{42} & b_{43} & b_{44} \end{bmatrix}$$

specifying D_1, define the completion $A(t)$ of B by choosing each unspecified entry to be the sign of its arc times t, i.e., $x_{13} = x_{24} = t, x_{31} = x_{42} = -t$. Then

$$\begin{aligned} &S_1(A(t)) = b_{11} + b_{22} + b_{33} + b_{44}, \\ &S_2(A(t)) = 2t^2 + q_2(t) && \text{where } \deg q_2(t) \le 1, \\ &S_3(A(t)) = (b_{11} + b_{22} + b_{33} + b_{44})\, t^2 + q_3(t) && \text{where } \deg q_2(t) \le 1, \\ &S_4(A(t)) = t^4 + q_4(t) && \text{where } \deg q_4(t) \le 2. \end{aligned}$$

Since B is a partial Q-matrix, $b_{11} + b_{22} + b_{33} + b_{44} > 0$. Therefore, for t large enough, $S_k(A(t)) > 0$ for all $k = 1, 2, 3, 4$.

THEOREM 2.3. *Let D be a digraph such that $\overline{D}$ is stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is $+$, then D has Q-completion.*

Proof. Given a partial Q-matrix B, a completion $A(t)$ is constructed by setting the unspecified entry $x_{ij} = \text{sgn}(i,j)t$ (using the sign of the arc in $\overline{D}$). Then for each $k = 2, \ldots, n$,

$$S_k(A(t)) = c_k t^k + q(t)$$

where c_k is the number of permutation digraphs of order k in $\overline{D}$ and $q(t)$ is a polynomial of degree less than k. If D contains all loops, then the trace of any partial Q-matrix specifying D is positive; if D omits a loop, then $S_1(A(t)) = c_1 t + q_0$ where c_1 is the number of loops in $\overline{D}$. So choosing t sufficiently large results in a Q-matrix. ☐

The next theorem shows that Q-completion is inherited by spanning subdigraphs of any digraph that is not complete and has Q-completion.

THEOREM 2.4. *If D is a spanning subdigraph of an order n digraph $\hat{D} \neq K_n$ that has Q-completion, then D has Q-completion. Equivalently, if D is a digraph of order n, and a digraph $\hat{D} \neq K_n$ obtained from D by adding one or more arcs to D has Q-completion, then D has Q-completion.*

Proof. Suppose $\hat{D}$ is obtained from D by adding the one arc (i, j), $\hat{D} \neq K_n$, and $\hat{D}$ has Q-completion (note that $j = i$ is permitted). Let B be a partial Q-matrix specifying D. Construct a partial matrix $\hat{B}$ that specifies $\hat{D}$ by choosing the i, j-entry x_{ij} of B as follows:

$$x_{ij} = \begin{cases} 0 & \text{if} \quad j \neq i \text{ or } \hat{D} \text{ omits a loop} \\ 1 - \sum_{k \neq i} b_{kk} & \text{if} \quad j = i \text{ and } \hat{D} \text{ includes all loops} \end{cases}$$

Then $\hat{B}$ is a partial Q-matrix, and any completion of $\hat{B}$ to a Q-matrix completes B to a Q-matrix. □

We obtain the following corollary from Theorem 2.4 and Theorem 2.3.

COROLLARY 2.5. *If D is a digraph and $\overline{D}$ has a stratified spanning subdigraph that has a signing in which the sign of every cycle is $+$, then D has Q-completion.*

EXAMPLE 2.6. The digraph D_2 in Figure 2.2 has Q-completion by Corollary 2.5,

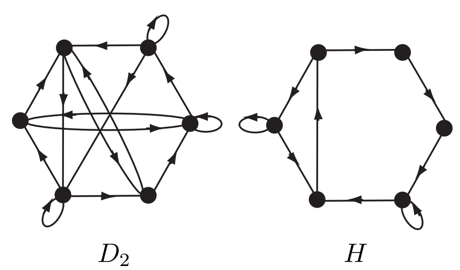

FIG. 2.2. *The digraph D_2 and stratified spanning subdigraph H of $\overline{D_2}$*

because H is a stratified spanning subdigraph of $\overline{D_2}$ that can be signed so that each cycle is $+$ (e.g., by signing all arcs $+$).

Additional techniques for establishing Q-completion of digraphs that have minimal complements are discussed in Subsection 2.4.

2.2. Necessary conditions for Q-completion. In our examples (and some of our results) on digraphs that omit a loop and have Q-completion, $\overline{D}$ is stratified. The next theorem shows that for a digraph D that omits at least one loop, stratification of $\overline{D}$ is a necessary condition for D to have Q-completion

THEOREM 2.7. *Let D be a digraph of order n that omits at least one loop. If D has Q-completion, then $\overline{D}$ is stratified.*

Proof. Suppose D has Q-completion. Let $k \geq 2$, and assume $\overline{D}$ has no order k permutation digraph. If B_0 is the partial matrix that specifies D with all specified entries zero, and A is a completion of B_0, then all $k \times k$ principal minors of A are zero, so A is not a Q-matrix. This implies that $\overline{D}$ must be stratified. □

The converse of Theorem 2.7 is true for the examples we have checked, including all digraphs of order at most four (cf. subsection 2.3). We do not know whether it is true in general.

QUESTION 2.8. *Let D be a digraph that omits a loop such that $\overline{D}$ is stratified. Must D have Q-completion?*

COROLLARY 2.9. *Let D be an order n digraph that omits at least one loop and such that $|A_D| > n(n-1)$. Then D does not have Q-completion.*

Proof. If D has more than $n(n-1)$ arcs (including loops), then $\overline{D}$ has fewer than $n^2 - n(n-1) = n$ arcs. Thus $\overline{D}$ does not contain an order n permutation digraph. Therefore by Theorem 2.7, D does not have Q-completion. □

It is possible to find a digraph D that omits a loop, has $|A_D| = n(n-1)$, and has Q-completion.

EXAMPLE 2.10. Let D_3 be the digraph shown in Figure 2.3. Then $|A_{D_3}| = 3(3-1)$, $\overline{D_3}$ is stratified, and it is easy to sign $\overline{D_3}$ so that all cycles are positive. Thus D_3 has Q-completion.

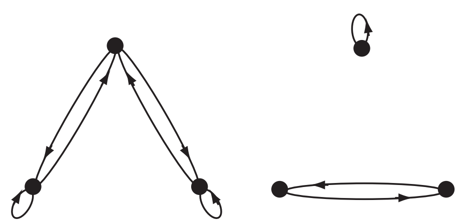

FIG. 2.3. *An order 3 digraph that has Q-completion and has 6 arcs*

In the case of a digraph D that contains all loops, it is not necessary that $\overline{D}$ be stratified, as can be seen by considering the digraph D_1 In Example 2.2. The next theorem gives a necessary but not sufficient condition for a digraph containing all loops to have Q-completion.

THEOREM 2.11. *Let $D \neq K_n$ be an order n digraph that includes all loops and has Q-completion. Then for each $k = 2, 3, \ldots, n$, either*

(i) $\overline{D}$ has a permutation digraph of order k, or

(ii) for each $v \in V_D, \overline{D} - v$ has a permutation digraph of order $k-1$.

Proof. Let $k \geq 2$, and assume $\overline{D}$ has no order k permutation digraph and there exists a vertex $v \in V_D$ such that $\overline{D} - v$ does not have an order $k-1$ permutation digraph. Let B be a matrix that specifies D, with $b_{vv} = 1$ and all other specified entries equal to zero, and suppose A is a completion of B. Then all $k \times k$ principal minors of A are zero, and thus, A is not a Q-matrix. □

The converse to Theorem 2.11 is false, as the next example demonstrates.

EXAMPLE 2.12. Let D_4 be the digraph shown in Figure 2.4. Then $\overline{D_4}$ has permutation subdigraphs of orders $2, 4$, and 5, and for each $v = 1, \ldots, 5$, $\overline{D_4} - v$ has a permutation digraph of order 2, so D_4 satisfies the conclusion of Theorem 2.11.

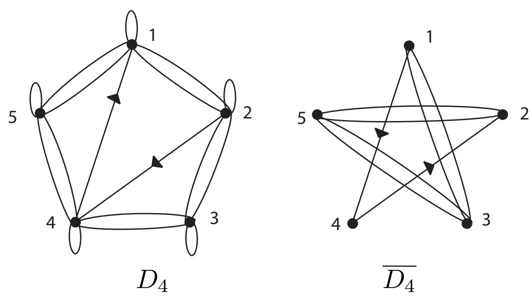

FIG. 2.4. *A counterexample to the converse of Theorem 2.11*

However, the partial Q-matrix $B = \begin{bmatrix} 2 & 0 & x & y & 0 \\ 0 & 2 & 0 & 0 & w \\ u & 0 & 2 & 0 & v \\ 0 & t & 0 & -7 & 0 \\ 0 & r & s & 0 & 2 \end{bmatrix}$ specifies D_4 and cannot be completed to a Q-matrix, because the sum of the 2×2 principal minors is $-32 - sv - rw - ux$, forcing $sv + rw + ux$ to be negative but the sum of the 3×3 principal minors is $-136 + 3sv + 3rw + 3ux$, forcing $sv + rw + ux$ to be positive.

COROLLARY 2.13. *Let $D \neq K_n$ be an order n digraph that includes all loops and such that D has more than $(n-1)^2 - 1$ non-loop arcs. Then D does not have Q-completion.*

Proof. If D has more than $(n-1)^2 - 1$ non-loop arcs, then $\overline{D}$ has fewer than $n(n-1) - (n-1)^2 + 1 = n$ arcs. Thus $\overline{D}$ does not contain an order n permutation digraph and contains at most one order $n-1$ permutation digraph. Therefore by Theorem 2.11, D does not have Q-completion. □

The digraph D_1 in Figure 2.1 shows it is possible to find a digraph D that has Q-completion, contains all loops, and has $(n-1)^2 - 1$ non-loop arcs.

COROLLARY 2.14. *Let D be an order n digraph that includes all loops and has Q-completion. Then $\overline{D}$ has a 2-cycle.*

Proof. If $\overline{D}$ does not have a 2-cycle, then for each $v \in V_D$, $\overline{D} - v$ has a permutation digraph of order 1, that is, $\overline{D} - v$ has a loop, which is false because D includes all loops. □

The next corollary follows from Theorem 2.7 and Corollary 2.14.

COROLLARY 2.15. *If D is a tournament of order n that includes all loops or omits exactly one loop, then D does not have Q-completion.*

The following result is also a corollary of Theorem 2.11 but the direct proof is as easy.

PROPOSITION 2.16. *If D contains all loops and has a vertex v such that for all $w, (v, w) \in A_D$ (respectively, for all $w, (w, v) \in A_D$), then D does not have Q-completion.*

Proof. Choose $w \neq v$. Construct a partial Q-matrix B specifying D by setting $b_{ww} = 1$ and all other specified entries equal to 0. Since B has a row of zeros (respectively, a column of zeros), $\det(A) = 0$ for any completion A of B, so B cannot be completed to a Q-matrix. □

2.3. Classification of small digraphs as to Q-completion. We can apply the previous theorems to classify the digraphs of order at most four that include all loops as to Q-completion.

A family of digraphs that is useful in this classification is the family of fan digraphs. The *fan* of order n is $F_n = (\langle n \rangle, P_n \cup S_n)$ where P_n is the path $1 \to 2 \to \cdots \to n)$ and S_n is the star with arcs $(k, 1), k = 2, \ldots, n$. See Figure 2.5.

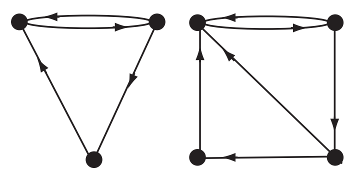

FIG. 2.5. *The fans F_3 and F_4*

PROPOSITION 2.17. *If D is a digraph of order n and F_n is a subdigraph of $\overline{D}$, then D has Q completion.*

Proof. Note that F_n is stratified and if we sign the arcs of F_n as $\operatorname{sgn}(k, k+1) = +$ for $k = 1, \ldots, n-1$ and $\operatorname{sgn}(k, 1) = (-)^{k+1}, k = 2, \ldots, n$, then the sign of every cycle is $+$. So D has Q-completion by Theorem 2.3. □

As noted earlier, when trying to find a Q-completion of a digraph D, we can always add arcs to D (equivalently, remove arcs from $\overline{D}$) as long as the remaining digraph has Q-completion and is not K_n (the complement is not the empty graph). A digraph $\overline{D}$ is a *Q-minimal complement* if D has Q-completion and the deletion of any arc from $\overline{D}$ (addition of any arc to D) results in a digraph that does not have Q-completion. The fan digraphs F_n are Q-minimal complements, by Proposition 2.17 and Theorem 2.11. The complete matching digraph (the digraph D66 in Figure 2.6 below) is also a Q-minimal complement.

THEOREM 2.18. *Let D be a digraph that contains all loops.*

1. *Let $|D| = 2$. Then D has Q-completion if and only $\overline{D} = K_2$ or $D = K_2$.*
2. *Let $|D| = 3$. Then D has Q-completion if and only if its complement $\overline{D}$ has F_3 as a subdigraph or $D = K_3$.*
3. *Let $|D| = 4$. Then D has Q-completion if and only if its complement $\overline{D}$ has one of the six digraphs shown in Figure 2.6 as a subdigraph or $D = K_4$.*

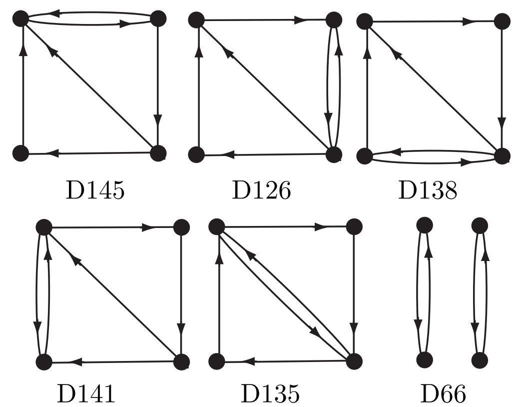

FIG. 2.6. *Order four Q-minimal complements (with names from* [10]*)*

Proof. For order less than four the stated results are an immediate consequence of Proposition 2.17, Theorem 2.4, and Theorem 2.11.

We show that any digraph whose complement has one of the digraphs in Figure 2.6 as a subdigraph has Q-completion. This has already been established for the first and last of these digraphs. Each of the remaining digraphs is stratified and for each it is easy to choose a signing that makes the sign of every cycle + (for D135, all arcs can be signed + except one of the 2-cycle arcs and another arc on its 3-cycle should be signed $-$).

With the exception of the complete digraph, any order four digraph whose complement does not contain one of the digraphs in Figure 2.6 fails the necessary condition given in Theorem 2.11 (in some cases this is more easily seen by applying one of its corollaries or Proposition 2.16). □

Of the 16 digraphs of order three that include all loops, four have Q completion ($D = K_3$ and D such that $\overline{D}$ is F_3, the digraph with 5 arcs, or K_3). Of the 218 digraphs on four vertices that include all loops, 72 have Q-completion (including K_4) and 146 do not have Q-completion. The order four digraphs whose complements have Q-completion are (in the nomenclature of [10]): D21, D66, D100, D126, D127, D134, D135, D138, D140, D141, D144, D145, D148, D150, D163, D164, D167, D168, D170, D171, D173, D174, D178 - D180, D183 - D188, D190, D191, D195 - D205, D207 - D211, D214 - D218, D220 - D230, D232 - D238.

2.4. Additional examples and strategies for signing digraph complements. In most of the previous examples where signing arcs of $\overline{D}$ is used to obtain Q-completion of D, it is easy to choose a signing that makes the sign of each cycle positive, because $\overline{D}$ has few arcs and each cycle has an arc that is not an arc in other cycle (hereafter called a *free arc*). Arc $(k, 1)$ is a free arc of cycle $(1\,2\,\ldots\,k)$ in the fan digraph F_n (see Figure 2.5).

A digraph is *minimally stratified* if is stratified and the deletion of any arc results in a digraph that is not stratified. The fan digraphs F_n (see Figure 2.5) are minimally stratified. Unfortunately, it is not always the case that in a minimally stratified digraph each cycle has a free arc, as we see from the digraph D135 in Figure 2.6 and the digraph T in Example 2.19 below (both do have signings with all cycles positive; cf. Example 2.21 for T).

EXAMPLE 2.19. The digraph T shown in Figure 2.7 is minimally stratified but no arc is free. If one attempts to go through one cycle at a time and sign the arcs of each cycle so the sign of the cycle is $+$, one ends up with a cycle whose sign has been determined without consideration of whether that cycle has been signed $+$.

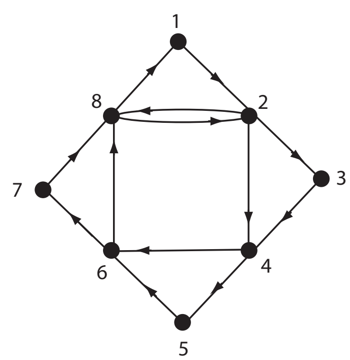

FIG. 2.7. *T is minimally stratified and difficult to sign*

We now introduce a strategy that can be used to sign the arcs of T (and other digraphs) so that the sign of every cycle is $+$. Let $P = P\langle v, w\rangle$ be a path in a digraph H. A *bypass* of P is a path $P' = P'\langle v, w\rangle$ such that $V_H \cap V_{P'} = \{v, w\}$ and $A_H \cap A_{P'} = \emptyset$. For the induced subdigraph $T[\{1, 2, 4, 5, 6, 7, 8\}]$, the path $2 \to 3 \to 4$ in T is a bypass of path $2 \to 4$.

THEOREM 2.20. *Let H be a digraph such that the sign of every cycle in H is $+$. Let $P = P\langle v, w\rangle$ be a path in H such that for any path R from w to v in H, $R \cup P$ is a cycle. Let $P' = P'\langle v, w\rangle$ be a bypass of path $P = P\langle v, w\rangle$ and $H' = H \cup P'$. Sign the arcs of P' so that $\text{sgn}(P') = (-)^{|P'|-|P|}\text{sgn}(P)$. Then the sign of every cycle in H' is $+$.*

Proof. Suppose C' is a cycle in H' that is not contained in H. Let $W = V_{C'} \cap V_H$. Then C' is the union of the path $C'[W]$ from w to v and the path P' from v to w, and $|C'| = |C'[W]| + |P'| - 2$. By hypothesis, $C = C'[W] \cup P$ is a cycle in H, and $|C| = |C'[W]| + |P| - 2$. Since every cycle of H has sign $+$,

$$\begin{aligned}
\text{sgn}(C') &= (-)^{|C'[W]|+|P'|-1}\text{sgn}(C'[W])\text{sgn}(P') \\
&= (-)^{|C'[W]|+|P'|-1}\text{sgn}(C'[W])(-)^{|P'|-|P|}\text{sgn}(P) \\
&= (-)^{|C'[W]|+|P|-1}\text{sgn}(C'[W])\text{sgn}(P) \\
&= \text{sgn}(C) \\
&= +
\end{aligned}$$

$\square$

Note that if the path P (that is bypassed by P') is a single arc, then the hypothesis that for any path R from w to v in H, $R \cup P$ is a cycle is automatically true.

EXAMPLE 2.21. We can sign the digraph T in Figure 2.7 using bypasses. Start by signing the arcs of $T[\{2,4,6,8\}$ as follows:

$$\operatorname{sgn}(2,4) = \operatorname{sgn}(4,6) = \operatorname{sgn}(6,8) = \operatorname{sgn}(2,8) = +; \operatorname{sgn}(8,2) = -.$$

Note that the signs of the 2-cycle and 4-cycle are both $+$. Add the following bypasses, signing all arcs as shown, which accords with Theorem 2.20:

- Bypass $P' = 2 \to 3 \to 4$ for $P = 2 \to 4$ ($\operatorname{sgn}(2,3) = -$, $\operatorname{sgn}(3,4) = +$).
- Bypass $P' = 4 \to 5 \to 6$ for $P = 4 \to 6$ ($\operatorname{sgn}(4,5) = -$, $\operatorname{sgn}(5,6) = +$).
- Bypass $P' = 6 \to 7 \to 8$ for $P = 6 \to 8$ ($\operatorname{sgn}(6,7) = -$, $\operatorname{sgn}(7,8) = +$).
- Bypass $P' = 8 \to 1 \to 2$ for $P = 8 \to 2$ ($\operatorname{sgn}(8,1) = + = \operatorname{sgn}(1,2)$).

Then the sign of every cycle in T is $+$. Hence $\overline{T}$ has Q-completion by Theorem 2.3.

If we can find a signing for $\overline{D}$ that makes the sign of every cycle $+$, then the digraph D has Q-completion, and since we can make the problem easier by removing arcs from $\overline{D}$, it makes sense to focus our attention on the problem of finding a signing of a minimally stratified $\overline{D}$ in which every cycle is signed $+$. However, the next example shows that it may not be possible to find such a signing even when $\overline{D}$ is minimally stratified and D has Q-completion.

EXAMPLE 2.22. Let D_8 and its complement $\overline{D_8}$ be the digraphs shown in Figure 2.8.

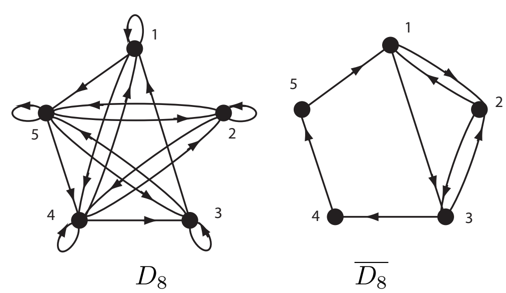

FIG. 2.8. $\overline{D_8}$ *is minimally stratified and impossible to sign so that the sign of every cycle is* $+$

Let

$$M = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & 0 & e & 0 & 0 \\ 0 & f & 0 & g & 0 \\ 0 & 0 & 0 & 0 & h \\ k & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We use the matrix M for the determination of cycles and signing, to show that $\overline{D_8}$ is minimally stratified but cannot be signed so every cycle is positive, even though M is not a Q-matrix since its trace is zero. Assuming all the parameters are nonzero, $\mathcal{D}(M) = \overline{D_8}$.

$$S_2(M) = -ac - ef \tag{2.1}$$
$$S_3(M) = bcf \tag{2.2}$$
$$S_4(M) = -bghk \tag{2.3}$$
$$S_5(M) = aeghk \tag{2.4}$$

Examination $S_k(M), k = 2, 3, 4, 5$ shows $\overline{D_8}$ is stratified and examination of $S_3(M)$, $S_4(M)$, and $S_5(M)$ shows that this stratification is minimal.

To sign all cycles +, the following signs are necessary:

$$bcf > 0 \text{ from (2.2)} \tag{2.5}$$
$$bghk < 0 \text{ from (2.3)} \tag{2.6}$$
$$cfghk < 0 \text{ from (2.5) and (2.6)} \tag{2.7}$$
$$aeghk > 0 \text{ from (2.4)} \tag{2.8}$$
$$acef < 0 \text{ from (2.7) and (2.8)} \tag{2.9}$$
$$ac < 0, ef < 0 \text{ from (2.1)} \tag{2.10}$$
$$acef > 0 \text{ from (2.10)} \tag{2.11}$$

But (2.9) and (2.11) are contradictory so it is not possible to sign this digraph so every cycle is signed +.

Finally, complete a partial Q-matrix $B = [b_{ij}]$ specifying D_8 as

$$A(t) = \begin{bmatrix} b_{11} & t^2 & -t^2 & b_{14} & b_{15} \\ t^2 & b_{22} & t^2 & b_{24} & b_{25} \\ b_{31} & -t^3 & b_{33} & t^2 & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & t^2 \\ t^2 & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}.$$

Then

$$S_1(A(t)) = b_{11} + b_{22} + b_{33} + b_{44} + b_{55} > 0$$
$$S_2(A(t)) = t^5 + q_2(t) \qquad \text{where } \deg q_2(t) \leq 4$$
$$S_3(A(t)) = t^7 + q_3(t) \qquad \text{where } \deg q_3(t) \leq 5$$
$$S_4(A(t)) = t^8 + q_4(t) \qquad \text{where } \deg q_4(t) \leq 7$$
$$S_5(A(t)) = t^{10} + q_5(t) \qquad \text{where } \deg q_5(t) \leq 9$$

So by choosing t sufficiently large we obtain a Q-completion of B. Thus D_8 has Q-completion.

2.5. Comparison of P-completion and Q-completion. Although every P-matrix is a Q-matrix, and every partial P-matrix is a partial Q-matrix, the completion problem for each of these classes is quite different as the following example shows.

EXAMPLE 2.23. We will establish the following:

1. The digraph D_1 in Figure 2.1 has both P-completion and Q-completion.
2. The digraph D_9 in Figure 2.9 has P-completion, but does not have Q-completion.
3. The digraph D_{10} in Figure 2.10 has Q-completion, but does not have P-completion.
4. The digraph D_{11} in Figure 2.11 has neither P-completion nor Q-completion.

Since the digraphs D_1 and D_9 in Figures 2.1 and 2.9 are symmetric, they have P-completion (see [9]). It was established in Example 2.2 that D_1 has Q-completion.

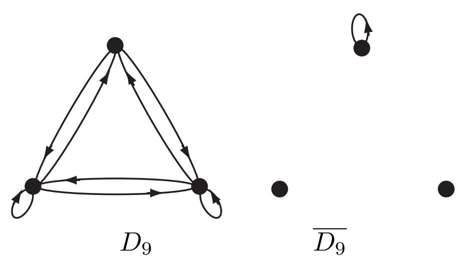

FIG. 2.9. *D_9: P-completion, no Q-completion*

The digraph D_9 in Figure 2.9 does not have Q-completion by Theorem 2.7, since $\overline{D_9}$ is not stratified.

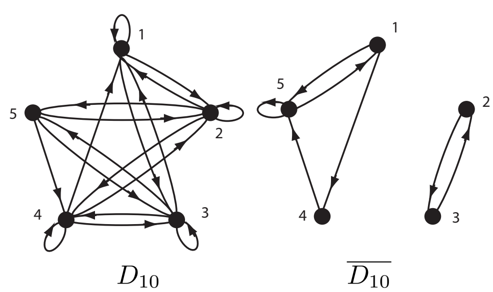

FIG. 2.10. *D_{10}: No P-completion, Q-completion*

The digraph D_{10} in Figure 2.10 does not have P-completion as can be seen by deleting vertex 5 (see [4]). Note that $\overline{D_{10}}$ is stratified. Sign the arcs of $\overline{D_{10}}$ as follows: $\mathrm{sgn}(5,5) = \mathrm{sgn}(1,4) = \mathrm{sgn}(4,5) = \mathrm{sgn}(5,1) = \mathrm{sgn}(2,3) = +$ and $\mathrm{sgn}(3,2) = \mathrm{sgn}(1,5) = -$. Then the sign of each cycle in $\overline{D_{10}}$ is $+$. So by Theorem 2.3, the digraph D_{10} has Q-completion.

The digraph D_{11} in Figure 2.11 does not have P-completion (see [9]). It does not have Q-completion by Theorem 2.11.

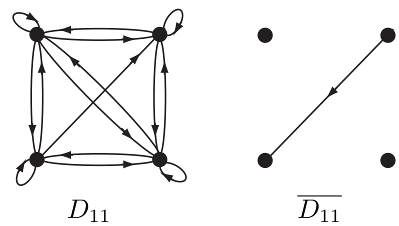

FIG. 2.11. *D_{11}: No P-completion, no Q-completion*

Since a principal submatrix of a P-matrix is a P-matrix, any induced subdigraph of a digraph having P-completion also has P-completion. This is not the case for Q-completion.

EXAMPLE 2.24. The digraph D_{12} in Figure 2.12 has Q-completion, because $\overline{D_{12}}$ is stratified and can be signed so the sign of each cycle is $+$ (e.g., $\text{sgn}(1,1) = \text{sgn}(1,3) = \text{sgn}(3,2) = \text{sgn}(2,1) = +$ and $\text{sgn}(3,1) = -$). But $\overline{D_{12}[\{1,2\}]}$ is not stratified, hence $D_{12}[\{1,2\}]$ does not have Q-completion.

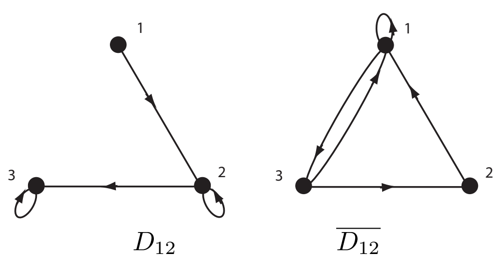

FIG. 2.12. *D_{12} has Q-completion but $D_{12}[\{1,2\}]$ does not*

3. Conclusion. For a digraph D that omits a loop, we have obtained a condition that is necessary for D to have Q-completion (Theorem 2.7); the question of whether this condition is sufficient remains open (Question 2.8). For a digraph D that contains all loops, we have obtained a condition that is necessary for D to have Q-completion (Theorem 2.11); this condition is not sufficient (see Example 2.12).

We have also obtained conditions that are sufficient to ensure D has Q-completion (Theorem 2.3); these conditions are not necessary for digraphs that contain all loops, in the sense that the complement need not be stratified (cf. Example 2.1), and even if it is minimally stratified there need not be a signing that makes all cycles of the complement positive (cf. Example 2.8).

REFERENCES

[1] J. Bang-Jensen and G. Gutin. *Digraphs.* Springer, London, 2002.
[2] R. A. Brualdi and H. J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, Cambridge, 1991.

[3] D. Hershkowitz. On the spectra of matrices having nonnegative sums of principal minors. *Linear Algebra and its Applications*, 55:81–86, 1983.
[4] L. M. DeAlba and L. Hogben. Completions of P-matrix patterns. *Linear Algebra and Its Applications*, 319 (2000), 83-102.
[5] D. Hershkowitz and C. R. Johnson. Spectra of matrices with P-matrix powers. *Linear Algebra and its Applications*, 80:159–171, 1986.
[6] D. Hershkowitz and N. Keller. Spectral properties of sign symmetric of matrices. *Electronic Journal of Linear Algebra*, 13:90–110, 2005.
[7] L. Hogben. Graph theoretic methods for matrix completion problems. *Linear Algebra and its Applications*, 328:161–202, 2001.
[8] L. Hogben, A. Wagnsness, Matrix completion problems. In *Handbook of Linear Algebra*, L. Hogben, Editor, Chapman & Hall/CRC Press, Boca Raton, 2007.
[9] C. R. Johnson and B. K. Kroschel. The Combinatorially Symmetric P-Matrix Completion Problem. *Electronic Journal of Linear Algebra*, 1:59–63, 1996.
[10] R. C. Read and R. J. Wilson. *An Atlas of Graphs.* Oxford University Press, Oxford, 1998.
[11] Jeffrey L. Stuart. Digraphs and Matrices. In *Handbook of Linear Algebra*, L. Hogben, Editor, Chapman & Hall/CRC Press, Boca Raton, 2007.
[12] Douglas West. *Introduction to Graph Theory.* Second edition, Prentice Hall, 2001.