Orthogonal representations, minimum rank, and graph complements

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Abstract

Orthogonal representations are used to show that complements of certain sparse graphs have (positive semidefinite) minimum rank at most 4. This bound applies to the complement of a 2-tree and to the complement of a unicyclic graph. Hence for such graphs, the sum of the minimum rank of the graph and the minimum rank of its complement is at most two more than the order of the graph. The minimum rank of the complement of a 2-tree is determined exactly.

Keywords. minimum rank, orthogonal representation, 2-tree, unicyclic graph, graph complement.

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1 Introduction

A graph is a pair $G = (V, E)$, where $V$ is the (finite, nonempty) set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and $E$ is the set of edges (an edge is a two-element subset of vertices); what we call a graph is sometimes called a simple undirected graph. The order of a graph $G$, denoted $|G|$, is the number of vertices of $G$. The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets of $V$ that are not in $E$. The subgraph $G[R]$ of $G = (V, E)$ induced by $R \subseteq V$ is the subgraph with vertex set $R$ and edge set $\{\{i, j\} \in E \mid i, j \in R\}$.

The set of $n \times n$ real symmetric matrices will be denoted by $S_n$. For $A \in S_n$, the graph of $A$, denoted $G(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of $A$ is ignored in determining $G(A)$.

The set of real symmetric matrices of $G$ is

$$S(G) = \{A \in S_n : G(A) = G\}$$
and the set of real positive semidefinite matrices of $G$ is

$$S_+(G) = \{ A \in S_n : \text{A positive semidefinite and } G(A) = G \}.$$  

The *minimum rank* of a graph $G$ is

$$\text{mr}(G) = \min \{ \text{rank}(A) : A \in S(G) \},$$

and the *positive semidefinite minimum rank* of $G$ is

$$\text{mr}_+(G) = \min \{ \text{rank}(A) : A \in S_+(G) \}.$$  

Clearly

$$S_+(G) \subseteq S(G) \quad \text{and} \quad \text{mr}(G) \leq \text{mr}_+(G).$$

The *minimum rank problem* (of a graph, over the real numbers) is to determine $\text{mr}(G)$ for any graph $G$. See [10] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there. A minimum rank graph catalog [2] is available on-line, and will be updated regularly. Positive semidefinite minimum rank, both of real symmetric matrices as just defined, and of possibly complex Hermitian matrices, has been studied in [5], [6], [7], [11], [13].

At the AIM workshop [3] the relationship between the minimum rank of a graph and its complement was explored. The following question was asked:

**Question 1.1.** [8, Question 0.16] How large can $\text{mr}(G) + \text{mr}(\overline{G})$ be?

It was noted there that for the few graphs for which the minimum rank of both the graph and its complement was known,

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2$$  \hspace{1cm} (1)

and equality in this bound is achieved by a path.

In [1] it was shown that the (positive semidefinite) minimum rank of the complement of a tree is at most 3, and thus a tree satisfies the bound (1). We will show that some other families of sparse graphs, including unicyclic graphs (a graph is *unicyclic* if it contains exactly one cycle) and 2-trees (defined below) also satisfy the bound (1).

Unfortunately, there are conflicting uses of the term 2-tree in the literature. Here we follow [15] in defining a *k-tree* to be a graph that can be built up from a $k$-clique by adding one vertex at a time adjacent to exactly the vertices in an existing $k$-clique. Thus a tree is a 1-tree and a 2-tree can be thought of as a graph built up one triangle at a time by identifying an edge of a new triangle with an existing edge. A 2-tree is *linear* if it has exactly two vertices of degree 2. Thus a linear 2-tree is a “path” of triangles built up one triangle at a time by identifying an edge of a new triangle with an edge that has a vertex of degree 2. In [14] a *linear singly edge-articulated cycle graph* or *LSEAC graph* is (essentially) defined to be a “path” of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle. Such a graph can be obtained by deleting
interior edges from a outerplanar drawing of a linear 2-tree. In [12], the term “linear 2-tree” was used for an LSEAC graph, although an equivalent definition using the dual of an outerplanar drawing was given.

Examples of a linear 2-tree, an LSEAC graph that is not a 2-tree, and a 2-tree that is not linear are shown in Figure 1.

![Figure 1: Linear 2-tree, an LSEAC graph that is not a 2-tree, and a nonlinear 2-tree](image)

In [12] (and independently in [14]) it is shown that a 2-connected graph \( L \) is an LSEAC graph if and only if \( \text{mr}(L) = |L| - 2 \). Hence the minimum rank of the left and center graphs shown in Figure 1 is 9.

For a tree, unicyclic graph, or 2-tree \( G \), the number of edges of \( G \) is \(|G| - 1\), \(|G|, 2|G| - 3\), respectively, so trees, unicyclic graphs, and 2-trees are all sparse.

Suppose \( G = (V, E) \) is a graph. Then a \textit{d-dimensional orthogonal representation} of \( G \) is a function \( v \rightarrow \vec{v} \) from \( V \) to \( \mathbb{R}^d \) such that \( \vec{u} \) and \( \vec{v} \) are orthogonal if and only if \( u \) and \( v \) are nonadjacent vertices. For a subspace \( U \) of \( \mathbb{R}^d \), let \( U^\perp \) be the subspace of \( \mathbb{R}^d \) of vectors orthogonal to \( U \), and \( v^\perp = \text{Span}(v)^\perp \). The following observations will be used repeatedly.

\textbf{Observation 1.2.} [9] Let \( d(G) \) denote the smallest dimension \( d \) over all orthogonal representations of \( G \). Then \( d(G) \) is equal to \( \text{mr}_+(G) \).

\textbf{Observation 1.3.} No subspace \( W \) of \( \mathbb{R}^d \) is a union of a finite number of proper subspaces of \( W \).

\textbf{Observation 1.4.} For any three pairwise independent vectors \( v, u, w \in \mathbb{R}^4 \), the following are equivalent.

1. \( \dim \text{Span}(v, u, w) = 3 \).
2. \( v^\perp \cap u^\perp \nsubseteq w^\perp \).

\section{Orthogonal representations of dense graphs}

In [1] orthogonal representations were used to prove that the complement of a tree has positive semidefinite minimum rank at most 3. A complement of a tree can be constructed by adding one vertex at a time, with each new vertex adjacent to all
but one of the prior vertices. In this section we extend this technique to certain (very) dense graphs constructed by adding vertices adjacent to all but one or two prior vertices. These results will be used in the next section to study complements of certain sparse graphs and the relationship between mr($G$) and mr($\overline{G}$).

The following is an easy generalization of the proof of [1, Theorem 3.16].

**Theorem 2.1.** Let $Y = (V_Y, E_Y)$ be a graph of order at least two such that there is an orthogonal representation in $\mathbb{R}^d, d \geq 3$ satisfying
\[ \vec{v} \notin \text{Span}(\vec{u}) \quad \text{for } u \neq v \] for all vertices in $V_Y$. Let $X$ be a graph that can be constructed by starting with $Y$ and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most one vertex. Then there is a $d$-dimensional orthogonal representation of $X$ satisfying (2); in particular, $\text{mr}(X) \leq \text{mr}_+(X) \leq d$.

**Proof.** Let $V_Y = \{v_1, \ldots, v_k\}$. Let $X$ be constructed from $Y$ by adding vertices $v_{k+1}, \ldots, v_n$ such that for $m > k$, $v_m$ is adjacent to all but at most one of $v_1, \ldots, v_{m-1}$. Assuming that an orthogonal representation of $X[v_1, \ldots, v_{m-1}]$ in $\mathbb{R}^d$ has been constructed satisfying (2), we show there is an orthogonal representation of $X[v_1, \ldots, v_m]$ in $\mathbb{R}^d$ satisfying (2). If $v_m$ is adjacent to $v_1, \ldots, v_{m-1}$ then choose as $\vec{v}_m$ any vector not in $\bigcup_{i=1}^{m-1} \vec{v}_i^- \cup \bigcup_{i=1}^{m-1} \text{Span}(\vec{v}_i)$.

Otherwise, let $v_s$ be the only vertex of $X[v_1, \ldots, v_{m-1}]$ not adjacent to $v_m$ in $X[v_1, \ldots, v_m]$. We want to choose a vector $\vec{v}_m$ such that
\[ \vec{v}_m \in \vec{v}_s^- \]
\[ \vec{v}_m \notin \vec{v}_i^- \quad \forall i \neq s, i < m \]
\[ \vec{v}_m \notin \text{Span}(\vec{v}_i) \quad \forall i < m. \]

By applying Observation 1.3 to $W = \vec{v}_s^-$ and subspaces
\[ A_i = W \cap \vec{v}_i^- \quad i \neq s \]
\[ B_i = W \cap \text{Span}(\vec{v}_i) \quad i \neq s, i < m \]
we can conclude the desired vector exists, since clearly none of the subspaces $A_i, B_i$ is equal to $W$. Thus we have constructed an orthogonal representation of $X$ in $\mathbb{R}^d$ such that $\vec{u}$ and $\vec{v}$ are linearly independent for any distinct vertices $u, v$ of $X$.

**Theorem 2.2.** Let $Y = (V_Y, E_Y)$ be a graph of order at least two such that there is an orthogonal representation in $\mathbb{R}^4$ satisfying
\[ \vec{v} \notin \text{Span}(\vec{u}) \quad \text{for } u \neq v \] \[ \dim \text{Span}(\vec{v}, \vec{u}, \vec{w}) = 3 \quad \text{for all distinct } v, u, w \text{ such that } v \not\sim u \] for all vertices in $V_Y$. Let $X$ be a graph that can be constructed by starting with $Y$ and adding one vertex at a time, such that the newly added vertex is adjacent to all prior vertices except at most two nonadjacent vertices. Then there is an orthogonal representation of $X$ satisfying (3) and (4); in particular, $\text{mr}(X) \leq \text{mr}_+(X) \leq 4$. 

Proof. Let \( V_Y = \{v_1, \ldots, v_k\} \). Let \( X \) be constructed from \( Y \) by adding vertices \( v_{k+1}, \ldots, v_n \) such that for \( m > k \), \( v_m \) is adjacent to all but at most two of \( v_1, \ldots, v_{m-1} \). Assuming that an orthogonal representation of \( X[v_1, \ldots, v_{m-1}] \) has been constructed satisfying (3) and (4), we show there is an orthogonal representation of \( X[v_1, \ldots, v_m] \) satisfying (3) and (4).

If \( v_m \) is adjacent to all vertices except \( v_s, v_t \), it suffices to choose \( \vec{v}_m \) such that

\[
\vec{v}_m \in \overrightarrow{v}_s \cap \overrightarrow{v}_t \quad (5)
\]

\[
\vec{v}_m \notin \overrightarrow{v}_i \quad \forall i \neq s, t, i < m \quad (6)
\]

\[
\vec{v}_m \notin \text{Span}(\vec{v}_r, \vec{v}_r) \quad r = s, t, \forall i \neq r, i < m \quad (7)
\]

\[
\vec{v}_m \notin \text{Span}(\vec{v}_i, \vec{v}_j) \quad \forall i, j \text{ such that } 1 \leq i < j < m \text{ and } v_i \neq v_j \quad (8)
\]

We will show that it is always possible to make such a choice.

By applying Observation 1.3 to

\[
W = \overrightarrow{v}_s \cap \overrightarrow{v}_t
\]

and subspaces

\[
A_i = W \cap \overrightarrow{v}_i \quad i \neq s, t, i < m
\]

\[
B_i = W \cap \text{Span}(\vec{v}_r, \vec{v}_r) \quad r = s, t, \forall i \neq r, i < m
\]

\[
C_{ij} = W \cap \text{Span}(\vec{v}_i, \vec{v}_j) \quad i < j < n \text{ such that } v_i \neq v_j
\]

we can conclude the desired vector exists, provided none of the subspaces \( A_i, B_i, C_{ij} \) is equal to \( W \).

By condition (4) and Observation 1.4, \( \overrightarrow{v}_i \cap \overrightarrow{v}_j \not\subseteq \overrightarrow{v}_i \) for all \( i \neq s, t \), so \( A_i \neq W \).

For three pairwise independent vectors \( v, u, w \in \mathbb{R}^4 \), \( \dim(v^\perp) = 3 \), and similarly for \( u, w \), so

\[
\dim(v^\perp \cap u^\perp) = \dim(v^\perp) + \dim(u^\perp) - \dim(v^\perp + u^\perp) = 3 + 3 - 4 = 2.
\]

Thus \( \dim W = 2 = \dim \text{Span}(\vec{v}_r, \vec{v}_r) \), so if \( W = B_i \), \( W = \text{Span}(\vec{v}_r, \vec{v}_r) \). But \( \vec{v}_r \in \text{Span}(\vec{v}_r, \vec{v}_r) \) and \( \vec{v}_r \notin W \) (since \( r = s \) or \( r = t \)). Thus \( B_i \neq W \).

Finally, consider \( C_{ij} \). If \( W = C_{ij} \), then again both \( W \) and \( \text{Span}(\vec{v}_i, \vec{v}_j) \) are of dimension two, so

\[
\text{Span}(\vec{v}_i, \vec{v}_j) = W = \overrightarrow{v}_s \cap \overrightarrow{v}_t = \text{Span}(\overrightarrow{v}_s, \overrightarrow{v}_t).
\]

Hence, none of \( v_i, v_j, v_s, v_t \) is adjacent to any of the others. But then if \( p = \max\{i, j, s, t\} \), when \( v_p \) was added, it would have been nonadjacent to three prior vertices, which is prohibited.

If \( v_m \) is adjacent to all vertices except \( v_s \), choose \( \vec{v}_m \) satisfying

\[
\vec{v}_m \in \overrightarrow{v}_s \quad (9)
\]

\[
\vec{v}_m \notin \overrightarrow{v}_i \quad \forall i \neq s \quad (10)
\]

\[
\vec{v}_m \notin \text{Span}(\vec{v}_s, \vec{v}_r) \quad \forall i \neq s, m \quad (11)
\]

\[
\vec{v}_m \notin \text{Span}(\vec{v}_i, \vec{v}_j) \quad \forall i, j \text{ such that } 1 \leq i < j < m \text{ and } v_i \neq v_j \quad (12)
\]
The existence of an acceptable choice is guaranteed by using $W = \vec{v}_s^\perp$ in the previous argument and examining dimensions to show none of $A_i, B_i, C_i$ equals $W$. If $v_m$ is adjacent to all vertices the choice is even easier. \hfill \Box

The next example shows that the hypothesis, \emph{the two vertices to which the new vertex will not be adjacent must themselves be nonadjacent}, is necessary for Theorem 2.2.

**Example 2.3.** Let $G$ be the graph shown in Figure 2. It is straightforward to verify that if $G$ is constructed by adding vertices in order, each vertex added is adjacent to all but at most two prior vertices. Since $G$ is a linear 2-tree, $\mr(G) = |G| - 2 = 5$.

![Figure 2: A graph of minimum rank 5 constructed with each added vertex adjacent to all but at most two prior vertices.](image)

### 3 Minimum rank of graph complements

Theorems 2.1 and 2.2 can be applied to the complements of several families of sparse graphs. Since the complement of a 2-tree satisfies the hypotheses of Theorem 2.2, we have

**Corollary 3.1.** If $H$ is a 2-tree, then $\mr_+(\overline{H}) \leq 4$.

In fact, we can determine exactly what the minimum rank of the complement of a 2-tree is.

**Theorem 3.2.** If $H$ is a 2-tree, then

$$\mr(\overline{H}) = \begin{cases} 0 & \text{if } |H| = 3; \\ 1 & \text{if } |H| \geq 4 \text{ and } H \text{ has two vertices of degree } |H| - 1; \\ 3 & \text{if } |H| \geq 5 \text{ and } H \text{ has exactly one vertex of degree } |H| - 1; \\ 4 & \text{if } |H| \geq 5 \text{ and } H \text{ does not have a vertex of degree } |H| - 1. \end{cases}$$

**Proof.** If $H$ has two vertices of degree $|H| - 1$, then $H = (|H| - 2)K_1 \lor K_2$. The complement of $K_3$ is $3K_1$ and the complement of $(|H| - 2)K_1 \lor K_2$ is $K_{|H|-2} \lor 2K_1$, so the cases $\mr(\overline{H}) = 0, 1$ are clear. If $H$ is not one of these graphs, then $H$ contains an induced $L_3$ (see Figure 3). Since $L_3 = P_1 \lor K_1, \mr(\overline{H}) \geq 3$.

When a vertex is added to $L_3$, either it is adjacent to $v$ (which is adjacent to all other vertices), or it is not. If it is not, an induced $L_4$ or $T_3$ (see Figure 3) is created.
The complements of $L_4$ and $T_3$ are the unicyclic graphs shown in Figure 4. It is easy to show by any of a variety of methods, including cut-vertex reduction (cf. [4] or [10]) or the use of zero forcing sets (cf. [1]), that $\text{mr}(L_4) = 4$ and $\text{mr}(T_3) = 4$.

If $H$ is a 2-tree created by always adding vertices adjacent to vertex $v$, then $v$ is adjacent to every vertex other of $H$, and $H - v$ is a tree. Thus $\overline{H} = \overline{H} - v \cup \{v\}$ and $\text{mr}(H) = 3$ by [1].

To apply Theorem 2.1 to complements of unicyclic graphs, we need to show that for any cycle there is an orthogonal representation of dimension at most 4.

**Theorem 3.3.** For all $n \geq 3$, there is an orthogonal representation of $\overline{C_n}$ in $\mathbb{R}^4$ satisfying condition (2).

**Proof.** By Theorem 2.2, we can find an orthogonal representation of $\overline{P_{n-1}}$ in $\mathbb{R}^4$ satisfying conditions (3) and (4). Then arguing as in the proof of Theorem 2.2 we can add the remaining vertex adjacent to all but two vertices. Note that (the here false) hypothesis that these two vertices are not adjacent in $\overline{C_n}$ is not needed to obtain this representation. It is not needed to establish the existence of a vector meeting criteria (5) and (6), which is all that is necessary here, but is needed to continue the process used in the proof of Theorem 2.2 by establishing criterion (8). \hfill $\square$

**Corollary 3.4.** Let $H$ be a unicyclic graph. Then $\text{mr}_+(\overline{H}) \leq 4$.

**Proof.** A unicyclic graph can be constructed from a cycle by adding one vertex at a time, with the new vertex adjacent to at most one prior vertex. Thus the complement of a unicyclic graph has an orthogonal representation of dimension at most 4 by Theorems 3.3 and 2.1. \hfill $\square$
To ensure condition (2) for $C_4$, $\mathbb{R}^4$ is needed, even though $\text{mr}(C_4) = 2$, because $C_4 = 2K_2$, which does not have an orthogonal representation satisfying (2) in $\mathbb{R}^3$. Furthermore, there are examples of unicyclic graphs whose complements have minimum rank 4:

**Example 3.5.** The unicyclic graph $\overline{L_4}$ shown in Figure 4 has the linear 2-tree $L_4$ (shown in Figure 3) as its complement, and $\text{mr}(L_4) = 6 - 2 = 4$.

We have established the bound (1) for 2-trees and unicyclic graphs.

**Corollary 3.6.** Let $G$ be a graph such that $\text{mr}(G) \leq 4$. Then

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2.$$ 

In particular, trees, unicyclic graphs and 2-trees satisfy this bound.

**Proof.** If $G = P_n$ is a path, it was shown in [1] that $\text{mr}(P_n) \leq 3$, so

$$\text{mr}(P_n) + \text{mr}(\overline{P_n}) \leq n - 1 + 3 = n + 2.$$ 

If $G$ is not a path,

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| - 2 + 4 = |G| + 2. \quad \square$$

**References**


