# SIGN PATTERNS THAT REQUIRE OR ALLOW POWER-POSITIVITY*

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**Abstract.** A matrix $A$ is power-positive if some positive integer power of $A$ is entrywise positive. A sign pattern $\mathcal{A}$ is shown to require power-positivity if and only if either $\mathcal{A}$ or $-\mathcal{A}$ is nonnegative and has a primitive digraph, or equivalently, either $\mathcal{A}$ or $-\mathcal{A}$ requires eventual positivity. A sign pattern $\mathcal{A}$ is shown to be potentially power-positive if and only if $\mathcal{A}$ or $-\mathcal{A}$ is potentially eventually positive.

**Key words.** Power-positive matrix, eventually positive matrix, requires power-positivity, potentially power-positive, potentially eventually positive, sign pattern

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**1. Introduction.** A matrix $A \in \mathbb{R}^{n\times n}$ is called *power-positive* [2, 10] if there is a positive integer $k$ such that $A^k$ is entrywise positive ($A^k > 0$). Note that if $A$ is a power-positive matrix, then $-A$ is also power-positive, because $A^k > 0$ implies $(-A)^{2k} > 0$. If there is an odd positive integer $k$ such that $A^k > 0$, then $A$ is called *power-positive of odd exponent*. Power-positive matrices have applications to the study of stability of competitive systems in economics; see, e.g., [7, 8, 9]. A real square matrix $A$ is *eventually positive* if there exists a positive integer $k_0$ such that $A^k > 0$ for all $k \geq k_0$. An eventually positive matrix and its negative are both obviously power-positive.

A *sign pattern matrix* (or *sign pattern*) is a matrix having entries in $\{+, -, 0\}$. For a real matrix $A$, $\operatorname{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in $A$. If $\mathcal{A}$ is an $n \times n$ sign pattern, the *sign pattern*

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*class* (or *qualitative class*) of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\text{sgn}(A) = \mathcal{A}$.

If $\mathcal{P}$ is a property of a real matrix, then a sign pattern $\mathcal{A}$ *requires* $\mathcal{P}$ if every real matrix $A \in \mathcal{Q}(\mathcal{A})$ has property $\mathcal{P}$, and $\mathcal{A}$ *allows* $\mathcal{P}$ or is *potentially* $\mathcal{P}$ if there is some $A \in \mathcal{Q}(\mathcal{A})$ that has property $\mathcal{P}$. Sign patterns that require eventual positivity have been characterized in [4], and sign patterns that allow eventual positivity have been studied in [1]. Here we characterize patterns that require power-positivity (Theorem 2.6 and Corollary 2.7) and show that a sign pattern $\mathcal{A}$ allows power-positivity if and only if $\mathcal{A}$ or $-\mathcal{A}$ allows eventual positivity (Theorem 3.1).

**1.1. Definitions and notation.** Let $\mathcal{A} = [\alpha_{ij}]$ and $\hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ be sign patterns. If $\alpha_{ij} \neq 0$ implies $\alpha_{ij} = \hat{\alpha}_{ij}$, then $\mathcal{A}$ is a *subpattern* of $\hat{\mathcal{A}}$. For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *positive part* of $\mathcal{A}$ is $\mathcal{A}^+ = [\alpha^+_{ij}]$ where $\alpha^+_{ij}$ is $+$ if $\alpha_{ij} = +$ and 0 if $\alpha_{ij} = 0$ or $\alpha_{ij} = -$; the *negative part* of $\mathcal{A}$ is defined analogously (see [1]). Note that $\mathcal{A}^- = (-\mathcal{A})^+$. We use $[-]$ (respectively, $[+]$) to denote a (rectangular) sign pattern consisting entirely of negative (respectively, positive) entries. The *characteristic matrix* $C_{\mathcal{A}}$ of the sign pattern $\mathcal{A}$ is the $(0, 1, -1)$-matrix obtained from $\mathcal{A}$ by replacing $+$ by 1 and $-$ by $-1$.

For an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *signed digraph of* $\mathcal{A}$ is

$$\Gamma(\mathcal{A}) = (\{1, \ldots, n\}, \{(i, j) : \alpha_{ij} \neq 0\})$$

where an arc $(i, j)$ is positive (respectively, negative) if $\alpha_{ij} = +$ (respectively, $-$). Conversely, for a signed digraph $\Gamma$ on the vertices $\{1, \ldots, n\}$, the *sign pattern of* $\Gamma$ is $\text{sgn}(\Gamma) = [s_{ij}]$ where $s_{ij} = +$ (respectively, $-$) if there is a positive (respectively, negative) arc from vertex $i$ to vertex $j$, and $s_{ij} = 0$ otherwise. There is a one-to-one correspondence between sign patterns and signed digraphs on the vertices $\{1, \ldots, n\}$ and we adopt some sign pattern notation for signed digraphs. For example, $\mathcal{Q}(\Gamma) = \mathcal{Q}(\text{sgn}(\Gamma))$ and $C_\Gamma = C_{\text{sgn}(\Gamma)}$.

A signed digraph $\Gamma$ is called *primitive* if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. This definition applies the standard definition of "primitive" for a digraph that is not signed to a signed digraph by ignoring the signs. Clearly for a sign pattern $\mathcal{A}$, $\Gamma(\mathcal{A})$ is primitive if and only if $\Gamma(-\mathcal{A})$ is primitive.

A *signed subdigraph* of a signed digraph is a subdigraph in which the arcs retain the signs of the original signed digraph. Let $\Gamma'$ be a signed digraph on $n$ vertices, and let $\Gamma$ be a signed subdigraph of $\Gamma'$ on $k$ vertices. Without loss of generality (by relabeling the vertices of $\Gamma'$) assume that the vertices of $\Gamma$ are $\{1, \ldots, k\}$. For $A = [a_{ij}] \in \mathcal{Q}(\Gamma)$, define the $n \times n$ matrix $B = [b_{ij}]$ by $b_{ij} = a_{ij}$ if $(i, j) \in \Gamma'$, and 0 otherwise. Then we call $B$ the $\Gamma'$*-embedding* of $A$. Note that the sign pattern $\mathcal{B} = \text{sgn}(B)$ is a subpattern of $\text{sgn}(\Gamma')$. When a $\Gamma'$-embedding is used in Section

2, $\Gamma'$ is the signed digraph $\Gamma(\mathcal{A})$ of a sign pattern $\mathcal{A}$, and we assume the necessary relabeling has been done.

**1.2. Power-positive and eventually positive matrices.** This subsection contains some known results about power-positive matrices and their applications. Any matrix in the sign pattern class of the sign pattern in Example 3.4 below illustrates the well known fact that there exist power-positive matrices that are not eventually positive.

An eigenvalue $\lambda_0$ of a matrix $A$ is *strictly dominant* if $|\lambda_0| = \rho(A)$ and for every eigenvalue $\lambda \neq \lambda_0$, $|\lambda| < |\lambda_0|$. Every power-positive matrix $A$ has a unique real simple strictly dominant eigenvalue $\lambda_0$ having positive left and right eigenvectors [10]. Furthermore, if $A$ is power-positive of odd exponent, then $\lambda_0 = \rho(A)$; otherwise, $\lambda_0$ may be negative. For example, any negative matrix $A$ is power-positive (with only the even powers being positive), and in this case $\lambda_0 = -\rho(A)$. The next theorem can be deduced from [2] and the discussions on pages 43-47 in [10].

Theorem 1.1. [2, Theorem 3] *If $A$ is a power-positive matrix, then either $A$ or $-A$ has a positive simple strictly dominant eigenvalue having positive left and right eigenvectors.*

Theorem 1.2. [6, p. 329] *The matrix $A$ is eventually positive if and only if $A$ is power-positive of odd exponent.*

Theorem 1.3. [6, Theorem 1] *The matrix $A$ is eventually positive if and only if $A$ has a positive simple strictly dominant eigenvalue having positive left and right eigenvectors.*

Corollary 1.4. *$A$ is a power-positive matrix if and only if either $A$ or $-A$ is eventually positive.*

Remark 1.5. *Note that Corollary 1.4 is not in general true if positive is replaced by nonnegative. For example, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a power-nonnegative matrix since $A^2 \geq 0$, but neither $A$ nor $-A$ is eventually nonnegative.*

In economics, power-positive matrices arise in the context of stability of competitive systems. Let $A = B - sI,\ s > 0$. A system of dynamic equations [9] such as

$$\frac{dx}{dt} = Ax,\ x(0) = x_0 \tag{1.1}$$

can be interpreted as a system of price adjustment equations of competitive markets in a general equilibrium analysis.

Theorem 1.6. [9, Theorem 1] *The competitive system (1.1) is dynamically stable if and only if* $s > \rho(B)$ *and* $B$ *satisfies one of the following conditions:*

1. *$B$ is a power-positive matrix of odd exponent, or*
2. *$B$ is a power-positive matrix and the entries of a row or of a column of $B$ are all nonnegative.*

Note that the first of the two conditions on $B$ given in Theorem 1.6 is equivalent to the eventually positivity of $B$, while the second implies that $B$ is eventually positive. Furthermore, the matrix $-A = sI - B$ in such a dynamically stable system is a pseudo-$M$-matrix as defined in [6].

**2. Sign patterns that require power-positivity.** In [4] it is shown that $\mathcal{A}$ requires eventual positivity if and only if $\mathcal{A}$ is nonnegative and $\Gamma(\mathcal{A})$ is primitive. In this section we use similar perturbation techniques to show that a sign pattern $\mathcal{A}$ requires power-positivity if and only either $\mathcal{A}$ or $-\mathcal{A}$ is nonnegative and $\Gamma(\mathcal{A})$ is primitive.

Observation 2.1. *Let $\mathcal{A}$ be an $n \times n$ sign pattern, $\Gamma$ a signed subdigraph of $\Gamma(\mathcal{A})$, $A \in \mathcal{Q}(\Gamma)$ and $B$ the $\Gamma(\mathcal{A})$-embedding of $A$. Then the nonzero eigenvalues of $B$ are the nonzero eigenvalues of $A$, and the eigenvectors for the nonzero eigenvalues of $B$ are the eigenvectors of the corresponding eigenvalues of $A$, suitably embedded.*

It is well known that for any matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of $A$ are continuous functions of the entries of $A$. For a simple eigenvalue, the same is true of the eigenvector (see, for example, [5, p. 323]).

Lemma 2.2. *Let $\mathcal{A}$ be an $n \times n$ sign pattern, $\Gamma$ a signed subdigraph of $\Gamma(\mathcal{A})$ and $A \in \mathcal{Q}(\Gamma)$.*

1. *If every nonzero eigenvalue of $A$ is simple and $A$ does not have a nonnegative eigenvector, then $\mathcal{A}$ does not require power-positivity.*
2. *If $A$ has a simple strictly dominant eigenvalue $\rho(A)$ that does not have a nonnegative eigenvector, then $\mathcal{A}$ does not require power-positivity.*

*Proof.* Let $B$ be the $\Gamma(\mathcal{A})$-embedding of $A$. In either case, by Observation 2.1, the matrix $B$ retains the property of not having a nonnegative eigenvector for the relevant eigenvalue(s). Let $B(\varepsilon) = B + \varepsilon C_{\mathcal{A}}$, where $\varepsilon$ is chosen positive so that $B(\varepsilon) \in \mathcal{Q}(\mathcal{A})$, and sufficiently small so that for every simple eigenvalue of $B$, the corresponding eigenvalue and eigenvector of $B(\varepsilon)$ are small perturbations of the eigenvalue and eigenvector of $B$. In case 2, the spectral radius of $B(\varepsilon)$ is a perturbation of $\rho(A)$ because $\rho(A)$ is a strictly dominant eigenvalue. In either case, by continuity, the spectral radius of $B(\varepsilon)$ is a perturbation of one of the (nonzero) simple eigenvalues of $A$ that did not have a nonnegative eigenvector. Thus the matrix $B(\varepsilon)$ retains the property of not having a nonnegative eigenvector for its spectral radius, showing (by Theorem 1.1) that $B(\varepsilon)$ is not power-positive. ∎

LEMMA 2.3. *Let $\mathcal{A}$ be an $n \times n$ sign pattern. If $\Gamma(\mathcal{A})$ has a signed subdigraph $\Gamma$ that is a cycle having both a positive and a negative arc, then $\mathcal{A}$ does not require power-positivity.*

*Proof.* Suppose that the cycle $\Gamma$ is of length $k$ and has a positive arc $(p, q)$ and a negative arc $(r, s)$. Note that the characteristic polynomial of $C_\Gamma$ is $p_{C_\Gamma}(x) = x^k \pm 1$, so the eigenvalues of $C_\Gamma$ are all nonzero and simple. Furthermore, no eigenvector can have a zero coordinate, so any nonnegative eigenvector must be positive. Suppose that $C_\Gamma$ has a positive eigenvector $x = [x_i]$ corresponding to an eigenvalue $\lambda$. Then the equation $C_\Gamma x = \lambda x$ gives

$$x_q = \lambda x_p \quad \text{and} \quad -x_s = \lambda x_r.$$

As $x_p, x_q > 0$, it follows that $\lambda > 0$, but on the other hand, $x_r, x_s > 0$ implies that $\lambda < 0$, a contradiction. Thus, $C_\Gamma$ cannot have a nonnegative eigenvector corresponding to a nonzero eigenvalue. The result then follows from the first statement in Lemma 2.2. ∎

LEMMA 2.4. *Let $\mathcal{A}$ be an $n \times n$ sign pattern. If $\Gamma(\mathcal{A})$ contains a figure-eight signed subdigraph $\Gamma(s,t) = \Gamma_s \cup \Gamma_t$ (see Figure 2.1), where $\Gamma_s$ is a cycle of length $s \geq 2$ with all arcs signed positively and $\Gamma_t$ is a cycle of length $t \geq 2$ with all arcs signed negatively, and $\Gamma_s$ and $\Gamma_t$ intersect in a single vertex, then $\mathcal{A}$ does not require power-positivity.*

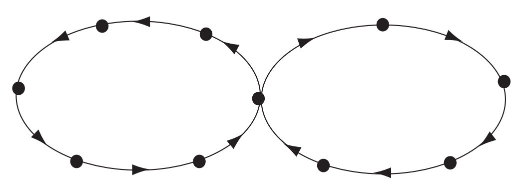

FIG. 2.1. *The figure-eight* $\Gamma(5,6)$

*Proof.* Without loss of generality, let $2 \leq s \leq t$. If $s < t$ or $s = t$ is even, the characteristic polynomial of $C_{\Gamma(s,t)}$ is

$$p_{C_{\Gamma(s,t)}}(x) = x^{s-1} g(x), \text{ where } g(x) = x^t - x^{t-s} + (-1)^{t+1}.$$

Note that $g(x)$ and $g'(x)$ have no common roots, so every nonzero eigenvalue of $C_{\Gamma(s,t)}$ is simple. Furthermore, as in the proof of Lemma 2.3, the cyclic nature of the digraph $\Gamma$ prevents any zeros in an eigenvector for a nonzero eigenvalue of $C_{\Gamma(s,t)}$, and the opposite signs prevent a positive eigenvector for a nonzero eigenvalue. The result now follows from the first statement in Lemma 2.2.

For the case $s = t$, where $s$ is odd, let $A \in \text{sgn}(\Gamma(s,s))$ be obtained from $C_{\Gamma(s,s)}$ by replacing one entry equal to 1 (in the positive cycle) by 2. Then $p_A(x) = x^{s-1}(x^s - 1)$ and the result follows by the same argument as above. □

COROLLARY 2.5. *If $\mathcal{A}$ requires power-positivity, then all off-diagonal entries are nonnegative, or all off-diagonal entries are nonpositive.*

*Proof.* If $\mathcal{A}$ requires power-positivity, then $\Gamma(\mathcal{A})$ is strongly connected and thus every arc in $\Gamma(\mathcal{A})$ lies in a cycle. Suppose that $\mathcal{A}$ has both a positive and a negative off-diagonal entry. Then $\Gamma(\mathcal{A})$ has a positive and a negative arc that lie on the same cycle, or $\Gamma(\mathcal{A})$ has two different cycles with arcs of opposite sign that intersect at a vertex. Lemma 2.3 or Lemma 2.4 implies that $\mathcal{A}$ does not require power-positivity. □

THEOREM 2.6. *The sign pattern $\mathcal{A}$ requires power-positivity if and only if either $\mathcal{A}$ or $-\mathcal{A}$ is nonnegative and $\Gamma(\mathcal{A})$ is primitive.*

*Proof.* Assume that $\mathcal{A}$ requires power-positivity. Then $\Gamma(\mathcal{A})$ is strongly connected. By Corollary 2.5, the off-diagonal entries are either all nonnegative or all nonpositive. Suppose that there is a diagonal entry of opposite sign from the nonzero off-diagonal entries. Without loss of generality, suppose that the off-diagonal entries are nonpositive and that the $(1,1)$ entry of $\mathcal{A}$ is $+$. Let $\Gamma$ be a signed subdigraph of $\Gamma(\mathcal{A})$ consisting of a cycle of length at least two that includes vertex 1 and the loop at vertex 1. Consider $A = C_\Gamma + 2E_{11} \in \text{sgn}(\Gamma)$, where $E_{11}$ has $(1,1)$ entry equal to one and zeros elsewhere. By Gershgorin's Theorem applied to $A$, there is a unique (necessarily real) eigenvalue $\rho$ in the unit disk centered at 3, and all other eigenvalues are in the unit disk centered at the origin, so $\rho = \rho(A)$ is simple and strictly dominant. Furthermore, no eigenvector of $A$ can have a zero coordinate. But the negative cycle entries do not allow a positive eigenvector for a positive eigenvalue. Thus by the second statement of Lemma 2.2, $\mathcal{A}$ does not require power-positivity, a contradiction. Thus either $\mathcal{A}$ or $-\mathcal{A}$ is nonnegative, and so $\Gamma(\mathcal{A})$ must be primitive [3, Theorem 3.4.4]. The converse is clear. □

COROLLARY 2.7. *The sign pattern $\mathcal{A}$ requires power-positivity if and only if either $\mathcal{A}$ or $-\mathcal{A}$ requires eventual positivity.*

*Proof.* The necessity follows from Theorem 2.6 and [4, Theorem 2.3] and the sufficiency is clear. □

**3. Sign patterns that allow power-positivity.** A square sign pattern $\mathcal{A}$ is called *potentially power-positive* (PPP) if there exists an $A \in \mathcal{Q}(\mathcal{A})$ that is power-positive. If $A \in \mathcal{Q}(\mathcal{A})$ exists such that $A$ is eventually positive, then the sign pattern $\mathcal{A}$ is called *potentially eventually positive* (PEP) [1]. Note that $\mathcal{A}$ is PPP if and only if $-\mathcal{A}$ is PPP. The following characterization of PPP sign patterns follows from Corollary 1.4.

THEOREM 3.1. *The sign pattern $\mathcal{A}$ is potentially power-positive if and only if $\mathcal{A}$ or $-\mathcal{A}$ is potentially eventually positive.*

Recall that $\mathcal{A}^+$ is the positive part of $\mathcal{A}$. Theorem 2.1 of [1] and Theorem 3.1 above give the following result.

THEOREM 3.2. *If $\Gamma(\mathcal{A}^+)$ or $\Gamma(\mathcal{A}^-)$ is primitive, then $\mathcal{A}$ is potentially power-positive.*

We next provide examples, including a sign pattern $\mathcal{A}$ such that both $\mathcal{A}$ and $-\mathcal{A}$ are PPP, sign patterns $\mathcal{A}$ that are PPP but not PEP, and a sign pattern that is not PPP.

EXAMPLE 3.3. The sign pattern $\mathcal{A} = \begin{bmatrix} + & + & - \\ - & 0 & + \\ + & - & - \end{bmatrix}$ is PPP, as is $-\mathcal{A}$, because both $\Gamma(\mathcal{A}^+)$ and $\Gamma(\mathcal{A}^-)$ are primitive.

EXAMPLE 3.4. The block sign pattern

$$\mathcal{A} = \begin{bmatrix} [-] & [-] \\ [-] & [+] \end{bmatrix},$$

(where the diagonal blocks are square and the diagonal $[-]$ block is nonempty) is PPP, because $\Gamma(\mathcal{A}^-)$ is primitive. However, $\mathcal{A}$ is clearly not PEP because the first row does not have a $+$ [6, p. 327].

EXAMPLE 3.5. A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is a $Z$ *sign pattern* if $\alpha_{ij} \neq +$ for all $i \neq j$. An $n \times n$ $Z$ sign pattern $\mathcal{A}$ with $n \geq 2$ cannot be PEP [1, Theorem 5.1], but if $\Gamma(\mathcal{A}^-)$ is primitive, then by Corollary 3.2, $\mathcal{A}$ is PPP. For $n \geq 3$, if $\mathcal{A}$ is an $n \times n$ $Z$ sign pattern having every off-diagonal entry nonzero, then $\Gamma(\mathcal{A}^-)$ is primitive and thus $\mathcal{A}$ is PPP.

Note that when $n = 2$, $\mathcal{A} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$ is not PPP, as in the next example, where Theorem 3.1 is used to show that a generalization of this sign pattern is not PPP.

EXAMPLE 3.6. Let

$$\mathcal{A} = \begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the diagonal blocks are square and there are at least 2 diagonal blocks. Then no subpattern of $\mathcal{A}$ is PPP because by [1, Theorems 5.3 and 3.2], no subpattern of $\mathcal{A}$ or $-\mathcal{A}$ is PEP. Thus by Theorem 3.1, no subpattern of $\mathcal{A}$ is PPP.

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