# Completions of P-Matrix Patterns

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**Abstract** A list of positions in an $n \times n$ real matrix (a pattern) is said to *have P-completion* if every partial P-matrix that specifies exactly these positions can be completed to a P-matrix. We extend work of Johnson and Kroschel [JK] by proving a larger class of patterns has P- completion, including any $4 \times 4$ pattern with eight or fewer off-diagonal positions. We also show that any pattern whose digraph contains a minimally chordal symmetric-Hamiltonian induced subdigraph does not have P-completion.

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## 1. Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix. A partial matrix *specifies the pattern* if its specified entries are exactly those listed in the pattern. A *matrix completion problem* for patterns asks what patterns of positions have the property that any specification of these positions that is a partial matrix of desired type can be completed to a matrix of that type. Matrix completion problems for patterns have been studied for positive definite matrices [GJSW], inverse M-matrices [JS], [Ho1], M-matrices [Ho2], P-matrices [JK], and P-matrices with various sign symmetry conditions [FJTU].

A real matrix is a *P-matrix* if the determinant of every principal submatrix is positive [HJ2]. The class of P-matrices generalizes many important classes of matrices, such as positive definite matrices, M-matrices, and inverse M-matrices, and arises in applications. In the case of positive definite matrices, P-matrices, and M-matrices, membership in the class is inherited by principal submatrices. Thus, to have a completion of a certain type it is certainly necessary that every completely specified principal submatrix be of the desired type. A partial

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matrix is a *partial P-matrix* if every fully specified principal submatrix is a P-matrix. A pattern *has P-completion* if every partial P-matrix that specifies the pattern can be completed to a P-matrix.

A pattern is *positionally symmetric* if whenever the position (i,j) is in the pattern, then (j,i) is in the pattern. (Although the usage of "positionally symmetric" is common, e.g., [JS], [Ho1], in [JK] the term "combinatorially symmetric" is used instead.) Johnson and Kroschel [JK] showed that any positionally symmetric pattern that contains the diagonal has P-completion, all $3 \times 3$ patterns have P-completion, and gave an example of a $4 \times 4$ pattern that does not have P-completion. We extend that work, enlarging the class of patterns known to have P-completion and providing a family of patterns that do not have P-completion.

Recall that the principal submatrix of the $n \times n$ matrix A defined by a subset T of {1,...,n} is the submatrix of A that lies in the rows and columns indexed by T and is denoted by A(T) (cf. [HJ1]). We will frequently apply a permutation similarity to a principal submatrix. A convenient way to denote this is to omit the set brackets and list the original row and column numbers in the order they appear after the permutation similarity. That is, if $i_1,...,i_k$ are distinct elements of {1,...,n}, the *reordered principal submatrix* $A(i_1,...,i_k)$ is obtained from $A(\{i_1,...,i_k\})$ by rearranging the rows and columns to the order $i_1,...,i_k$ (the p,q-entry of $A(i_1,...,i_k)$ is $a_{i_p i_q}$). Since the determinant is not affected by a permutation similarity, if A is a P-matrix then the determinant of any reordered principal submatrix must be greater than zero.

The *principal subpattern* Q(T) of a pattern Q for $n \times n$ matrices defined by a subset T of {1,...,n} is obtained from Q by deleting all positions whose row or column number is not in T. If a pattern Q has P-completion then so does any principal subpattern R, because any partial P-matrix A specifying R can be extended to a partial P-matrix A' specifying Q by setting additional specified diagonal elements equal to 1 and additional off-diagonal elements equal to 0. Since Q has P-completion, A' can be completed to a P-matrix, and this completion completes A to a P-matrix.

An immediate consequence of this is that if a pattern has P-completion then the principal subpattern defined by the diagonal positions of the pattern has P-completion. The converse of this statement is true for P-matrices: In fact, if B is a partial P-matrix and the principal submatrix A of B determined by the diagonal entries of B can be completed to a P-matrix, then so can B. First complete A, then set the remaining off-diagonal entries to zero and choose the remaining diagonal entries sufficiently large. Because all diagonal entries of a P-matrix are strictly positive, these can be chosen to assure that the determinant of any principal submatrix is positive. These remarks imply the following theorem.

**1.1 Theorem** A pattern with some diagonal positions unspecified has P-completion if and only if the principal subpattern defined by the diagonal positions of Q has P-completion.

Thus, it is necessary to consider only patterns that contain all the diagonal positions.

As with completions of patterns of inverse M-matrices, positive definite matrices and P-matrices, the graph plays a crucial role. Since the patterns here are not positionally symmetric, the digraph must be used instead of the graph. All the digraphs we use arise from patterns or matrices and, thus, are finite.

Let A be a (fully specified) $n \times n$ matrix. The *nonzero directed graph (nonzero-digraph)* of A is the digraph having vertex set $\{1,...,n\}$ and, as its set of arcs (directed edges), the set of ordered pairs $(i,j)$ where $i \neq j$ and $a_{ij} \neq 0$. The *characteristic matrix* of a pattern for $n \times n$ matrices is the $n \times n$ matrix C such that $c_{ij} = 1$ if the position $(i,j)$ is in the pattern and $c_{ij} = 0$ if $(i,j)$ is not in the pattern. The *directed graph of a pattern (pattern-digraph)* is the nonzero-digraph of its characteristic matrix. Equivalently, it is the digraph on $\{1,...,n\}$ with arc set $\{(i,j): i \neq j$ and position $(i,j)$ is in the pattern$\}$. The *order* of a digraph is the number of vertices.

The *underlying* graph G of a digraph D is the graph obtained by replacing each arc $(i,j)$ or pair of arcs $(i,j)$ and $(j,i)$ if both are present by the one edge $\{i,j\}$. Arc $(i,j)$ (or arcs $(i,j)$ and $(j,i)$ if both are present) of D and edge $\{i,j\}$ of G are said to *correspond*. It should be noted that any nonzero-digraph or pattern-digraph does not contain multiple arcs or loops, and neither does its underlying graph contain multiple edges or loops. As we use the terms "graph" and "digraph", multiple edges/arcs and loops are not allowed. A *pseudograph* is like a graph except it may contain loops or multiple edges.

A *subdigraph* of the digraph $D = (V,E)$ is a digraph $H = (V_H,E_H)$, where $V_H$ is a subset of V and $E_H$ is a subset of E. (Note: $(i,j)$ $E_H$ requires $i,j$ $V_H$.) If W is a subset of V, the *subdigraph induced by W, <W>,* is the digraph $(W,E_W)$ with $E_w = \{$all the arcs of D between the vertices in W$\}$. A subdigraph induced by a subset of vertices is also called an *induced subdigraph.*

A subdigraph is *complete* if it contains all possible arcs between its vertices. The principal subpattern $Q(T)$ is complete if $Q(T) = T \times T$.

For any arc (i,j), the *reverse arc* is (j,i) (whether it is present in the digraph or not). An arc (i,j) in a digraph D is *symmetric in D* if its reverse arc is also in D; otherwise (i,j) is *asymmetric in D*. A digraph D is *symmetric* if every arc of D is symmetric in D.

A *path* in a digraph (respectively graph) is a sequence of arcs $(i_1,i_2)$, $(i_2,i_3)$, ..., $(i_{k-1},i_k)$ (respectively edges $\{i_1,i_2\}$, $\{i_2,i_3\}$, ..., $\{i_{k-1},i_k\}$) in which the vertices are distinct (except that possibly the first vertex is the same as the last). The length of a path is the number of arcs (edges) in the path. A *cycle* is a path with the last vertex equal to the first.

A graph is connected if there is a path from any vertex to any other vertex. A digraph is *strongly connected* if there is a path from any vertex to any other vertex. Clearly, the underlying graph of a strongly connected digraph is connected, although the converse is false.

Renaming the vertices of a digraph corresponds to performing a permutation similarity on the matrix or pattern.

A pattern Q is called *reducible* if its characteristic matrix C is reducible, i.e., if there is a permutation matrix P such that

$PCP^T = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$, where $C_{11}, C_{22}$ are square and 0 denotes a matrix consisting entirely of 0s. This is also written as $PQP^T = \begin{bmatrix} Q_{11} & ? \\ Q_{21} & Q_{22} \end{bmatrix}$, where "?" indicates a rectangular set of positions not included in the pattern. Note that it is possible that $Q_{11}, Q_{22}$, and $Q_{21}$ are missing some positions. A pattern is *irreducible* if it is not reducible, or equivalently, if its characteristic matrix is irreducible.

Any matrix is permutation similar to a block triangular matrix with irreducible diagonal blocks [HJ1]. This is useful because a block triangular matrix with irreducible diagonal blocks is a P-matrix if and only if each of the diagonal blocks is a P-matrix.

Since the reducibility of a pattern is defined in terms of the reducibility of its characteristic matrix, any pattern is permutation similar to a block triangular pattern with each diagonal block irreducible.

It is well known [see, e.g., V, p. 19] that a matrix is irreducible if and only if its nonzero-digraph is strongly connected. Since the digraph of a pattern is the nonzero-digraph of its characteristic matrix, an induced subdigraph of a pattern is strongly connected if and only if the principal subpattern defined by its vertices is irreducible.

**1.2 Lemma** Let Q be a pattern that contains all diagonal positions. If every irreducible principal subpattern of Q is complete, then Q has P-completion.

**Proof**. Let Q be a pattern. There is a permutation matrix P such that

$$PQP^T = \begin{bmatrix} Q_{11} & ? & \cdots & ? \\ Q_{21} & Q_{22} & \cdots & ? \\ . & . & . & . \\ Q_{K1} & Q_{K2} & \cdots & Q_{KK} \end{bmatrix},$$

where "?" denotes a completely unspecified block and each diagonal block is irreducible and therefore complete.

For any partial P-matrix A that specifies Q,

$$PAP^T = \begin{bmatrix} A_{11} & ? & \cdots & ? \\ A_{21} & A_{22} & \cdots & ? \\ . & . & . & . \\ A_{K1} & A_{K2} & \cdots & A_{KK} \end{bmatrix}.$$

If a diagonal block $A_{II}$ of $PAP^T$ is completely specified then it is a P-matrix (since $PAP^T$ is a partial P-matrix). Thus, if all the diagonal blocks are completely specified, the matrix B, obtained from $PAP^T$ by setting all unspecified entries to zero, is a block triangular matrix in which each diagonal block is a P-matrix. Thus, B is a P-matrix and $P^TBP$ is an P-matrix that completes A.
■

**1.3 Corollary** Let Q be a pattern**.** Then Q has P-completion if and only if every irreducible principal subpattern of Q has P-completion.

**Proof.** Let Q be a pattern. Suppose every irreducible principal subpattern of Q has P-completion. Let A be a partial P-matrix specifying Q. Let A' be the result of completing each principal submatrix associated with an irreducible principal subpattern. Then A' specifies the pattern Q' that is obtained from Q by completing every irreducible principal subpattern. By Lemma 1.2, Q' has P-completion, so A' and thus A can be completed to a P-matrix, and Q has P-completion. The converse follows from the fact that any principal subpattern of a pattern with P-completion inherits P-completion. ■

Thus, it is necessary for us to consider only irreducible patterns, or equivalently, strongly connected digraphs.

## 2. Patterns that have P-Completion

Johnson and Kroschel [JK] considered the P-matrix completion problem for the first time. Their results are based on the assumption that the diagonal entries of a square partial P-matrix are specified, and that the partial matrices are positionally symmetric, that is, the j, i-entry is specified if and only if the i, j-entry is specified. Their main result is the following theorem:

**Theorem [JK]** Every positionally symmetric partial P-matrix has P-completion.

The following lemma is based on Proposition 1 of [JK].

**2.1 Lemma [JK]** Any $3 \times 3$ partial P-matrix can be completed to a P-matrix. Any pattern for $3 \times 3$ matrices has P-completion.

**Proof.** The first statement appears as Proposition 1 in [JK] but omits some cases. As noted there, the only case to be considered is where only one entry is unspecified (choose values for any others, subject only to the restriction that any $2 \times 2$ principal submatrices completed in this manner are P-matrices), and it is an off-diagonal entry (or else apply Theorem 1.1). Any such $3 \times 3$ partial P-matrix can be reduced by permutation similarity and multiplication by a positive diagonal matrix to $\begin{bmatrix} 1 & s & c \\ a & 1 & t \\ ? & b & 1 \end{bmatrix}$,

where "?" denotes the unspecified entry. Provided $s \neq 0$ and $t \neq 0$, this matrix can be further reduced by diagonal similarity to $\begin{bmatrix} 1 & 1 & c \\ a & 1 & 1 \\ ? & b & 1 \end{bmatrix}$, and [JK] establishes that this matrix can be completed to a P-matrix. However, consideration of the case $s=0$ or $t=0$ was omitted.

So consider the matrix $B = \begin{bmatrix} 1 & 0 & c \\ a & 1 & t \\ y & b & 1 \end{bmatrix}$ which completes the partial P-matrix A. Since A is a partial P-matrix, $bt < 1$. For B to be a P-matrix, y must satisfy $0 < 1 - cy$ and $0 < \det B = 1 + abc - bt - cy$. If $c = 0$, B is a P-matrix (for any y). If $c \neq 0$, choosing y opposite in sign to c and sufficiently large in absolute value will ensure B is a P-matrix. ■

We extend the results of Johnson and Kroschel in two directions: In this section we enlarge the class of patterns known to have P-completion. In Section 3 we enlarge the class of patterns known to not have P-completion. In Section 4 we apply the results of Sections 2 and 3 to classify most patterns of $4 \times 4$ matrices according as to whether the pattern has P-completion.

Here we provide examples of several strategies that we find practical for completing a partial P-matrix to a P-matrix. Given a partial P-matrix that is not positionally symmetric, we complete it to a positionally symmetric partial P-matrix. We can then apply the Theorem [JK] to complete the new positionally symmetric partial P-matrix to a P-matrix.

Some of our general techniques consist of completing one 2 × 2 partial P-matrix (Example 2.2 below), or one 3 × 3 partial P-matrix (Examples 2.4, 2.5) at a time, in succession—these constructions are possible as described in Lemma 2.1.

Another technique used to show a pattern has P-completion (Example 2.3) is to note that triangular or block triangular partial P-matrices, which include reducible partial P-matrices, are completable with zeros (this is the idea behind Lemma 1.2). Finally, many special cases that do not fall under any of the preceding techniques can be handled by judicious assignment of unspecified entries.

The techniques showcased in our examples below apply to matrices of any order; however, since part of our goal was to characterize all digraphs on four vertices for which any partial P-matrix, with a given digraph, has P-completion, we illustrate these strategies with 4 × 4 partial P-matrices. In the figures, a double line indicates both arcs are present.

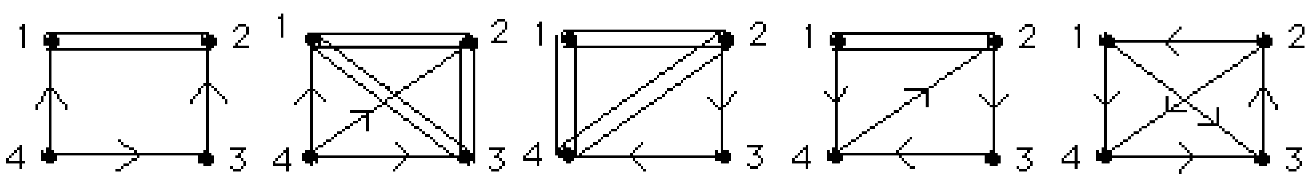


Figure 1 (a) (b) (c) (d) (e)

Our first example illustrates the completion of 2 × 2 partial P-submatrices producing a positionally symmetric partial P-matrix.

**2.2 Example** Consider the pattern Q = {(1,1), (1,2), (2,1), (2,2), (3,2), (3,3), (4,1), (4,3), (4,4)}, whose digraph appears in Figure 1(a). Let

$$A = \begin{bmatrix} 1 & a & ? & ? \\ b & 1 & ? & ? \\ ? & c & 1 & ? \\ d & ? & e & 1 \end{bmatrix}$$

be a partial P-matrix specifying Q. If we complete the superdiagonal, and the northeast corner of the matrix with zeros, that is complete the 2 × 2 principal submatrices A({1, 4}), A({2, 3}), A({3, 4}), then the resulting matrix is a positionally symmetric partial P-matrix. Thus, Q has P-completion.

**2.3 Example** Consider the pattern Q whose digraph appears in Figure 1(b). Let A $= \begin{bmatrix} 1 & a & b & ? \\ c & 1 & d & ? \\ f & g & 1 & ? \\ i & j & k & 1 \end{bmatrix}$ be a partial P-matrix specifying Q. Since A is a block triangular partial P-matrix, we can complete A to a P-matrix by assigning zeros to the fourth column of A. Thus, Q has P-completion.

The completion of Example 2.3 used the property that the digraph has all edges pointing away from vertex 4. (The digraph of Example 2.2 also has that property but it was not used in the completion given in Example 2.2.) Whenever all edges point away from or towards a particular vertex, we can complete the row or column that corresponds to that vertex with zeros, thus reducing the problem to a smaller dimension. Such a pattern is reducible.

In the next two examples we demonstrate the technique of completing partial 3 × 3 submatrices in order to arrive at a positionally symmetric partial P-matrix that is completable by Theorem [JK].

**2.4 Example** For Figure 1(c), with partial P-matrix A = $\begin{bmatrix} 1 & a & ? & b \\ c & 1 & d & e \\ ? & ? & 1 & f \\ g & h & ? & 1 \end{bmatrix}$, completion of the lower 3 × 3 principal submatrix to a P-matrix, which can be accomplished by Lemma 2.1, will yield a positionally symmetric partial P-matrix. Thus, the pattern with digraph Figure 1(c) has P-completion.

**2.5 Example** The techniques for completing partial P-matrices sometimes involve completing two 3 × 3 partial P-matrices in succession. For Figure 1(d), with partial P-matrix A = $\begin{bmatrix} 1 & a & ? & b \\ c & 1 & d & ? \\ ? & ? & 1 & e \\ ? & f & ? & 1 \end{bmatrix}$, a P-matrix completion can be carried out by first completing the lower 3 × 3 principal submatrix A({2,3,4}), and then the 3 × 3 principal submatrix A({1, 2 , 4}). This results in a positionally symmetric P-matrix, which has P-completion. Thus, the pattern with digraph Figure 1(d) has P-completion.

**2.6 Example** One case in which the results of [JK] cannot be used is when the digraph is a tournament (i.e., a digraph with the property that for every two distinct vertices i and j, exactly one of (i,j), (j,i) is an arc).

For Figure 1(e), with partial P-matrix A = $\begin{bmatrix} 1 & ? & a & b \\ c & 1 & ? & d \\ ? & e & 1 & ? \\ ? & ? & f & 1 \end{bmatrix}$, a P-matrix completion can be carried out in the following manner:

If c = 0, complete the lower principal 3 × 3 matrix (A({2,3,4})) to a P-matrix, fill column one with zeros, and the 1,2 entry may be any number. Then this will produce a P-matrix.

If c ≠ 0, we can assume, without loss of generality, that c = 1; this can be achieved by diagonal similarity. We can choose to complete the rest of column one, and specify the two lower superdiagonal elements as zeros, thus yielding the matrix

B = $\begin{bmatrix} 1 & x & a & b \\ 1 & 1 & 0 & d \\ 0 & e & 1 & 0 \\ 0 & y & f & 1 \end{bmatrix}$, where x and y are to be determined. The lower 3 × 3 principal submatrix is a partial P-matrix and can be completed to a P-matrix, so that y is a known value at this point, and x is to be determined in such a way that the competed matrix is a P-matrix. It follows that x has to satisfy the following:

- From the 2 × 2 leading principal submatrix, 1 – x > 0
- From the 3 × 3 principal submatrices containing x, B({1, 2, 3}) and B({1, 2, 4}), the two determinants below are to be positive:

$$\begin{vmatrix} 1 & x & a \\ 1 & 1 & 0 \\ 0 & e & 1 \end{vmatrix} = 1 - x + ae > 0$$

$$\begin{vmatrix} 1 & x & b \\ 1 & 1 & d \\ 0 & y & 1 \end{vmatrix} = 1 - yd - (x - by) = 1 - x + y(b - d) > 0$$

- The determinant of the whole matrix must also be positive:

$$\begin{vmatrix} 1 & x & a & b \\ 1 & 1 & 0 & d \\ 0 & e & 1 & 0 \\ 0 & y & f & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & d \\ e & 1 & 0 \\ y & f & 1 \end{vmatrix} - \begin{vmatrix} x & a & b \\ e & 1 & 0 \\ y & f & 1 \end{vmatrix} = \begin{vmatrix} 1-x & -a & d-b \\ e & 1 & 0 \\ y & f & 1 \end{vmatrix} = 1 - x + ae + (d - b)(ef - y) > 0$$

Thus, the requirements on x are
1 – x > 0, 1 – x > –ae, 1 – x > –y(b – d), and 1 – x > –ae –(b – d)(y – ef).

All of these inequalities can be made true by selecting $x < 0$ with large enough absolute value, therefore, A has P-completion. Thus, the pattern with digraph Figure 1(e) has P-completion.

Using the techniques showcased in Examples 2.2 - 2.6, we can prove that many patterns have P-completion. In particular, any pattern for $4 \times 4$ matrices with eight or fewer off-diagonal positions has P-completion (cf. Section 4).

## 3. Patterns that do not have P-completion

Johnson and Kroschel's example $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ y & -10 & -1 & 1 \end{bmatrix}$ in Proposition 2 of [JK] is a $4 \times 4$ partial P-matrix with exactly one unspecified entry (y) that cannot be completed to a P-matrix. The key idea is to use two $3 \times 3$ matrices that include the unspecified entry to give contradictory requirements. This type of difficulty was discussed in [J] in reference to the positive definite completion problem. We use the $3 \times 3$ matrices in the next two lemmas to derive contradictory requirements that allow us to extend Johnson and Kroschel's example to a family of digraphs that prevent P-completion.

**3.1 Lemma** For each of the following matrices $A_i$, suppose a, b, q, u, v satisfy $0 < q < \frac{1}{2}$, $0 < a, b < \frac{1}{8} q$, and $|u|,|v| < \frac{1}{4} q$. If $A_i$ is a P-matrix, then x must satisfy $|x| < 2q$.

$$A_1 = \begin{bmatrix} 1 & 1\text{-}u & 1\text{-}q \\ 1\text{-}a & 1 & 1\text{-}v \\ 1\text{-}x & 1\text{-}b & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1\text{-}a & 1\text{-}q \\ 1\text{-}u & 1 & 1\text{-}b \\ 1\text{-}x & 1\text{-}v & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1\text{-}u & 1\text{-}q \\ 1\text{-}a & 1 & 1\text{-}b \\ 1\text{-}x & 1\text{-}v & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 1\text{-}b & 1\text{-}q \\ 1\text{-}v & 1 & 1\text{-}u \\ 1\text{-}x & 1\text{-}a & 1 \end{bmatrix}$$

Proof: We show that $-2q < x < 2q$. The first inequality is established by considering the principal submatrix of $A_i$ defined by {1,3} and requiring that its determinant, $1-(1-x)(1-q)$, be greater than zero. Thus, $\frac{1}{1-q} > 1-x$ and $x > 1 - \frac{1}{1-q}$. Since $0 < q < \frac{1}{2}$, $2q^2 < q$ and $1 < (1-q)(1+2q)$. Therefore, $\frac{1}{1-q} < 1 + 2q$, and $1 - \frac{1}{1-q} > -2q$.

$$0 < \det A_1 = ab + aq + bq - abq - au - bv + uv - qx + ux + vx - uvx$$

$= -x(q-u-v+uv) + (a+b-ab)q + (a-v)(b-u).$

$x(q-u-v+uv) < (a+b-ab)q + (a-v)(b-u).$

$$x < \frac{(a+b-ab)q + (a-v)(b-u)}{q-u-v+uv}$$

$$< \frac{(a+b)q + (a+|v|)(b+|u|)}{q-(|u|+|v|+|uv|)}$$

$$< \frac{(\frac{1}{8}q+\frac{1}{8}q)q + (\frac{1}{8}q+\frac{1}{4}q)(\frac{1}{8}q+\frac{1}{4}q)}{q - (\frac{1}{4}q+\frac{1}{4}q+\frac{1}{4}q\frac{1}{8})} = \frac{5}{6} q < 2q.$$

$0 < \det A_2 = ab - au + qu - bv + qv + uv - uvq + ax + bx - abx - qx$

$= -x(q-a-b+ab) + (u+v-uv)q + (a-v)(b-u).$

$x(q-a-b+ab) < (u+v-uv)q + (a-v)(b-u).$

$$x < \frac{(u+v-uv)q + (a-v)(b-u)}{q-a-b+ab}$$

$$< \frac{(|u|+|v|+|uv|)q + (a+|v|)(b+|u|)}{q-(a+b)}$$

$$< \frac{(\frac{1}{4}q+\frac{1}{4}q+\frac{1}{4}q\frac{1}{8})q + (\frac{1}{8}q+\frac{1}{4}q)(\frac{1}{8}q+\frac{1}{4}q)}{q - \frac{1}{8}q - \frac{1}{8}q} = \frac{43}{48} q < 2q.$$

$0 < \det A_3 = \det A_4$

$= aq - au + bu + av - bv + qv - aqv + bx - qx + ux - ubx$

$= -x(q-u-b+ub) + (a-b)(v-u) + (a+v-av)q.$

$x(q-u-b+ub) < (a-b)(v-u) + (a+v-av)q.$

$$x < \frac{(a-b)(v-u) + (a+v-av)q}{q-u-b+ub}$$

$$< \frac{\max(a,b)(|v|+|u|) + (a+|v|+a|v|)q}{q-(|u|+b+|u|b)}$$

$$< \frac{\frac{1}{8}q(\frac{1}{4}q+\frac{1}{4}q) + (\frac{1}{8}q+\frac{1}{4}q+\frac{1}{8}q\frac{1}{8})q}{q - (\frac{1}{8}q+\frac{1}{4}q+\frac{1}{8}q\frac{1}{8})} = \frac{29}{39} q < 2q.$$ ■

**3.2 Lemma** If the matrix $A_5 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0.9 \\ -10 & y & 1 \end{bmatrix}$ is a P-matrix, then $y < -8$.

**Proof.** $0 < \det A_5 = -8 - 0.9y$, so $y < \frac{-8}{0.9} < -8$ ■

We can use the matrices $A_1$ - $A_5$ to "squeeze" a digraph D, producing contradictory requirements on the values for the unspecified entries corresponding to the reverse arcs of the arcs of D that are asymmetric in D and preventing completion of a partial matrix specifying D. This is illustrated in the next example, and then proven in general.

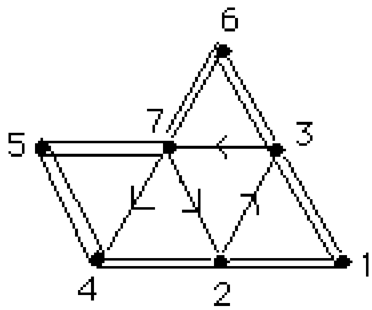


Figure 2

**3.3 Example** Let Q be the pattern {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (1,2), (2,1), (2,4), (4,2), (4,5), (5,4), (5,7), (7,5), (7,6), (6,7), (6,3), (3,6), (3,1), (1,3), (2,3), (3,7), (7,2), (7,4)}, whose digraph D is illustrated in Figure 2. Because the partial P-matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & ? & ? & ? & ? \\ 0 & 1 & 0.9 & 0.999 & ? & ? & ? \\ -10 & ? & 1 & ? & ? & 1 & 0.99 \\ ? & 1 & ? & 1 & 1 & ? & ? \\ ? & ? & ? & 0.9999 & 1 & 0 & 0.9999 \\ ? & ? & 0.999 & ? & ? & 1 & 0.999 \\ ? & 0.99 & ? & 0.999 & 1 & 1 & 1 \end{bmatrix}$$

cannot be completed to a P-matrix, the pattern Q does not have P-completion.

Suppose B is a partial matrix that completes A as a P-matrix. The principal submatrix $B(\{1,2,3\}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0.9 \\ -10 & b_{32} & 1 \end{bmatrix} = A_5$, so

· $b_{32} < -8$.

The reordered principal submatrix $B(7,5,4) = \begin{bmatrix} 1 & 1 & 0.999 \\ 0.9999 & 1 & 0.9999 \\ b_{47} & 1 & 1 \end{bmatrix}$, which is $A_3$ with $q = 0.001$, $a = b = 0.0001 < \frac{1}{8}q$, $u = v = 0 < \frac{1}{4}q$. Thus,

· $|1-b_{47}| < 2q = 0.002$.

The principal submatrix B({3,6,7}) = $\begin{bmatrix} 1 & 1 & 0.99 \\ 0.999 & 1 & 0.999 \\ b_{73} & 1 & 1 \end{bmatrix}$, which is $A_3$ with

$q = 0.01$, $a = b = 0.001 < \frac{1}{8}q$, $u = v = 0 < \frac{1}{4}q$. Thus,

· $|1-b_{73}| < 2q = 0.02$.

The reordered principal submatrix B(7,4,2) = $\begin{bmatrix} 1 & 0.999 & 0.99 \\ b_{47} & 1 & 1 \\ b_{27} & 0.999 & 1 \end{bmatrix}$, which is $A_4$

with $q = 0.01$, $a = b = 0.001 < \frac{1}{8}q$ , $u = 0$ and $v = 1-b_{47}$ with

$|v| < 0.002 < \frac{1}{4}q$. Thus,

· $|1-b_{27}| < 2q = 0.02$.

The reordered principal submatrix B(2,7,3) = $\begin{bmatrix} 1 & b_{27} & 0.9 \\ 0.99 & 1 & b_{73} \\ b_{32} & 0.99 & 1 \end{bmatrix}$, which is $A_1$

with $q = 0.1$, $a = b = 0.01 < \frac{1}{8}q$ , $u = 1-b_{27}$, $v = 1-b_{73}$ with

$|u|, |v| < 0.02 < \frac{1}{4}q$. Thus,

· $|1-b_{32}| < 2q = 0.2$.

So $b_{32} > 0.8$, contradicting $b_{32} < -8$. Therefore, A cannot be completed to P-matrix.

To describe the process used in this example more generally, we need to introduce some terminology and establish some graph theoretic properties.

A graph is *planar* if it can be drawn in the plane without any edges crossing; such a drawing is called a *plane diagram*. Note that any cycle in a plane diagram is a simple closed curve. Let G^ be a connected plane diagram. Then G^ divides the plane into regions, all but one of which are bounded. The *exterior* region is the unbounded region. Its boundary is a simple closed curve, and all the other regions are in the interior of this simple closed curve, and are called *interior* regions of G^. A region is *triangular* if its boundary has exactly 3 edges.

A *tree* is a connected graph that does not contain any cycles. Between any two vertices in a tree there is exactly one path [11.2,GP]. The *distance* between any two vertices in a tree is the length of the (unique) path between the vertices. If one vertex is distinguished as the *root,* the distance between a vertex v and the root is called the *height* of v.

A chord of a cycle is an edge joining two non-consecutive vertices of the cycle. A graph G is *chordal* if any cycle of length > 3 in G has a chord.

A cycle of a graph G is a *Hamiltonian cycle* of G if it contains all the vertices of G.

**3.4 Definition** The *interior dual pseudograph* $G^\wedge_d$ of a plane diagram G^ is obtained by placing a vertex of the dual in each interior region and creating an edge of the dual for each original edge common to the boundaries of the two interior regions.

**3.5 Definition** The graph G is a *minimally chordal Hamiltonian* graph if

1) G has a Hamiltonian cycle H.
2) G is chordal.
3) If any non-empty set S of chords of H is removed from G, the resulting graph G-S is not chordal.

**3.6 Lemma** Let G be a planar graph with a Hamiltonian cycle H such that there is a plane diagram G^ of G in which all the chords of H are in the interior of H and all the interior regions of G^ are triangular. Then

1) G is chordal
2) H is the only Hamiltonian cycle of G.
3) The interior dual pseudograph $G^\wedge_d$ is a tree
4) H has at least 2 vertices of degree 2

**Proof.** Let G, H and G^ satisfy the hypotheses.

1) Any cycle C in G is a simple closed curve in G^ and thus has an interior that is contained in the interior of G^. The interior of C does not contain any vertices. If the order of C is more than 3 then the interior of C must contain more than one of the triangular interior regions and so C has a chord.

2) The removal of the two endpoints of any chord (and all edges incident with these two vertices disconnects G^. Any cycle that contains a chord must remain in the subdigraph induced by the union of one piece and the chord, and thus cannot be Hamiltonian.

3) The drawing G^ can be produced by first drawing the unique Hamiltonian cycle H as a simple closed curve and then adding one chord at a time (in the interior of H). Throughout this process, the interior regions and chords will satisfy the relationship:

number of interior regions = number of chords + 1.

Initially, (before any chords are drawn) there is one interior region and no edges. Drawing a new chord of H divides one interior region into two regions with one chord in common, so drawing a new chord adds one region and one chord, and the plane diagram still satisfies the relationship:

number of interior regions = number of chords + 1.

When the plane diagram G^ is complete, construct the interior dual pseudograph $G^\wedge_d$. Each region in G^ corresponds to a vertex of $G^\wedge_d$, and each chord corresponds to an edge of $G^\wedge_d$, so $G^\wedge_d$ satisfies the relationship: number of vertices = number of edges + 1. The diagram G^ is connected so $G^\wedge_d$ is connected. Therefore, $G^\wedge_d$ is a tree [Theorem 11.5, GP] (although the theorem assumes a graph rather than a pseudograph, the proof shows any connected pseudograph satisfying number of edges = number of vertices - 1 is a tree).

4) If the order of G is 3, all three vertices have degree 2. If the order of G is greater than 3, there must be a chord of H, so the interior dual graph $G^\wedge_d$ is a tree with more than one vertex. Therefore, $G^\wedge_d$ must have at least 2 vertices of degree 1 [11.6,GP]. A vertex of degree 1 in $G^\wedge_d$ corresponds to a triangular region with only one side a chord in the plane diagram G^. The vertex between the other two sides of this region is, therefore, of degree 2.

■

The interior dual pseudograph of a plane diagram G^ meeting the conditions of the preceding lemma is called the *interior dual tree*.

**3.7 Lemma**[2] A graph G is a minimally chordal Hamiltonian graph if and only if G has a Hamiltonian cycle H, is planar and there is a plane diagram G^ of G such that all the chords of H are in the interior of H and all the interior regions of G^ are triangular.

**Proof.** Let G be a minimally chordal Hamiltonian graph with Hamiltonian cycle H. Draw H as a circle with the vertices equally spaced. If H has only 3 vertices, this is a plane drawing of G with no chords and the interior a triangle. If H has more than 3 vertices, then (since G is chordal) there must be a chord, which we can draw as a straight line in the interior. This results in two smaller cycles. Continue this process. If a cycle has more than 3 vertices, it has a chord, which can be drawn in its interior. The end result will be a plane diagram of a subgraph F of G with H as a circle, all chords of F in the interior of H and all regions of the interior triangular. This subgraph F satisfies the hypotheses of the preceding lemma, and is, therefore, chordal. Furthermore, F is either G itself or can be obtained from G by deleting the set S of chords not shown in the drawing. But

[2]The authors wish to thank Michael M. Parmenter of Memorial University of Newfoundland for providing the proof of Lemma 3.7 and for an enjoyable correspondence concerning graph theory.

deleting a nonempty set S would violate (c) of the definition of a minimally chordal Hamiltonian graph. Thus, F = G and G has a plane diagram with all the chords in the interior and all interior regions triangular. Conversely, let G be a planar graph with a Hamiltonian cycle H such that there is a plane diagram G^ of G in which all the chords of H are in the interior of H and all the interior regions of G^ are triangular. Then G is chordal by the preceding lemma. The removal of one or more chords of H from G^ clearly creates a cycle of length > 3 with no chords, so G is a minimally chordal Hamiltonian graph. ■

The preceding definitions and lemmas describe what the underlying graph must look like in order to apply the "squeezing" process used in Example 3.3 to a pattern-digraph. To apply this process, it is clearly also necessary that the Hamiltonian cycle be symmetric and that the digraph not contain the reverse arc of any chord.

**3.8 Definition** A digraph D is a *minimally chordal symmetric-Hamiltonian* digraph if

1) The underlying graph G of D is a minimally chordal Hamiltonian graph.
2) The arcs corresponding to the unique Hamiltonian cycle of G are symmetric in D.
3) Any arc corresponding to a chord of the unique Hamiltonian cycle of G is asymmetric in D.

We can use the matrices $A_1$ - $A_4$ and $A_5$ to "squeeze" a minimally chordal symmetric-Hamiltonian digraph to produce a contradiction preventing completion of a partial matrix specifying a pattern with the digraph in the same way we did Example 3.3. The labeling described in the proof of the next theorem is illustrated in Figure 3. The labels of the regions are circled and the label of the edges are written on the edges.

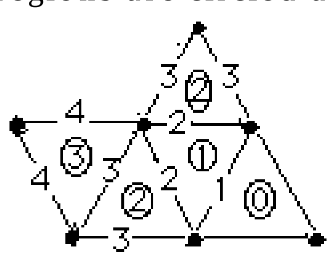


Figure 3

**3.9 Theorem** A pattern that contains all diagonal positions and whose pattern-digraph is a minimally chordal symmetric-Hamiltonian digraph with at least 4 vertices does not have P-completion.

**Proof.** Let Q be a pattern whose pattern-digraph D is a minimally chordal symmetric-Hamiltonian digraph with at least 4 vertices. Let G be the underlying graph of D and let G^ be a plane diagram of G with the unique Hamiltonian cycle H drawn as a simple closed curve and the chords drawn in the interior (so all interior regions are triangular). Throughout this proof, "region" will mean interior (triangular) region of G^.

By Lemma 3.6, G has at least 2 vertices of degree 2. A vertex of degree 2 is on the boundary of only one region of G^, because if a vertex is on the boundary between two regions then a chord is incident with the vertex and so the vertex must have degree at least 3. Select a vertex of degree 2 in G. Since G has at least 1 chord, by renaming the vertices of D if necessary, we may assume the selected vertex is 1 and the two vertices adjacent to it are 2 and 3 with (2,3) being the arc of D corresponding to the chord {2,3} of G. Designate the dual vertex of the (unique) region whose boundary contains vertex 1 as the root of the interior dual tree. Label each region with the height of its dual vertex in the interior dual tree. Note that a region may be bounded by 1, 2, or 3 other regions. For every region except the region containing the root (which is labeled 0), exactly one of these regions is of lesser height (otherwise there would be more than one path from the dual vertex to the root of the dual tree, and in a tree there is a unique path between any 2 vertices). Every chord of G lies between two interior regions in G^, which are labeled with consecutive whole numbers. Label the chord between k and k+1 as k+1. Every edge in the Hamiltonian cycle is on the boundary of only one interior region. Label every edge of the Hamiltonian cycle as k+1 where k is the label of the interior region the edge bounds, except edges bordering the region labeled 0 (i.e., except edges incident with vertex 1). All the edges of the graph G are now labeled except {1,2} and {1,3}.

Define a partial matrix A as follows: All diagonal entries are 1. Set $a_{12} = 1$, $a_{13} = 0$, $a_{21} = 0$, and $a_{31} = -10$. Every entry whose arc in D corresponds to a chord in G is set to $1-(0.1)^k$, where k is the label of the corresponding chord. For a symmetric pair of entries whose arcs correspond to an edge of the Hamiltonian cycle, set one entry to 1 and the other to $1-(0.1)^k$ where k is the label of the corresponding edge (it does not matter which is which). The partial matrix A is a partial P-matrix because the only complete principal submatrices are of size 1 or 2, and these have positive determinants.

Let B be a matrix that completes A. The principal submatrix B(1,2,3) is $A_5$, so by Lemma 3.2, $b_{32} < -8$. However, we can obtain a contradictory result by starting with the rest of the Hamiltonian cycle and working toward the root: Let m be the greatest label of any chord of G. Begin with chords labeled m and work down to the chord labeled 1. Let k satisfy $m \geq k \geq 1$. Assume that for every arc (s,t) corresponding to a chord labeled $h > k$, $b_{ts}$ satisfies $|1-b_{ts}| < 2(0.1)^h$ (note that for k=m there is nothing required). Let (p,q) be an arc corresponding to a chord labeled k. Then in G, {p,q} is on the boundary of a region R that is labeled k. The region R is triangular. All the possibilities for parts of the digraph corresponding to R and its boundaries are shown in Figure 4. In each case, each of the other two sides of R is either a chord and the region on the other side is labeled k+1 or part of the Hamiltonian cycle. Therefore, the other two edges are both labeled k+1. Then B(p,r,q) is one of $A_1$ - $A_4$ with $q=(0.1)^k$, $a=b=(0.1)^{k+1}$, and $|u|, |v| < 2(0.1)^{k+1}$. The statements about q, a, and b are immediate from specified entries of B. The statement about u is true by our assumption if u corresponds to an edge that is a chord, and because u = 0 if the corresponding edge is in the Hamiltonian cycle. (The same reasoning applies to v. Thus, by Lemma 3.1, $b_{qp}$ satisfies $|1-b_{qp}| < 2(0.1)^k$. Applying this to $b_{32}$, which is labeled 1, $|1-b_{32}| < 0.2$, so $b_{32} > 0.8$, contradicting $b_{32} < -8$. ■

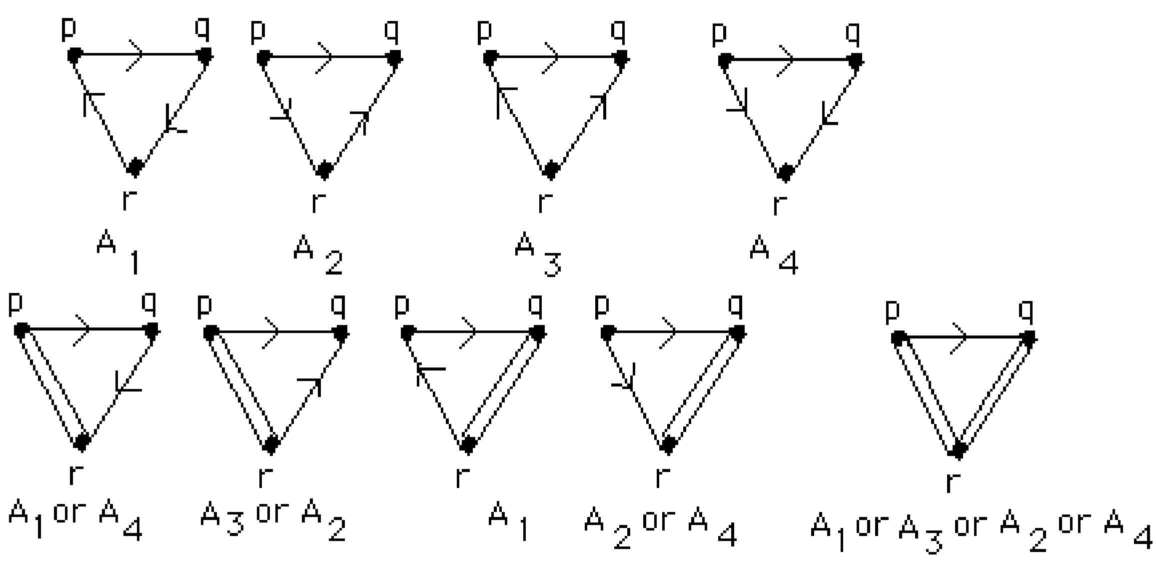


Figure 4

**3.10 Corollary** Let Q be pattern and let D be its pattern digraph. If D contains a subdigraph F that is a minimally chordal symmetric-Hamiltonian digraph containing at least four vertices, and such that the subdigraph of D induced by the vertices of F does not contain any arc symmetric in D except those in the Hamiltonian cycle of F, and Q contains all the diagonal positions of F, then Q does not have P-completion.

**Proof.** Let Q, D and F satisfy the hypotheses. Define a partial P-matrix specifying the subdigraph of D induced by F, <F>, as in the previous theorem, and set any additional arcs that are chords of the cycle to 0. Since any arcs of <F> that are not in F are asymmetric in D, there are no complete principal submatrices of A of size greater than 2, so A is a partial P-matrix that cannot be completed to a P-matrix by the same argument used in the proof of Theorem 3.9. Then Q does not have P-completion because the principal subpattern corresponding to <F> does not.
■

## 4. Patterns of 4 × 4 matrices

Using various techniques such as those discussed in Section 2 and 3, in Theorem 4.1 below we classify 207 of 218 patterns for 4 × 4 matrices as having P-completion or not having P-completion. In particular, all the patterns with associated digraphs having eight or fewer arcs are classified as having P-completion. The only remaining unclassified 4 × 4 digraphs (as labeled in [Ha]) are q = 9, n= 4, 5, 6, 7, 9, 10, 12, 13 and q = 10, n= 2, 3, 4. (Note that q is the number of arcs in the digraph and n is the digraph number.)

**4.1 Theorem** The patterns for 4 × 4 matrices having eight or fewer off-diagonal positions all have P-completion, as do the patterns associated with digraphs q = 9, n = 1, 2, 8, 11; q = 10, n = 1; and q = 12. The patterns associated with digraphs q = 9, n = 3; q = 10, n = 5; and q = 11 do not have P-completion.

**Proof.** The following digraphs represent positionally symmetric patterns (and thus have P-completion by [JK]): q=0; q=2, n = 1; q = 4, n = 1, 2; q = 6, n = 1, 2, 3; q = 8, n = 1, 2; q = 10, n = 1; and q = 12.

The following digraphs represent patterns that can be made positionally symmetric by completing one or more 2 × 2 matrices (without completing anything larger), as was done in Example 2.2. Thus, they have P-completion: q = 1, n = 1; q = 2, n = 2, 3, 4, 5; q = 3, n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11; q = 4, n = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17, 18, 19; q = 5, n = 1, 2, 3, 4, 5, 7, 8, 9, 10; q = 6, n = 4, 5, 6, 7, 8; q = 7, n = 2, 4, 5.

The following digraphs represent patterns that can be made positionally symmetric by completing one 3 × 3 matrix and possibly one or

more 2 × 2 matrices (without completing any other 3 × 3 or anything larger), as was done in Example 2.4. Thus, they have P-completion:
q = 3, n = 12, 13; q = 4, n = 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27;
q = 5, n = 6, 11, 12, 13, 14, 15, 16, 17, 18, 19, 10, 21, 22, 23, 24, 25, 26, 27, 28; q = 6, n = 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21;
q = 7, n = 1, 3, 6; q = 8, n = 10, 11, 12; q = 9, n = 1, 2.

The following digraphs represent patterns that can be made positionally symmetric by completing first one 3 × 3 matrix and then another 3 × 3 matrix, (without completing more than one 3 × 3 at a time or anything larger), as was done in Example 2.5. Thus, they have P-completion: q = 5, n = 29, 30, 31, 32, 33, 34, 35, 36, 37, 38; q = 6, n = 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44; q = 7, n = 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28; q = 8, n = 3, 4, 5, 6, 7, 8, 9, 13, 14, 15.

The following digraphs represent patterns that are reducible, and thus can be completed by assigning a row or column of zeros, as in Example 2.3, or first completing one 3 × 3 matrix and then assigning a row or column of zeros. Thus, they have P-completion: q = 6, n = 46, 47, 48;
q = 7, n = 29, 31, 34, 36, 37; q = 8, n = 21, 27; q = 9, n = 8, 11.

The following digraphs represent patterns that have P-completion, but each case has been done individually, as Example 2.6. The details are omitted: q = 6, n = 45; q = 7, n = 30, 32, 33, 35, 38; q = 8, n = 16, 17, 18, 19, 20, 22, 23, 24, 25, 26.

The following digraphs represent patterns that do not have P-completion, by modification of the partial P-matrix $A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ y & -10 & -1 & 1 \end{bmatrix}$ in Proposition 2 of [JK] (note that the 2,3- and 3,2-entries are irrelevant to that example, so one or both may be unspecified): q = 9, n = 3;
q = 10, n = 5; q = 11. In fact, q = 9, n = 3 is also a minimally chordal symmetric-Hamiltonian digraph. ■

**References**

[FJTU] S. M. Fallat, C. R. Johnson, J. R. Torregrosa, and A. M. Urbano, P-matrix Completions under Weak Symmetry Assumptions, *Linear Algebra and Its Applications* (to appear).

[GP] E. G. Goodaire and M. M. Parmenter, *Discrete Mathematics with Graph Theory*, Prentice-Hall, Upper Saddle River, NJ, 1998.

[GJSW] R. Grone, C. R. Johnson, E. M. Sá, and H. Wolkowicz, Positive Definite Completions of Partial Hermitian Matrices, *Linear Algebra and Its Applications* 58 (1984), 109-124.

[Ha] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA,1969.

[Ho1] L. Hogben, Completions of Inverse M-matrix Patterns. *Linear Algebra and Its Applications* 282 (1998): 145-160.

[Ho2] L. Hogben, Completions of M-matrix Patterns. *Linear Algebra and Its Applications* 285 (1998): 143-152.

[HJ1] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[HJ2] R. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[J] C. R. Johnson, Matrix Completion Problems: A Survey, *Proceedings of Symposia of Applied Mathematics 40* , American Mathematical Society, Providence, RI, 1990, 171-198.

[JK] C. R. Johnson and B. K. Kroschel, The Combinatorially Symmetric P-Matrix Completion Problem, *Electronic Journal of Linear Algebra* 1 (1996), 59-63.

[JS] C. R. Johnson and R. L. Smith, The Completion Problem for M-Matrices and Inverse M-matrices, *Linear Algebra and Its Applications* 241-243(1996): 655-667.

[V] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Inc., Englewood, NJ, 1962.