

1 **ON THE GRAPH COMPLEMENT CONJECTURE FOR MINIMUM**
2 **RANK***

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6 **Abstract.** The minimum rank of a graph has been an interesting and well studied parameter
7 investigated by many researchers over the past decade or so. One of the many unresolved questions on
8 this topic is the so-called graph complement conjecture, which grew out of a workshop in 2006. This
9 conjecture asks for an upper bound on the sum of the minimum rank of a graph and the minimum rank
10 of its complement, and may be classified as a Nordhaus-Gaddum type problem involving the graph
11 parameter minimum rank. The conjectured bound is the order of the graph plus two. Other variants
12 of the graph complement conjecture are introduced here for the minimum semidefinite rank and the
13 Colin de Verdière type parameter ν . We show that if the ν -graph complement conjecture is true for
14 two graphs then it is true for the join of these graphs. Related results for the graph complement
15 conjecture (and the positive semidefinite version) for joins of graphs are also established. We also
16 report on the use of recent results on partial k -trees to establish the graph complement conjecture
17 for graphs of low minimum rank.

18 **Key words.** Minimum rank, Nordhaus-Gaddum type, graph, graph complement, maximum
19 multiplicity, join of graphs, Colin de Verdière type parameters, minimum semidefinite rank, k -trees,
20 Hadwiger number.

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22 **1. Introduction.** All matrices discussed are real and symmetric; the set of $n \times n$
 23 real symmetric matrices will be denoted by $S_n(\mathbb{R})$. A *graph* $G = (V, E)$ means a simple
 24 undirected graph (no loops, no multiple edges) with a nonempty set of vertices V and
 25 edge set E (an edge is a two-element subset of vertices). Recall that the *complement*
 26 *of a graph* $G = (V, E)$ is the graph \overline{G} on the same set of vertices, and with edges
 27 $\{i, j\}$, $i \neq j$ exactly when $\{i, j\} \notin E$. The *order of* G , denoted by $|G|$, is simply the
 28 cardinality of its vertex set V .

29 Studying collections of matrices associated to a combinatorial object, such as a
 30 graph, has long been a topic of interest to both the linear algebra community and
 31 to the combinatorial community. One instance of this general study is the so-called
 32 *minimum rank problem for graphs*. In general, the minimum rank problem for a graph
 33 G asks to determine the smallest possible rank over the collection of all real symmetric
 34 matrices $A = [a_{ij}]$ with the adjacency property that for each $i \neq j$, $a_{ij} \neq 0$ if and only
 35 if $\{i, j\}$ is an edge in G . In general, for $A \in S_n(\mathbb{R})$, the *graph of* A , denoted $\mathcal{G}(A)$, is
 36 the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$.

Let G be a graph. The *set of symmetric matrices described by* G is defined to be

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\},$$

and the quantity we study is the *minimum rank of* G , defined and denoted by

$$\text{mr}(G) = \min\{\text{rank } A : A \in \mathcal{S}(G)\}.$$

37 Since the main diagonal of any member in $\mathcal{S}(G)$ is ignored in determining $\mathcal{G}(A)$,
 38 it is clear that $0 \leq \text{mr}(G) \leq n - 1$; and if G is nontrivial we have $1 \leq \text{mr}(G) \leq n - 1$.
 39 A basic example along these lines is the path on n vertices, which is known to be the
 40 only graph (on n vertices) with minimum rank equal to $n - 1$ (see [13]).

41 The general and important matter of resolving the minimum rank of an arbitrary
 42 graph is a very difficult and open problem. However, considerable research on this
 43 issue has led to significant progress on many facets of it. For example, general
 44 formulas are known for the minimum rank of trees and unicyclic graphs, and complete
 45 descriptions of the graphs G for which $\text{mr}(G) = 1, 2, n - 2, n - 1$ have been recorded
 46 in the literature (see, for example, [12] and the references therein).

47 The topic of minimum rank of graphs has garnered sufficient attention in the
 48 literature to a point where it became a core subject at an American Institute of
 49 Mathematics workshop “Spectra of Families of Matrices described by Graphs, Di-
 50 graphs, and Sign Patterns [2]. A result of this workshop was a number of suggested
 51 problems one of which has become known as the “Graph Complement Conjecture”
 52 or GCC for short. The GCC can be stated as the following conjecture about the
 53 minimum rank of G and its complement.

54 CONJECTURE 1.1 (GCC Conjecture). *For any graph G ,*

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2.$$

55

56 For example, if $G = C_5$, the cycle on 5 vertices, then $\text{mr}(C_5) = 3$ and $\text{mr}(\overline{C_5}) =$
57 $\text{mr}(C_5) = 3$. Hence, $\text{mr}(G) + \text{mr}(\overline{G}) = 3 + 3 < 5 + 2$.

58 It is worth noting that the actual question posed at this 2006 AIM workshop was:
59 How large can $\text{mr}(G) + \text{mr}(\overline{G})$ be? From this two possibilities arise (see also [8]):

60 Question 1) Does there exist a constant $c \geq 2$ such that $\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + c$?
61 If so, what is the smallest such c ?

62 Question 2) Find the smallest constant $d \leq 2$ such that $\text{mr}(G) + \text{mr}(\overline{G}) \leq d|G|$.

63 The condition $c \geq 2$ in Question 1 follows from examination of the path on 4
64 vertices, written as P_4 . Observe that $\text{mr}(P_4) + \text{mr}(\overline{P_4}) = 6 = 4 + 2$, which implies
65 $c \geq 2$. On the other hand, since $\text{mr}(G) \leq |G| - 1$ for any graph it follows that d (in
66 Question 2) can be chosen to be at most 2. It has been suspected for some time that
67 $c = 2$ is the correct bound for Question 1, hence the GCC.

68 Since the original GCC was recorded, further analysis has lead to stronger con-
69 jectures on the minimum rank of all positive semidefinite matrices in $\mathcal{S}(G)$. We let
70 $\text{mr}_+(G)$ denote the minimum rank of all matrices A in $\mathcal{S}(G)$ with the additional
71 constraint that A be positive semidefinite.

72 CONJECTURE 1.2 (GCC₊ Conjecture). *For any graph G ,*

$$\text{mr}_+(G) + \text{mr}_+(\overline{G}) \leq |G| + 2.$$

73

74 It is clear that GCC₊ represents a stronger inequality than does GCC, and thus
75 the bound of $|G| + 2$ is best possible for GCC₊ (note that for GCC₊, any tree T that
76 contains an induced P_4 has equality in the bound, because $\text{mr}_+(T) = |T| - 1$ and
77 $\text{mr}_+(\overline{T}) = 3$ (see [3]).

78 We note here that both GCC and GCC₊ (and later GCC _{ν}) fall into the class of
79 so-called Nordhaus-Gaddum type problems (see [1], for example) in that they involve
80 bounding the sum of a graph parameter evaluated at a graph G and its complement
81 \overline{G} . Nordhaus-Gaddum type problems have been studied for many different graph

82 parameters, including chromatic number, independence number, domination number
 83 and others such as the Hadwiger number (see [20]). Since the matrix community has
 84 been referring to these suspected inequalities as graph complement conjectures, we
 85 continue to use these names within this work as well.

Observe that if we define the *maximum nullity* of G as

$$M(G) = \max\{\text{null } A : A \in \mathcal{S}(G)\},$$

and the *maximum positive semidefinite nullity* of G as

$$M_+(G) = \max\{\text{null } A : A \in \mathcal{S}(G), A \text{ is positive semidefinite}\},$$

Conjectures 1.1 and 1.2 are equivalent to

$$M(G) + M(\overline{G}) \geq |G| - 2, \tag{1.1}$$

$$M_+(G) + M_+(\overline{G}) \geq |G| - 2. \tag{1.2}$$

86 A related conjecture (see Conjecture 1.7 below) was made in [21], using the Colin
 87 de Verdière number $\mu(G)$ that is equal to the maximum nullity among all matrices
 88 satisfying several conditions including the Strong Arnold Hypothesis (see definitions
 89 below). The parameter μ , which is used to characterize planarity, is the first of several
 90 parameters that require the Strong Arnold Hypothesis and bound the maximum nul-
 91 lity from below (called *Colin de Verdière type parameters*). A real symmetric matrix
 92 A satisfies the *Strong Arnold Hypothesis* provided there does not exist a nonzero real
 93 symmetric matrix X satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where \circ denotes
 94 the Hadamard (entry-wise) product and I is the identity matrix. The Strong Arnold
 95 Hypothesis is equivalent to the requirement that certain manifolds intersect transver-
 96 sally (see [18]). The parameter $\mu(G)$ is defined ([9] in English) to be the maximum
 97 nullity among symmetric matrices $A = [a_{ij}] \in \mathcal{S}(G)$ that satisfy:

- 98 • A satisfies the Strong Arnold Hypothesis.
- 99 • For all $i \neq j$, $a_{ij} \leq 0$.
- 100 • A has exactly one negative eigenvalue (counting multiplicity).

101 In [10] Colin de Verdière introduced the parameter $\nu(G)$, defined to be the maximum
 102 nullity among positive semidefinite matrices $A \in \mathcal{S}(G)$ that satisfy the Strong Arnold
 103 Hypothesis. Evidently, for every graph G , $\nu(G) \leq M_+(G) \leq M(G)$. So it is natural
 104 to ask whether GCC_+ can be extended to ν :

CONJECTURE 1.3 (GCC_ν Conjecture). *For any graph G ,*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2, \tag{1.3}$$

106 Thus (1.3) is stronger than GCC_+ (and hence GCC) in general (cf. (1.1) and
 107 (1.2)). Since some of the arguments later are done in terms of rank, it is instructive
 108 to associate a name to the rank parameter associated with the nullity parameter ν .

109 DEFINITION 1.4. For a graph G , define $\text{mr}_\nu(G) = |G| - \nu(G)$.

With this definition, Conjecture 1.3 becomes

$$\text{mr}_\nu(G) + \text{mr}_\nu(\overline{G}) \leq |G| + 2. \quad (1.4)$$

110 An important property of Colin de Verdière-type parameters is minor monotonic-
 111 ity. The *contraction* of edge $e = \{u, v\}$ of G is obtained by identifying the vertices u
 112 and v , deleting any loops that arise in this process, and replacing any multiple edges
 113 by a single edge. A *minor* of G arises by performing a sequence of deletions of edges,
 114 deletions of isolated vertices, and/or contractions of edges. A graph parameter β is
 115 *minor monotone* if for any minor H of G , $\beta(H) \leq \beta(G)$ and $\beta(G) = \beta(H)$ if G is
 116 isomorphic to H . In [9] and [10] it is shown that μ and ν are minor monotone.

117 For any graph G , the *Hadwiger number* $h(G)$ is the maximum size of a clique
 118 minor in G . It is straightforward to verify that $\nu(K_s) = s - 1$ whenever $s > 1$, so by
 119 minor monotonicity we have:

OBSERVATION 1.5. *Let G be a graph.*

$$M(G) \geq M_+(G) \geq \nu(G) \geq h(G) - 1.$$

120

Given this relationship (and the fact that it is common to use $h(G) - 1$ as a lower
 bound when establishing the value of $\nu(G)$), it is reasonable to ask whether a version of
 GCC is true for the Hadwiger number. The bound would be $(h(G) - 1) + (h(\overline{G}) - 1) \geq$
 $|G| - 2$ or equivalently,

$$h(G) + h(\overline{G}) \geq |G|. \quad (1.5)$$

121 There is a body of literature on Hadwiger number Nordhaus-Gaddum type problems,
 122 and it is known [20] that (1.5) is not true for all graphs G (or even for most graphs
 123 of large order). The next example gives a specific graph for which (1.5) fails. As is
 124 standard, we let $\kappa(G)$ denote the *vertex connectivity* of G , i.e., if G is not complete,
 125 it is the smallest number k such that there is a set of vertices S , with $|S| = k$, for
 126 which $G - S$ is disconnected (by convention, $\kappa(K_n) = n - 1$). As noted in [17], as a
 127 consequence of results in [22] and [23], for every graph G , $\kappa(G) \leq \nu(G)$.

128 EXAMPLE 1.6. Let G_{12} be the icosahedral graph, which has order 12, is 5-regular,
 129 and is planar. Thus G_{12} cannot have a K_5 minor. So in order for G_{12} to satisfy (1.5),

130 $\overline{G_{12}}$ would need to have a K_8 minor. This is impossible, since for any minor that has
 131 8 vertices, we must partition the 12 vertices of $\overline{G_{12}}$ into 8 sets (associated with the
 132 8 vertices of the minor), requiring that there be a set with only one vertex of $\overline{G_{12}}$,
 133 hence a vertex of degree at most 6 in the minor, because $\overline{G_{12}}$ is 6-regular.

134 Note that $\kappa(G_{12}) = 5$ and $\kappa(\overline{G_{12}}) = 6$, so $\nu(G_{12}) + \nu(\overline{G_{12}}) \geq 11 > |G_{12}| - 2$. So
 135 G_{12} satisfies GCC_ν and hence GCC and GCC_+ .

136 In 1997 the following related conjecture was made:

CONJECTURE 1.7 (GCC_μ Conjecture). [21, p. 512]¹ For any graph G ,

$$\mu(G) + \mu(\overline{G}) \geq |G| - 2. \quad (1.6)$$

137

138 It is a consequence of results in [21] and [9] that GCC_μ holds for all planar graphs,
 139 so the icosahedral graph in Example 1.6 does satisfy GCC_μ . Since, in general, μ is
 140 not comparable to ν or M_+ , it follows that GCC_μ does not imply either GCC_ν or
 141 GCC_+ , but it does imply GCC .

142 In Section 2 we turn to the case of k -trees, and making use of some recent analysis
 143 on the minimum ranks of the complements of k -trees, we will establish that the GCC ,
 144 GCC_+ , and GCC_ν are valid for this class of graphs as well. In Section 3 we consider
 145 joins of graphs. The first subsection involves GCC_ν and we prove, via induction, that
 146 if two graphs satisfy GCC_ν so will their join. In the next subsection we will verify
 147 that if a modified version of GCC (or GCC_+) holds for two graphs, then GCC (or
 148 GCC_+) holds for their join.

149 **2. k -trees and the Graph Complement Conjecture.** A graph G is called a
 150 k -tree if it can be constructed inductively by starting with K_{k+1} and connecting each
 151 new vertex to the vertices of an existing K_k (i.e., a k -clique). Every clique in a k -tree
 152 is part of a $(k+1)$ -clique, and a k -tree is a k -connected chordal graph with maximum
 153 clique size $k+1$. The graph depicted in Figure 2.1, known as the supertriangle, is
 154 an example of a 2-tree on 6 vertices. A graph G is called a *partial k -tree* if G is a
 155 subgraph of a k -tree. Observe that each graph is a partial k -tree for some value of
 156 k (for example, $k = |G| - 1$ always works). The minimum k for which G is a partial
 157 k -tree is equal to the tree-width $\text{tw}(G)$ of G (see, for example, [7, F12 p. 111]).

158 The main purpose of this section is to verify that the graph complement conjecture
 159 (and its variants GCC_+ and GCC_ν) hold for k -trees, for certain classes of partial k -
 160 trees, and for small graphs. Much of the following analysis relies on recent work by
 161 H. van der Holst and J. Sinkovic (see [19]), which we state here for completeness.

¹The reader is warned that in [21] the notation $\nu(G)$ means something entirely different from the Colin de Verdière parameter $\nu(G)$ used in this paper.

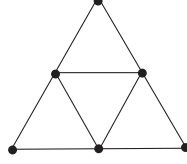


FIG. 2.1. The supertriangle T_3

THEOREM 2.1. [19] *If G is a partial k -tree, then*

$$\nu(\overline{G}) \geq |G| - k - 2.$$

162

THEOREM 2.2. [19] *If G is a partial 3-tree, then*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

163

164 Finally, in the same work they observe (using the above results and results from
165 [22], [23]) that GCC_ν holds for k -connected partial k -trees.

COROLLARY 2.3. [19] *If G is a k -connected partial k -tree, then*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

166 *In particular, if G is a k -tree, then G satisfies GCC_ν .*

167 As a consequence of Theorem 2.1 and the fact that $\text{tw}(G)$ is the minimum k such
168 that G is a partial k -tree, we have the following corollary.

169 COROLLARY 2.4. *If GCC_ν fails for a graph G , then $\kappa(G) < \text{tw}(G)$.*

170 Since the Hadwiger number minus one is a lower bound for ν , we have the fol-
171 lowing consequence of Theorem 2.1.

COROLLARY 2.5. *If G is a partial k -tree with $h(G) = k + 1$, then G satisfies*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

172

173 *Proof.* Apply Observation 1.5 and Theorem 2.1. \square

174 OBSERVATION 2.6. *If G is a graph for which $\nu(\overline{G}) \geq |G| - h(G) - 1$ (respectively,
175 $\text{mr}_+(\overline{G}) \leq h(G) + 1$, $\text{mr}(\overline{G}) \leq h(G) + 1$) then G will satisfy GCC_ν (respectively,
176 GCC_+ , GCC).*

177 Previously, GCC was known to hold for all graphs on seven or fewer vertices,
 178 since for all such graphs, the minimum ranks have been exhaustively computed [11].
 179 Here we extend this result (and eliminate the need for exhaustive computation) and
 180 determine properties of a minimum counterexample to GCC, GCC_+ , or GCC_ν .

181 COROLLARY 2.7. *If GCC_ν fails for some graph G , then $\nu(G) \geq 3$ and $\nu(\overline{G}) \geq 3$.*

182
 183 *Proof.* By Theorem 2.2, neither G nor \overline{G} can be a partial 2-tree. Since a graph is
 184 not a partial 2-tree if and only if it has a K_4 minor [7, F31, p. 112], $h(G), h(\overline{G}) \geq 4$
 185 and $\nu(G), \nu(\overline{G}) \geq 3$. \square

186 COROLLARY 2.8. *If G is a graph with $|G| \leq 8$, then GCC_ν holds for G .*

Proof. If GCC_ν fails, then by Corollary 2.7 we have

$$|G| - 2 > \nu(G) + \nu(\overline{G}) \geq 3 + 3.$$

187 This reduces to $|G| > 8$, as desired. \square

188 Note that GCC_μ (Conjecture 1.7) holds for any graph G of order at most 7, since
 189 for such a graph either G or \overline{G} must be planar, and, as observed in the paragraph
 190 following Conjecture 1.7, GCC_μ holds for all planar graphs (see also [21]).

191 Since it is established that any graph having tree-width at most three satisfies
 192 GCC_ν (and hence GCC), we can improve the bounds in Corollaries 2.7 and 2.8 for
 193 GCC by examining the forbidden minors for tree-width three, which are K_5 , the
 194 complete tripartite graph $K_{2,2,2}$, the graph V_8 shown in Figure 2.2 with the numbering
 195 that will be used throughout the discussion of this graph, and the Cartesian product
 196 $C_5 \square P_2$ (see [3] for the definition of Cartesian product). We use the minor monotone
 197 Colin de Verdière-type parameter ξ , introduced in [5] and defined to be the maximum
 198 nullity over all matrices A in $\mathcal{S}(G)$ that satisfy the Strong Arnold Hypothesis. Clearly
 $\nu(G) \leq \xi(G) \leq M(G)$ for all G .

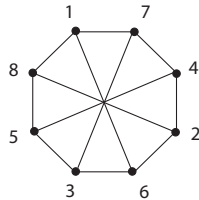


FIG. 2.2. The graph V_8

199
 200 PROPOSITION 2.9. $\nu(K_{2,2,2}) = \xi(K_{2,2,2}) = 4$, $\xi(V_8) = 4$, and $\xi(C_5 \square P_2) = 4$.

Proof. For $\nu(K_{2,2,2}) = \xi(K_{2,2,2}) = 4$, note that $\text{mr}(K_{2,2,2}) = 2$ and let

$$B = \begin{bmatrix} 1 & 1 & 2 & 1 & -2 & 1 \\ 1 & 2 & 1 & -1 & 1 & -2 \end{bmatrix}.$$

201 Then the positive semidefinite matrix $A = B^T B \in \mathcal{S}(K_{2,2,2})$, $\text{rank } A = 2$, and it is
 202 straightforward to verify that A satisfies the Strong Arnold Hypothesis (which can be
 203 checked using a computer symbolic package).

For $\xi(V_8) = 4$, note that $M(V_8) = 4$ (see the Möbius ladder in [3]) and let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 1 & 0 & -2 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & -2 & -1 & 0 \\ 0 & -2 & 0 & 2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 2 & 0 & 0 & -2 \\ -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & -2 & 0 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}(V_8).$$

204 Since $\text{rank } A = 4$ and A satisfies the Strong Arnold Hypothesis, $4 \leq \xi(V_8) \leq M(V_8) =$
 205 4.

For $\xi(C_5 \square P_2) = 4$, note that $M(C_5 \square P_2) = 4$ [3] and let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

206 Then $A \in \mathcal{S}(C_5 \square P_2)$, $\text{null}(A) = 4$, and A satisfies the Strong Arnold Hypothesis. \square

207 **COROLLARY 2.10.** *If GCC fails for some graph G , then $\text{mr}(G) \leq |G| - 4$ and*
 208 $\text{mr}(\overline{G}) \leq |G| - 4$.

209 *Proof.* By Theorem 2.2, neither G nor \overline{G} can be a partial 3-tree. A graph is not
 210 a partial 3-tree if and only if it has at least one of the graphs K_5 , $K_{2,2,2}$, V_8 , $C_5 \square P_2$
 211 as a minor [7, F33, p. 112]. Thus by Proposition 2.9, G has $4 \leq \xi(G) \leq M(G)$. Thus
 $\text{mr}(G) \leq |G| - 4$, and similarly for \overline{G} . \square

212

213 COROLLARY 2.11. *If G is a graph with $|G| \leq 10$, then GCC holds for G .*

214 The method used to establish Corollaries 2.10 and 2.11 does not work for GCC_v ,
 215 or GCC_+ , since $\text{mr}_+(V_8) = 5$. To see this, we attempt to construct a vector represen-
 216 tation in \mathbb{R}^4 of the vertices of V_8 (as labeled in Figure 2.2). Without loss of generality
 217 the first three vectors (representing vertices 1, 2, and 3) are the first three standard
 218 basis vectors. Then vector 4 is orthogonal to 1 and 3, but not to 2 and not a multiple
 219 of 2, so it is a multiple of $(0, 1, 0, a)$ for a nonzero a ; similarly 5 is $(0, 0, 1, b)$ with
 220 $b \neq 0$. Then it follows that vector 6 must be a multiple of $(c, a, b, -1)$, for c nonzero.
 221 Finally, vector 7 is in the null space of columns 2, 5, and 6, so it is a multiple of
 222 $(1 + b^2, 0, -bc, c)$, and vector 8 is a multiple of $(1 + a^2, -ac, 0, c)$. But vectors 7 and 8
 223 are required to be orthogonal, implying $1 + a^2 + b^2 + a^2b^2 + c^2 = 0$, a contradiction.

3. Joins of Graphs. All unions and joins in this paper involve disjoint graphs. Recall that, if G_1 and G_2 are disjoint graphs, the *union* and the *join* of G_1 and G_2 , denoted respectively by $G_1 \cup G_2$ and $G_1 \vee G_2$, are the graphs defined by

$$\begin{aligned} V(G_1 \cup G_2) &= V(G_1 \vee G_2) = V(G_1) \cup V(G_2); \\ E(G_1 \cup G_2) &= E(G_1) \cup E(G_2); \\ E(G_1 \vee G_2) &= E(G_1) \cup E(G_2) \cup E, \end{aligned}$$

where E consists of all the edges $\{u, v\}$ with $u \in V(G_1)$, $v \in V(G_2)$. A union or a join of r graphs is defined inductively by

$$\bigcup_{i=1}^r G_i = \left(\bigcup_{i=1}^{r-1} G_i \right) \cup G_r, \quad \bigvee_{i=1}^r G_i = \left(\bigvee_{i=1}^{r-1} G_i \right) \vee G_r.$$

224 Some of the results in this section rely on a ‘‘Rotation Lemma’’ as it was referred
 225 to in [4, Lemma 2.3] that pertains to the construction of certain types of isometries in
 226 an indefinite inner product space. To this end, we require some additional notation
 227 regarding the inertia of a symmetric matrix. For any $n \times n$ symmetric matrix A , we
 228 define the *inertia* of A as the triple $(i_+(A), i_-(A), i_0(A))$, consisting of the number
 229 of positive, negative, and zero eigenvalues (counting multiplicity) of A , respectively.
 230 Clearly, $i_0(A) = n - i_+(A) - i_-(A)$, and A is positive semidefinite if and only if
 231 $i_-(A) = 0$.

DEFINITION 3.1. Suppose is an $n \times n$ A symmetric matrix. A *nonzero (h, k) -representation of A* is a $(h + k) \times n$ matrix

$$\begin{bmatrix} P_A \\ N_A \end{bmatrix}$$

10

232 with no zero columns such that P_A has h rows, N_A has k rows and $A = P_A^T P_A -$
 233 $N_A^T N_A$.

234 Observe that for such a representation to exist, we must have that $h \geq i_+(A)$
 235 and $k \geq i_-(A)$. In fact, the matrix P_A represents the positive inertia of A , and N_A
 236 represents the negative inertia of A . Also note that if A is positive semidefinite, then
 237 N_A may be chosen to be the zero matrix.

238 Any symmetric matrix having all columns nonzero has a nonzero (h, k) -representa-
 239 tion whenever both $h \geq i_+(A)$ and $k \geq i_-(A)$. However, not every symmetric ma-
 240 trix has a nonzero $(i_+(A), i_-(A))$ -representation, due to the presence of zero columns.
 241 In particular if G is a graph with no isolated vertices, then any matrix $A \in \mathcal{S}(G)$ with
 242 rank $A = \text{mr}(G)$ has a nonzero (h, k) -representation with $h + k = \text{mr}(G)$. Finally,
 243 observe that if A has a nonzero (h, k) -representation, then, by padding both P_A and
 244 N_A with zero rows as needed, it follows that A has a nonzero (h', k') -representation
 245 for $h' \geq h$ and $k' \geq k$.

A matrix Q of order $h + k$ is said to be (h, k) -orthogonal if $Q^T \tilde{I} Q = \tilde{I}$, where

$$\tilde{I} = \begin{bmatrix} I_h & 0 \\ 0 & -I_k \end{bmatrix}$$

246 (I_s refers to the $s \times s$ identity matrix). Given a nonzero (h, k) -representation $\begin{bmatrix} P_A \\ N_A \end{bmatrix}$
 247 for A and a (h, k) -orthogonal matrix Q , it follows that $Q \begin{bmatrix} P_A \\ N_A \end{bmatrix}$ is also a nonzero
 248 (h, k) -representation for A . The previous fact can be verified by direct computation.

249 We are now in a position to state a revised version of the rotation lemma that
 250 was presented in [4, Lemma 2.3]. We remark here that the proof is basically the same
 251 as the one presented in [4] and is not repeated here.

LEMMA 3.2. *Let G and H be two graphs and let $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$.
 Suppose A and B each have nonzero (h, k) -representations $\begin{bmatrix} P_A \\ N_A \end{bmatrix}$ and $\begin{bmatrix} P_B \\ N_B \end{bmatrix}$,
 respectively with $h \geq 2$. Then there exists an (h, k) -orthogonal matrix Q such that*

$$\begin{bmatrix} P_A & P'_B \\ N_A & N'_B \end{bmatrix}$$

is a nonzero (h, k) -representation of a matrix in $\mathcal{S}(G \vee H)$ with

$$\begin{bmatrix} P'_B \\ N'_B \end{bmatrix} = Q \begin{bmatrix} P_B \\ N_B \end{bmatrix}$$

252

Note that in Lemma 3.2 we must have

$$h \geq \max\{i_+(A), i_+(B)\} \text{ and } k \geq \max\{i_-(A), i_-(B)\}.$$

253 Also observe that if $k = 0$, we obtain a result for positive semidefinite matrices in
254 $\mathcal{S}(G)$ and $\mathcal{S}(H)$

255 In Section 3.1 we prove that if G and H are graphs each satisfying GCC_ν , then
256 $G \vee H$ (or equivalently, $G \cup H$) satisfies GCC_ν . Related results for GCC and GCC_+ ,
257 which are substantially more complicated, are proved in Section 3.2.

258 **3.1. GCC_ν for joins of graphs.** The Colin de Verdière type parameters have
259 the important property that instead of summing over connected components (like
260 maximum nullity or minimum rank), they take the maximum.

THEOREM 3.3. [10] *For disjoint graphs G and H , $\nu(G \cup H) = \max\{\nu(G), \nu(H)\}$,
so*

$$\text{mr}_\nu(G \cup H) = |G| + |H| - \max\{\nu(G), \nu(H)\}.$$

261 For example, whereas $\text{mr}(G) = 1$ implies $G = K_r \cup \overline{K_s}$, $\text{mr}_\nu(G) = 1$ implies $G =$
262 K_r , $r \geq 2$ or $G = K_1 \cup K_1$.

263 DEFINITION 3.4. A ν -optimal matrix for a graph G is a positive semidefinite
264 matrix $A \in \mathcal{S}(G)$ that satisfies the Strong Arnold Hypothesis and has $\text{null } A = \nu(G)$
265 (or equivalently, $\text{rank } A = \text{mr}_\nu(G)$).

266 LEMMA 3.5. *If G has an edge then there exists a ν -optimal matrix A for G such
267 that every column of A has a nonzero entry.*

268 *Proof.* For any $B \in \mathcal{S}(G)$, B is a block diagonal matrix with diagonal blocks
269 associated with the connected components of G . If there is only one component, the
270 result is immediate. If $B \in \mathcal{S}(G)$ satisfies the Strong Arnold Hypothesis, then at
271 most one of the diagonal blocks of B is singular [5, Lemma 3.1]. A ν -optimal matrix
272 must have the singular block associated with a component having maximum value
273 of ν . Since $\nu(K_1) = 1$, we can choose to have the singular block associated with a
274 component that has an edge. \square

275 LEMMA 3.6. *Suppose H is an induced subgraph of G . Then $\text{mr}_\nu(H) \leq \text{mr}_\nu(G)$.*

Proof. Suppose A is a ν -optimal matrix for G , and let B be the principal subma-
trix of A that corresponds to the induced subgraph H . By renumbering if necessary
we may assume $A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$. From properties of positive semidefinite matrices, it
follows that B is a positive semidefinite matrix with graph H , so once we show that

B satisfies the Strong Arnold Hypothesis,

$$\text{mr}_\nu(H) \leq \text{rank } B \leq \text{rank } A = \text{mr}_\nu(G).$$

Note that the column inclusion property of positive semidefinite matrices guarantees that there exists a matrix E such that $C = BE$. So if Y is a symmetric matrix such that $BY = 0, B \circ Y = 0$ and $I \circ Y = 0$, define $X = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$AX = \begin{bmatrix} BY & 0 \\ C^T Y & 0 \end{bmatrix} = \begin{bmatrix} BY & 0 \\ E^T BY & 0 \end{bmatrix} = 0.$$

276 Since A satisfies the Strong Arnold Hypothesis, $X = 0$, so $Y = 0$ and B satisfies the
277 Strong Arnold Hypothesis. \square

278 **THEOREM 3.7.** *Let G and H be graphs. If*

- 279 1. G and H each have an edge, or
280 2. either G has an edge and $H = \overline{K_r}$, and $\text{mr}_\nu(G) \geq r$;
281 or the same is true with the roles of G and H reversed,

then

$$\text{mr}_\nu(G \vee H) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\}.$$

Otherwise,

$$\text{mr}_\nu(G \vee H) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\} + 1.$$

282

283 *Proof.* Assume first one of conditions (1) and (2) is true. In case (1) without
284 loss of generality $\text{mr}_\nu(G) \geq \text{mr}_\nu(H)$. In case (2) without loss of generality G has
285 an edge, $H = \overline{K_r}$, and $\text{mr}_\nu(G) \geq r$, so $\text{mr}_\nu(G) > r - 1 = \text{mr}_\nu(H)$. In either case,
286 $\text{mr}_\nu(G) \geq \text{mr}_\nu(H)$.

287 Assume first that $\text{mr}_\nu(G) = 1$. Since G has an edge, the case $G = K_1 \cup K_1$
288 is excluded and $G = K_t$ for some $t \geq 2$. Since $\text{mr}_\nu(G) \geq \text{mr}_\nu(H)$, $\text{mr}_\nu(H) \leq 1$.
289 Furthermore, either H has an edge, in which case $H = K_s$, or $H = K_1$ (because in
290 this case $1 = \text{mr}_\nu(G) \geq |H|$), so $H = K_s$ for some $s \geq 1$. Thus $G \vee H = K_{t+s}$ and
291 $\text{mr}_\nu(G \vee H) = 1 = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\}$.

So assume $\text{mr}_\nu(G) \geq 2$. Since G has an edge, by Lemma 3.5 we can choose
a ν -optimal matrix A for G such that every column of A has a nonzero entry. If
 $H \neq \overline{K_r}$, then we can also choose a ν -optimal matrix B for H such that every
column of B has a nonzero entry. If $H = \overline{K_r}$, then $r \leq \text{mr}_\nu(G)$ by hypothesis, so

we can choose a diagonal matrix $B \in \mathcal{S}(H)$ having all diagonal entries positive and $\text{rank } B = r \leq \text{mr}_\nu(G) = \text{rank } A$. Note that $i_+(A) = \text{rank } A = \text{mr}_\nu(G)$. Then, by Lemma 3.2, we may construct a positive semidefinite matrix $C = \begin{bmatrix} A & *^T \\ * & B \end{bmatrix}$ (where $*$ denotes a matrix all of whose entries are nonzero) with $\text{rank } C = i_+(A) = \text{mr}_\nu(G)$. Since A and B satisfy the Strong Arnold Hypothesis, any such matrix C satisfies the Strong Arnold Hypothesis. Thus it follows that

$$\text{mr}_\nu(G \vee H) = \text{mr}_\nu(G) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\},$$

292 by also applying Lemma 3.6. This completes the proof for the case in which G and
293 H satisfy condition (1) or (2).

For all remaining cases, we may assume that $H = \overline{K_r}$ and $r > \text{mr}_\nu(G)$. Then

$$\text{mr}_\nu(G \vee H) \geq r = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\} + 1$$

294 because if $C \in \mathcal{S}(G \vee H)$ is positive semidefinite, then C contains an $r \times r$ diagonal
295 matrix with positive diagonal (associated with H).

296 Either G has an edge or $G = \overline{K_s}$ with $s \leq r$. If G has an edge, choose a ν -
297 optimal matrix A for G such that every column of A has a nonzero entry. If $G = \overline{K_s}$,
298 chose a positive definite diagonal matrix $A \in \mathcal{S}(G)$. Then choosing a positive definite
299 diagonal matrix $B \in \mathcal{S}(H)$ and arguing as above shows $\text{mr}_\nu(G \vee H) \leq r$. \square

300 **THEOREM 3.8.** *If G and H are graphs that satisfy GCC_ν , then $G \vee H$ and $G \cup H$*
301 *satisfy GCC_ν .*

Proof. It suffices to prove the result for $G \vee H$. First assume $H = \overline{K_r}$ and $r > \text{mr}_\nu(G)$, so by Theorem 3.7, $\text{mr}_\nu(G \vee H) = r = |H|$. By Theorem 3.3,

$$\begin{aligned} \text{mr}_\nu(\overline{G} \cup K_r) &= |G| + r - \max\{\nu(\overline{G}), r - 1\} \\ &\leq |G| + r - (r - 1) \\ &= |G| + 1. \end{aligned}$$

Thus

$$\text{mr}_\nu(G \vee H) + \text{mr}_\nu(\overline{G} \cup \overline{H}) \leq |H| + |G| + 1,$$

302 and $G \vee H$ satisfies GCC_ν .

Now assume G and H satisfy GCC_ν and satisfy condition (1) or (2) of Theorem 3.7, so

$$\text{mr}_\nu(G \vee H) + \text{mr}_\nu(\overline{G} \cup \overline{H}) = \max\{\text{mr}_\nu(G), \text{mr}_\nu(H)\} + |G| + |H| - \max\{\nu(\overline{G}), \nu(\overline{H})\}.$$

303 Without loss of generality, $\text{mr}_\nu(G) \geq \text{mr}_\nu(H)$.

Suppose first that $\nu(\overline{G}) \geq \nu(\overline{H})$. Then

$$|G| + |H| - \max\{\nu(\overline{G}), \nu(\overline{H})\} = |G| - \nu(\overline{G}) + |H| = \text{mr}_\nu(\overline{G}) + |H|,$$

so

$$\begin{aligned} \text{mr}_\nu(G \vee H) + \text{mr}_\nu(\overline{G \vee H}) &= \text{mr}_\nu(G) + \text{mr}_\nu(\overline{G}) + |H| \\ &\leq 2 + |G| + |H| = 2 + |G \vee H|, \end{aligned}$$

304 using the fact that G satisfies GCC_ν . Thus $G \vee H$ satisfies GCC_ν .

Now suppose that $\nu(\overline{H}) > \nu(\overline{G}) \geq 1$. Thus

$$|\overline{H}| - \text{mr}_\nu(\overline{H}) > |\overline{G}| - \text{mr}_\nu(\overline{G}).$$

Using reasoning similar to that above,

$$\begin{aligned} \text{mr}_\nu(G \vee H) + \text{mr}_\nu(\overline{G \vee H}) &= \text{mr}_\nu(G) + \text{mr}_\nu(\overline{H}) + |G| \\ &< \text{mr}_\nu(G) + \text{mr}_\nu(\overline{G}) + |H| \\ &\leq 2 + |G| + |H|, \end{aligned}$$

so again $G \vee H$ satisfies GCC_ν . \square

305

306 A graph is said to be *decomposable* if it can be expressed as a sequence of joins
307 and unions of isolated vertices (these graphs are also known as cographs). We also
308 note that the complement of a decomposable graph is again decomposable.

309 **COROLLARY 3.9.** *If G is a decomposable graph, then G satisfies GCC_ν .*

310 **3.2. GCC and GCC_+ for joins of graphs.** In this section for convenience
311 we extend the definition of a graph to include a graph with no vertices, which will be
312 denoted by \emptyset . By definition, $\text{mr}(\emptyset) = \text{mr}_+(\emptyset) = 0$.

313 If $G = \bigcup_{i=1}^r G_i$, where each G_i is connected, the subgraph $\check{G} = \bigcup_{|G_i| > 1} G_i$ is
314 called the *core* of G , while $\ddot{G} = \bigcup_{|G_i|=1} G_i$ is called the *isolated part* of G . Note that
315 if G is connected, then $G = \check{G}$ if and only if $|G| = 1$. Also if G has no isolated vertices
316 then $G = \check{G}$, and G is said to be *isolated free*, while if G only consists of one or more
317 isolated vertices, then $\check{G} = \emptyset$.

318 **OBSERVATION 3.10.** *Let G be a graph. Then $\text{mr}(G) = \text{mr}(\check{G})$, $\text{mr}_+(G) =$
319 $\text{mr}_+(\check{G})$.*

320 The *join minimum rank* of $G \neq \emptyset$ is defined to be $\text{jmr}(G) = \text{mr}(K_1 \vee G)$ [4] and
321 $\text{jmr}(\emptyset) = 1$. Along similar lines, we define the notion of the join minimum rank within
322 the setting of positive semidefinite matrices.

DEFINITION 3.11. For any graph $G \neq \emptyset$, define

$$\text{jmr}_+(G) = \text{mr}_+(K_1 \vee G).$$

We also define $\text{jmr}_+(\emptyset) = 1$.

The notion of join minimum rank is needed here, as the minimum rank of the join can be adversely affected if at least one of the graphs contains isolated vertices (see the next result for example). However, incorporating the join minimum rank then introduces a complication when it is applied to unions.

The following result is [4, Prop. 3.6] adapted to account for our definition of the minimum rank of \emptyset .

PROPOSITION 3.12. For any graph G ,

$$\text{jmr}(G) = \begin{cases} \text{mr}(G) & \text{if and only if } |\check{G}| = 0 \text{ and } G \neq \emptyset, \\ \text{mr}(G) + 1 & \text{if and only if } |\check{G}| = 1 \text{ or } G = \emptyset, \\ \text{mr}(G) + 2 & \text{if and only if } |\check{G}| \geq 2. \end{cases}$$

LEMMA 3.13. Let $G \neq \emptyset$ with $r \geq 0$ isolated vertices. Then

1. $\text{jmr}(G) = \text{mr}(\check{G}) + \min\{2, r\}$,

2. $\text{jmr}_+(G) = \text{mr}_+(\check{G}) + r$.

Proof. The proof of (1) is a direct application of Proposition 3.12, as $\text{mr}(G) = \text{mr}(\check{G})$. For (2), let $A \in \mathcal{S}((\check{G} \cup \overline{K_r}) \vee K_1)$ be positive semidefinite and let B be the principal submatrix of A obtained by deleting the joined K_1 . Then $\text{rank } A \geq \text{rank } B \geq \text{mr}_+(\check{G}) + r$, because B is a block diagonal matrix with positive diagonal entries associated with $\overline{K_r}$. By choosing a matrix of minimum semidefinite rank in $\mathcal{S}(\check{G})$ and positive diagonal entries associated with $\overline{K_r}$, we construct a matrix $B' \in \mathcal{S}(\check{G} \cup \overline{K_r})$ having rank $\text{mr}_+(\check{G}) + r$ that does not have a zero column, and it is straightforward to use B' to construct a matrix $A' \in \mathcal{S}((\check{G} \cup \overline{K_r}) \vee K_1)$ of the same rank. \square

We now move onto further notions of the core that will be relevant for the following discussion.

DEFINITION 3.14. For any graph G , the *symmetric core*, denoted by \tilde{G} is defined as follows:

$$\tilde{G} = \begin{cases} \check{G} & \text{if } G \text{ has isolated vertices,} \\ \overline{(\check{G})} & \text{otherwise (i.e., complement of the core of the complement).} \end{cases}$$

347

348 Observe that $G = \widetilde{G}$ if and only if both G and \overline{G} are isolated free. For a given
 349 graph G , we define inductively, the graphs: $G_0 = G$, and for $i = 1, 2, \dots$, let $G_i =$
 350 $\widetilde{G_{i-1}}$.

DEFINITION 3.15. The *inductive core* of G , denoted by \check{G} is defined as

$$\check{G} = \bigcap_i G_i.$$

351

352 Evidently, $\check{G} = G_i$, where i is the first integer in which $G_i = \widetilde{G_i}$. Thus it follows
 353 that the core and the symmetric core of \check{G} coincide with \check{G} itself, that is, both \check{G}
 354 and its complement are isolated free. Note, it may be the case that \check{G} will have no
 355 vertices.

DEFINITION 3.16. Let G be a graph. The *j-gap* and *j-gap₊* of G are defined to
 be

$$\begin{aligned} \text{jgap}(G) &= \text{jmr}(G) + \text{jmr}(\overline{G}) - |G|, \\ \text{jgap}_+(G) &= \text{jmr}_+(G) + \text{jmr}_+(\overline{G}) - |G|. \end{aligned}$$

356

357 Clearly, we have that $\text{jgap}(\overline{G}) = \text{jgap}(G)$, for any graph G . Moreover, if $\text{jgap}(G) \leq$
 358 2 , then since $\text{jmr}(G) \geq \text{mr}(G)$ it follows that G must satisfy GCC, and analogously
 359 for jgap_+ and GCC_+ .

360 LEMMA 3.17. For any graph $G \neq \emptyset$:

- 361 1. $\text{jgap}(G) \leq \text{jgap}(\check{G})$,
- 362 2. $\text{jgap}_+(G) \leq \text{jgap}_+(\check{G})$.

363 *Proof.* If G is isolated free then there is nothing to show, as $G = \check{G}$. So, suppose
 364 $G = \check{G} \cup \overline{K_r}$. Then $\overline{G} = \overline{\check{G}} \vee K_r$, and we have $\text{mr}(G) = \text{mr}(\check{G})$, $\text{mr}(\check{G}) \leq \text{jmr}(\check{G})$ (and
 365 equality holds unless $\check{G} = \emptyset$). Finally, it follows that $\text{jmr}(\overline{G}) = \text{jmr}(\check{G})$, as $r \geq 1$ and
 366 $\text{jmr}(X) = \text{jmr}(X \vee K_1)$ for any graph X by Proposition 3.12.

Then we have

$$\text{jmr}(G) = \text{mr}(\check{G}) + \min\{2, r\} \leq \text{jmr}(\check{G}) + r.$$

Hence

$$\begin{aligned}
\text{jgap}(G) &= \text{jmr}(G) + \text{jmr}(\overline{G}) - |G| \\
&\leq \text{jmr}(\check{G}) + r + \text{jmr}(\overline{\check{G}}) - (|\check{G}| + r) \\
&= \text{jgap}(\check{G}).
\end{aligned}$$

367 The proof of (2) is similar and is omitted here. \square

368 COROLLARY 3.18. For any graph $G \neq \emptyset$:

- 369 1. $\text{jgap}(G) \leq \text{jgap}(\check{G})$,
370 2. $\text{jgap}_+(G) \leq \text{jgap}_+(\check{G})$.

371 LEMMA 3.19. If G is a graph such that \check{G} satisfies GCC (respectively, GCC_+),
372 then $\text{jgap}(G) \leq 2$ (respectively, $\text{jgap}_+(G) \leq 2$). Then G satisfies GCC (respectively,
373 GCC_+).

374 *Proof.* In view of Corollary 3.18 it is sufficient to show that $\text{jgap}(\check{G}) \leq 2$. If
375 $\check{G} = \emptyset$, then so is $\overline{\check{G}}$, and hence $\text{jgap}(\emptyset) = 2$. So assume that $\check{G} \neq \emptyset$. Since \check{G} and
376 $\overline{\check{G}}$ are isolated free we have $\text{jmr}(\check{G}) = \text{mr}(\check{G})$ and $\text{jmr}(\overline{\check{G}}) = \text{mr}(\overline{\check{G}})$. Since \check{G} satisfies
377 GCC, we have $\text{jgap}(\check{G}) \leq 2$. The proof for GCC_+ is similar. \square

LEMMA 3.20. Let G and H be graphs. Then

$$\text{jmr}(G \cup H) \leq \text{jmr}(G) + \text{jmr}(H),$$

where the inequality can be strict. In the positive semidefinite case,

$$\text{jmr}_+(G \cup H) \leq \text{jmr}_+(G) + \text{jmr}_+(H),$$

378 with equality provided both $G \neq \emptyset$ and $H \neq \emptyset$.

Proof. These results are immediate if either $G = \emptyset$ or $H = \emptyset$, so assume that both
 $G \neq \emptyset$ and $H \neq \emptyset$. In both cases we use Lemma 3.13. In the positive semidefinite
case, assume $G \neq \emptyset$ has r_1 isolated vertices and $H \neq \emptyset$ has r_2 isolated vertices. Then
 $\text{jmr}_+(G) = \text{mr}_+(\check{G}) + r_1$ and $\text{jmr}_+(H) = \text{mr}_+(\check{H}) + r_2$, by Lemma 3.13. Another
application of Lemma 3.13 yields

$$\begin{aligned}
\text{jmr}_+(G \cup H) &= \text{mr}_+(\check{G} \cup \check{H}) + (r_1 + r_2) \\
&= \text{mr}_+(\check{G}) + \text{mr}_+(\check{H}) + r_1 + r_2 \\
&= \text{jmr}_+(G) + \text{jmr}_+(H).
\end{aligned}$$

In the symmetric case, an inequality appears since

$$\text{jmr}(G) = \text{mr}(\check{G}) + \min\{2, r\} \leq \text{mr}(\check{G}) + r,$$

whenever $r \geq 2$. To verify an instance of a strict inequality, consider $G = H = \overline{K_2}$. Then $\text{jmr}(G) = \text{jmr}(H) = 2$ and $\text{jmr}(G \cup H) = 2$. \square

LEMMA 3.21. *Suppose G is a given graph. Then $\text{jmr}(G) = 1$ ($\text{jmr}_+(G) = 1$) if and only if $G = K_r$ for some integer $r \geq 1$ or $G = \emptyset$.*

Proof. Since, in general, $\text{jmr}(G) \geq \text{mr}(G)$ (respectively, $\text{jmr}_+(G) \geq \text{mr}_+(G)$) it follows $\text{jmr}(G) = 1$ if and only if $\text{mr}(G) = 0$ or 1 (respectively, $\text{jmr}_+(G) = 1$ if and only if $\text{mr}_+(G) = 0$ or 1). The conclusion, then, readily follows. \square

Before we come to our main results on the join of graphs, we recall the following fact that can be found in [4] and deduced from the work in [6]. If the minimum rank of a graph G is at most 2, then G must be a decomposable graph. For more information, on the minimum rank of the joins of graphs and of decomposable graphs, see [4]. In particular, let $G = \bigvee_{i=1}^r G_i$ be a decomposable graph. Then G is said to be *anomalous* if

1. for each i , $\text{jmr}(G_i) \leq 2$; and
2. $K_{3,3,3}$ is a subgraph of G .

In particular, in a non-anomalous decomposable graph G with $\text{mr}(G) \leq 2$, there are at most two i for which $|G_i| \geq 3$ and $G_i = \check{G}_i$.

We now need to state a result that was originally used in [4, Lemma 3.7] for the case of inertially balanced graphs, but in fact this result holds under more relaxed conditions.

LEMMA 3.22. *Let $G \neq \emptyset$ be a graph. There exists $A \in \mathcal{S}(G)$ such that A has a nonzero (h, k) -representation with $h + k = \text{jmr}(G)$. There exists $A \in \mathcal{S}_+(G)$ such that $\text{rank } A = \text{jmr}_+(G)$ and A has a nonzero $(\text{jmr}_+(G), 0)$ -representation.*

Proof. We only provide a proof in the positive semidefinite case, as the argument for the indefinite case is identical to the one provided in [4, Lemma 3.7].

For the positive semidefinite case, suppose $\text{jmr}_+(G) = \text{mr}_+(G)$. Then G has no isolated vertices and any matrix $A \in \mathcal{S}_+(G)$ with $\text{rank } A = \text{mr}_+(G)$ has a $(\text{jmr}_+(G), 0)$ -representation. On the other hand, if $\text{jmr}_+(G) = \text{mr}_+(\check{G}) + r$ where $r \geq 1$, then $G = \check{G} \cup \overline{K_r}$. Let $\begin{bmatrix} P \\ 0 \end{bmatrix}$ be a nonzero $(\text{mr}_+(\check{G}), 0)$ -representation for any optimal matrix in $\mathcal{S}(\check{G})$. Then

$$\begin{bmatrix} P & 0 \\ 0 & I_r \end{bmatrix}$$

404 is a nonzero $(\text{jmr}_+(G), 0)$ -representation for a matrix in $\mathcal{S}_+(G)$. \square

405 A related result on the join of two graphs appears in [14] in the context of Her-
 406 mitian positive semidefinite matrices. Since the analysis in Lemmas 3.2 and 3.22
 407 requires only working over the reals, we have the next result as a consequence.

COROLLARY 3.23. *Let $G \neq \emptyset$ and $H \neq \emptyset$ be two disjoint graphs. Then*

$$\text{jmr}_+(G \vee H) = \text{mr}_+(G \vee H) = \max\{\text{jmr}_+(G), \text{jmr}_+(H)\}.$$

408

409 *Proof.* Let $m = \max\{\text{jmr}_+(G), \text{jmr}_+(H)\}$. Then by Lemma 3.22 there exist
 410 $A \in \mathcal{S}(G), B \in \mathcal{S}(H)$ having nonzero $(m, 0)$ -representations, so by Lemma 3.2 there
 411 is a $(m, 0)$ -representation a matrix in $\mathcal{S}_+(G \vee H)$. Thus $\text{mr}_+(G \vee H) \leq m$, but since
 412 G and H are induced subgraphs of $G \vee H$, $\text{mr}_+(G \vee H) \geq m$ also. The result for join
 413 minimum rank follows from the fact that $G \vee H$ does not have isolated vertices. \square

414 THEOREM 3.24. *Let G and H be two disjoint graphs with $G \neq \emptyset$ and $H \neq \emptyset$.
 415 Then:*

- 416 1. *If $\text{jmr}(G) \geq \text{jmr}(H) \geq 1$ and*
 - 417 (a) *if $\text{jmr}(H) \geq 3$, then $\text{jmr}(G \vee H) \leq \text{jmr}(G) + \text{jmr}(H) - 2$;*
 - 418 (b) *if $\text{jmr}(H) \leq 2$, then $\text{jmr}(G \vee H) \leq \text{jmr}(G) + \text{jmr}(H) - 1$.*
- 419 2. *If $\text{jmr}_+(G) \geq \text{jmr}_+(H) \geq 1$ and*
 - 420 (a) *if $\text{jmr}_+(H) \geq 2$, then $\text{jmr}_+(G \vee H) \leq \text{jmr}_+(G) + \text{jmr}_+(H) - 2$;*
 - 421 (b) *if $\text{jmr}_+(H) = 1$, then $\text{jmr}_+(G \vee H) \leq \text{jmr}_+(G) + \text{jmr}_+(H) - 1$.*

422 *Proof.* For 1(a), suppose $\text{jmr}(H) \geq 3$. By Lemma 3.22 we can choose $A \in \mathcal{S}(G)$
 423 such that A has a nonzero (h_G, k_G) -representation, with $h_G + k_G = \text{jmr}(G)$. Since
 424 $\text{jmr}(G) \geq 3$, by replacing A by $-A$ if necessary, we may assume $h_G \geq 2$, so $k_G \leq$
 425 $\text{jmr}(G) - 2$. Similarly, choose $B \in \mathcal{S}(H)$ having a nonzero (h_H, k_H) -representation,
 426 with $h_H + k_H = \text{jmr}(H)$, $h_H \geq 2$, and $k_H \leq \text{jmr}(H) - 2$.

Define $h = \max\{h_G, h_H\}$ and $k = \max\{k_G, k_H\}$. Then, by padding with zero
 rows as needed, there exist nonzero (h, k) -representations for A and B , respectively.
 Then, by Lemma 3.2, we may construct a symmetric matrix in $\mathcal{S}(G \vee H)$ with rank
 at most $h + k$. Thus it follows that

$$\text{jmr}(G \vee H) = \text{mr}(G \vee H) \leq h + k.$$

427 Observe that among the four possible sums of $h + k$, the maximum is always bounded
 428 above by $\text{jmr}(G) + \text{jmr}(H) - 2$, as desired.

For 1(b), consider first the case when $\text{jmr}(G) \geq 3$ and $\text{jmr}(H) = 2$. As with
 the argument applied in the case above, choose $A \in \mathcal{S}(G)$ with a nonzero (h_G, k_G) -

representation in which $h_G + k_G = \text{jmr}(G)$, and $h_G \geq 2$, $k_G \leq \text{jmr}(G) - 2$; and choose $B \in \mathcal{S}(H)$ having a nonzero (h_H, k_H) -representation, with $h_H \geq 1$, $k_H \leq 1$, and $h_H + k_H = 2$. As above, we can construct, by Lemma 3.2 a matrix in $\mathcal{S}(G \vee H)$ with rank at most $h + k$, where $h = \max\{h_G, h_H\} = h_G$ and $k = \max\{k_G, k_H\}$. It follows that

$$\text{jmr}(G \vee H) = \text{mr}(G \vee H) \leq h + k \leq \text{jmr}(G) + \text{jmr}(H) - 1.$$

Under 1(b), the next case to consider is $\text{jmr}(G) \geq 3$ and $\text{jmr}(H) = 1$. Then, as in the previous case, we may choose $A \in \mathcal{S}(G)$ having a nonzero (h_G, k_G) -representation, with $h_G + k_G = \text{jmr}(G)$, and with $h_G \geq 2$, $k_G \leq \text{jmr}(G) - 2$. Further, since $\text{jmr}(H) = 1$, $H = K_r$ for some $r \geq 1$ (Lemma 3.21), so let $B \in \mathcal{S}(H)$ with $i_+(B) = 1$ and $i_-(B) = 0$. Applying Lemma 3.2, we can construct a matrix in $\mathcal{S}(G \vee H)$ with rank at most $h_G + h_K$. Then we have

$$\text{mr}(G \vee H) = \text{jmr}(G \vee H) = \text{jmr}(G) = \text{jmr}(G) + \text{jmr}(H) - 1.$$

The next case to consider under 1(b) is $\text{jmr}(G) = 2$ and $\text{jmr}(H) = 2$. Then both G and H are decomposable and hence so is $G \vee H$. By Theorem 4.5 [4] we have

$$\begin{aligned} \text{jmr}(G \vee H) &\leq \max\{\text{jmr}(G), \text{jmr}(H)\} + 1 \\ &= \text{jmr}(G) + 1 \\ &= \text{jmr}(G) + \text{jmr}(H) - 1. \end{aligned}$$

The final case under 1(b) is $\text{jmr}(G) \leq 2$ and $\text{jmr}(H) = 1$. Then, again, both G and H are decomposable as is $G \vee H$. The graph $K_{3,3,3}$ is not induced in G because $\text{mr}(G) \leq \text{jmr}(G) \leq 2 < 3 = \text{mr}(K_{3,3,3})$, so it is not induced in $G \vee H$, as H is a complete graph. Thus $G \vee H$ is not anomalous. Hence in this case we have

$$\begin{aligned} \text{jmr}(G \vee H) &\leq \max\{\text{jmr}(G), \text{jmr}(H)\} \\ &= \text{jmr}(G) \\ &= \text{jmr}(G) + \text{jmr}(H) - 1, \end{aligned}$$

429 where, again, the first inequality follows from [4, Thm. 4.5].

430 For (2), the positive semidefinite case, the arguments are very similar, and since
431 we require $i_+ \geq 2$ to apply Lemma 3.2 the inequality conditions in 2(a) and 2(b) have
432 been reduced by 1 as compared with item 1. \square

433 Specializing to the case of decomposable graphs, we have the next result, which
 434 not only demonstrates that they satisfy GCC (or GCC₊), but they satisfy a slightly
 435 stronger condition.

436 **THEOREM 3.25.** *If $G \neq \emptyset$ is a decomposable graph, then $\text{jgap}(G) \leq 1$ (respec-*
 437 *tively, $\text{jgap}_+(G) \leq 1$).*

Proof. The proof is by induction on the order of the decomposable graph. Observe that if $G = K_1$, then $\text{jmr}(G) = \text{jmr}(\overline{G}) = |G| = 1$. Hence $\text{jgap}(G) = 1$ for the base case. Now, consider two arbitrary decomposable graphs G and H , each with jgap at most one. For the decomposable graph $G \vee H$, assume first that $G \vee H$ is not anomalous and $\text{jmr}(G) \geq \text{jmr}(H)$. In this case,

$$\begin{aligned} \text{jgap}(G \vee H) &= \text{jmr}(G \vee H) + \text{jmr}(\overline{G \vee H}) - (|G| + |H|) \\ &\leq \max\{\text{jmr}(G), \text{jmr}(H)\} + \text{jmr}(\overline{G}) + \text{jmr}(\overline{H}) - (|G| + |H|) \\ &= \text{jmr}(G) + \text{jmr}(\overline{G}) - |G| + (\text{jmr}(\overline{H}) - |H|) \\ &\leq \text{jmr}(G) + \text{jmr}(\overline{G}) - |G| \\ &= \text{jgap}(G) \leq 1, \end{aligned}$$

where the first inequality follows from [4, Thm. 4.5] and Lemma 3.20. If, on the other hand, $G \vee H$ is anomalous, then $\text{jmr}(G \vee H) = 3$ by [4, Thm. 4.5], and

$$\begin{aligned} \text{jgap}(G \vee H) &= \text{jmr}(G \vee H) + \text{jmr}(\overline{G \vee H}) - (|G| + |H|) \\ &\leq 3 + \text{jmr}(\overline{G}) - |G| + \text{jmr}(\overline{H}) - |H|. \end{aligned}$$

438 Observe that for any graph X , $\text{jmr}(X) = |X|$ if and only if $X = K_1$ or $X = K_1 \cup K_1$.
 439 If both $\text{jmr}(\overline{G}) < |\overline{G}| = |G|$ and $\text{jmr}(\overline{H}) < |\overline{H}| = |H|$, then $\text{jgap}(G \vee H) \leq 1$. Further,
 440 since $G \vee H$ is anomalous, it is not possible for both equalities $\text{jmr}(\overline{G}) = |\overline{G}|$ and
 441 $\text{jmr}(\overline{H}) = |\overline{H}|$ to hold. Thus, without loss of generality, assume $\text{jmr}(\overline{H}) = |\overline{H}|$ and
 442 hence G must itself be anomalous (and decomposable). In this case $\text{jmr}(G) = 3$, and
 443 hence $\text{jgap}(G \vee H) = \text{jgap}(G) \leq 1$, by induction.

444 Since the parameter jgap is symmetric with respect to complementation, the case
 445 of the union of two decomposable graphs follows trivially. This completes the proof,
 446 as any decomposable graph can be decomposed as a union or join of two decomposable
 447 graphs.

448 The argument in the positive semidefinite case can be proved in a similar manner
 449 in the nonanomalous case by using Corollary 3.23. \square

450 We are now in a position to state and prove the main results of this subsection
 451 on the join and union of graphs and the GCC.

452 THEOREM 3.26. *Suppose G and H are two graphs. Then*

1. *if $\text{jgap}(G)$ and $\text{jgap}(H)$ are both at most two, then*

$$\text{jgap}(G \vee H) \leq 2;$$

2. *if $\text{jgap}_+(G)$ and $\text{jgap}_+(H)$ are both at most two, then*

$$\text{jgap}_+(G \vee H) \leq 2.$$

453 *Proof.* First, we assume without loss of generality, that $\text{jmr}(G) \geq \text{jmr}(H)$. For (1),
 454 suppose $\text{jgap}(G), \text{jgap}(H) \leq 2$. We separate the argument into two cases: $\text{jmr}(H) \geq 3$
 455 and $\text{jmr}(H) \leq 2$.

If $\text{jmr}(H) \geq 3$, then we have

$$\begin{aligned} \text{jgap}(G \vee H) &= \text{jmr}(G \vee H) + \text{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|) \\ &\leq \text{jmr}(G) + \text{jmr}(H) - 2 + \text{jmr}(\overline{G}) + \text{jmr}(\overline{H}) - (|G| + |H|) \\ &= \text{jgap}(G) + \text{jgap}(H) - 2 \\ &\leq 2 + 2 - 2 = 2, \end{aligned}$$

456 where the first inequality follows from 1(a) of Theorem 3.24 and Lemma 3.20.

If, otherwise, $\text{jmr}(H) \leq 2$, then H is decomposable, and so by Theorem 3.25 we have $\text{jgap}(H) \leq 1$. Thus it follows that,

$$\begin{aligned} \text{jgap}(G \vee H) &= \text{jmr}(G \vee H) + \text{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|) \\ &\leq \text{jmr}(G) + \text{jmr}(H) - 1 + \text{jmr}(\overline{G}) + \text{jmr}(\overline{H}) - (|G| + |H|) \\ &= \text{jgap}(G) + \text{jgap}(H) - 1 \\ &\leq 2 + 1 - 1 = 2. \end{aligned}$$

457 The first inequality above follows from 1(b) of Theorem 3.25 and Lemma 3.20.

458 The positive semidefinite case follows in a similar manner. \square

459 The following are immediate consequences of the above results.

460 COROLLARY 3.27. *If G and H are two given graphs, and both of their induc-*
 461 *tive cores satisfy GCC (respectively, GCC_+), then $G \vee H$ and $G \cup H$ satisfy GCC*
 462 *(respectively, GCC_+).*

463 *Proof.* If G and H are two given graphs such that both of their inductive cores
 464 satisfy GCC (or GCC_+), then $\text{jgap}(\overset{\circ}{G}) \leq 2$ and $\text{jgap}(\overset{\circ}{H}) \leq 2$ (if $\overset{\circ}{G} \neq \emptyset$, then $\text{jmr}(\overset{\circ}{G}) =$

465 $\text{mr}(\check{G})$, and if $\check{G} = \emptyset$, then $\text{jgap}(\check{G}) = 2$). By Lemma 3.18 both $\text{jgap}(G) \leq 2$ and
466 $\text{jgap}(H) \leq 2$ and similarly for the positive semidefinite case. Thus an application of
467 Theorem 3.26 implies that $G \vee H$ satisfies GCC (GCC₊). The fact that the union
468 of G and H satisfy GCC (or GCC₊) follows from complementation and the fact
469 that both the hypothesis and conclusion are symmetric under the operation of taking
470 complements. \square

471 **COROLLARY 3.28.** *If $\delta(G) \geq |G| - 3$, then G satisfies GCC₊ (and hence GCC),*
472 *where $\delta(G)$ represents the minimum degree of G .*

473 *Proof.* Observe that if $\delta(G) \geq |G| - 3$, then \overline{G} is a disjoint union of cycles and
474 paths. Then apply Corollary 3.27, as the inductive cores of both paths and cycles can
475 easily be seen to satisfy GCC (or GCC₊). \square

Requiring the inductive core to satisfy GCC seems critical. Suppose G is a graph
that does not satisfy GCC. Without loss of generality, we may assume that $G = \check{G}$
(by Lemma 3.19). For the purposes of this argument, we actually need to assume that

$$\text{mr}(G) + \text{mr}(\overline{G}) = |G| + 4$$

476 (if it is larger than 4, the argument below can be modified). Define the new graph
477 $G' = G \cup \overline{K_2}$. Since G has no isolated part ($G = \check{G}$), by Proposition 3.12 we have
478 $\text{mr}(G') = \text{mr}(G)$ and $\text{mr}(\overline{G'}) = \text{mr}(\overline{G})$. Then, it follows that G' satisfies GCC. So G'
479 is a graph that satisfies GCC but its inductive core does not. Now let $H = \{w\}$, and
480 form $\Gamma = G' \vee H$. Again, applying Proposition 3.12 shows that $\text{mr}(\Gamma) = \text{mr}(G') + 2$
481 and $\text{mr}(\overline{\Gamma}) = \text{mr}(\overline{G'})$. Thus we may conclude that Γ does not satisfy GCC.

482 **4. Conclusion.** We close this work by formulating some basic necessary condi-
483 tions on a potential counterexample for each conjecture. We begin with a discus-
484 sion of graphs that attain low minimum rank. Since for every graph H , $\text{mr}(H)$,
485 $\text{mr}_+(H), \text{mr}_\nu(H) \leq |H| - 1$, GCC (respectively, GCC₊, GCC_ν) is valid for graphs
486 G that satisfy $\text{mr}(G) \leq 3$ ($\text{mr}_+(G) \leq 3, \text{mr}_\nu(G) \leq 3$). The low minimum rank case
487 argument can be pushed a little further (as was done in [15] for GCC).

488 **PROPOSITION 4.1.** *Suppose $\text{mr}(G) \leq 4$ (respectively, $\text{mr}_+(G) = 4, \text{mr}_\nu(G) = 4$).*
489 *Then G satisfies GCC (GCC₊, GCC_ν).*

490 *Proof.* Assume that $\text{mr}(G) \leq 4$ and G does not satisfy GCC (GCC₊, GCC_ν).
491 Then it follows that $\text{mr}(\overline{G}) = n - 1$ ($\text{mr}_+(\overline{G}) = \text{mr}_\nu(\overline{G}) = n - 1$). Hence $\overline{G} = P_n$ (\overline{G}
492 is a tree, or a forest) (see [13, 16]). However, paths (trees or forests) on n vertices
493 satisfy GCC (GCC₊, GCC_ν), which is a contradiction. \square

Hence it follows that if GCC, GCC₊, or GCC_ν fails for a given graph G , then G

must satisfy:

$$5 \leq \text{mr}(G) \leq \text{mr}_+(G) \leq \text{mr}_\nu(G),$$

and

$$5 \leq \text{mr}(\overline{G}) \leq \text{mr}_+(\overline{G}) \leq \text{mr}_\nu(\overline{G}).$$

494 Concerning the GCC, by Corollaries 2.10 and 2.11 and Proposition 4.1 it follows
495 that the first possible counterexample for GCC is a graph on 11 vertices that satisfies
496 $\text{mr}(G) = \text{mr}(\overline{G}) = 7$.

497 Similarly for GCC_+ and GCC_ν we may deduce from Corollaries 2.7 and 2.8 and
498 Proposition 4.1 that a first potential counterexample for GCC_ν or GCC_+ would be a
499 graph G on 9 vertices that satisfies $\nu(G) = \nu(\overline{G}) = 3$ or $\text{M}_+(G) = \text{M}_+(\overline{G}) = 3$.

500 Furthermore, from the work in Section 3.1 we may conclude that a minimal
501 counterexample for GCC_ν must be a graph for which both it and its complement are
502 connected. A similar statement can be made for the conjectures GCC and GCC_+
503 involving inductive cores, but the actual details are omitted here.

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