Distance Spectra of Graphs

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Section I: Introduction

(A) Data communication problem: Loop switching
(B) Distance matrices
(C) Distance spectra results
[Graham, Pollak; 1971]

**Problem:** Find the best routing for messages through a communication network.

**Model:**
- Subscribers or computer terminals are on one-way loops (local loops) that are connected to each other and to regional loops, which are connected to each other and to a national loop.
- A message from a subscriber on one loop is usually destined for a subscriber on another loop.
- To move through the system, a message proceeds around its loop until it reaches a switching point, at which point it chooses to continue on its loop or switch to another loop.

**Question:** Find a simple efficient routing strategy for loop switching.
Graham and Pollak’s solution for loop switching

- Model the loop system with a graph.
- Use the distance matrix of the graph to find addresses.
- Use the addresses to choose at each junction whether or not to switch.
- Routing method is simple to apply.
- Message takes shortest path between loops in same region.
- Applies to any loop configuration.
Graph interpretation of loop system

http://www.math.ucsd.edu/~ronspubs/71_05_loop_switching.pdf
(B) Distance matrices

All graphs are simple, undirected, finite, and connected. $G = (V, E)$ is a graph of order $n$ with $V = \{v_1, \ldots, v_n\}$.

- The distance $d(v_i, v_j)$ between vertices $v_i$ and $v_j$ is the length of a shortest path between $v_i$ and $v_j$.

- $\mathcal{D}(G) = [d(v_i, v_j)]$ is the distance matrix of $G$ ($n \times n$ matrix).

\[
\begin{bmatrix}
0 & 1 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 1 \\
3 & 3 & 2 & 1 & 0
\end{bmatrix}
\]
Let $A$ be a real symmetric matrix.

- Real number $\lambda$ is an **eigenvalue** of $A$ if $A\mathbf{w} = \lambda \mathbf{w}$ for a nonzero vector $\mathbf{w}$.
- $p_A(x) = \det(xI - A)$ is the **characteristic polynomial** of $A$.
- $\lambda$ is an eigenvalue of matrix $A$ if and only if $p_A(\lambda) = 0$.
- The **multiplicity** $\text{mult}_A(\lambda)$ is the exponent of $(x - \lambda)$ in $p_A(x)$.
- Because $A$ is real and symmetric, $\text{mult}_A(\lambda) = \text{null}(A - \lambda I)$.
- The multiset of eigenvalues, appearing with multiplicity, is the **spectrum**, $\text{spec}(A)$, of $A$.
- The **inertia** of $A$ is the triple $(n_+, n_0, n_-)$, with the entries indicating the number of positive, zero, and negative eigenvalues (counting multiplicities).
Distance spectra: definitions and examples

- \( p_{\mathcal{D}(G)}(x) = \det(xI - \mathcal{D}(G)) \) is the distance characteristic polynomial of \( G \).
- A distance eigenvalue of \( G \) is an eigenvalue of \( \mathcal{D}(G) \).
- \( \text{spec}_{\mathcal{D}}(G) := \text{spec}(\mathcal{D}(G)) \) is the distance spectrum of \( G \) (the multiset of distance eigenvalues, appearing with multiplicity).

\[
\mathcal{D}(G) = \begin{bmatrix}
0 & 1 & 1 & 2 & 3 \\
1 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 1 \\
3 & 3 & 2 & 1 & 0 \\
\end{bmatrix}
\]

\[
\text{spec}_{\mathcal{D}}(G) = \{-4.26261, -1.17742, -1, -0.568579, 7.00861\}
\]

Distance Spectra Survey
Theorem (Graham, Pollak; 1971)

Given any system of \( n \) loops with maximum distance \( s \) between loops, there is a system of addresses of length no more than \( s(n - 1) \) with the property that a minimal path between loops can be obtained by switching to an adjacent loop if and only if the Hamming distance to the destination is decreased by one.

Theorem (Graham, Pollak; 1971)

If the graph of the loop system is one of the following graphs, then addresses of length \( |V(G)| - 1 \) suffice:

- Tree
- Complete graph
Distance spectra of trees

The proof that loop system addresses of length \( n - 1 \) suffice when the graph is a tree (of order \( n \)) relies on the graph having one positive (and no zero) distance eigenvalues.

Theorem (Graham, Pollak; 1971)

For a tree \( T \) of order \( n \geq 2 \),

\[
\text{det} \mathcal{D}(T) = (-1)^{n-1}(n-1)2^{n-2},
\]

and the inertia of \( \mathcal{D}(T) \) is \((n_+, n_0, n_-) = (1, 0, n - 1)\).

Generalization to graphs:

- The determinant of the distance matrix depends only on the blocks of the graph.
- Conjectured in [Hosoya, Murakami, Gotoh; 1973].
- Proved in [Graham, Hoffman, Hosoya; 1977].
Distance spectra of trees

- The distance determinant is a coefficient of the distance characteristic polynomial.
- Signs of the coefficients of the distance characteristic polynomial of a tree determined in [Edelberg, Garey, Graham; 1976]
- Coefficients of the distance characteristic polynomial of a tree computed in terms of subforests in [Graham, Lovász; 1978]

Theorem (Merris; 1990)

For $T$ a tree, $L(T)$ the line graph of $T$, $A(L(T))$ the adjacency matrix of $L(T)$, and $K := 2I + A(L(T))$. The eigenvalues of $-2K^{-1}$ interlace the eigenvalues of $D(T)$ (with $D(T)$ having one more).
Distance spectra of cartesian products

- \( G = (V, E) \) and \( G' = (V', E') \): \( G \square G' \) has vertex set \( V \times V' \), and vertices \((u, u')\) and \((v, v')\) are adjacent if \((u = v \text{ and } \{u', v'\} \in E')\) or \((u' = v' \text{ and } \{u, v\} \in E)\).

- \( G \) is transmission regular if \( D(G)1 = \rho 1 \) (where \( \rho \) is the constant row sum of \( D(G) \) and \( 1 \) is the all ones vector).

**Theorem (Indulal; 2009)**

*Suppose \( G \) and \( G' \) are transmission regular graphs with\n\[ \text{spec}_D(G) = \{\rho, \theta_2, \ldots, \theta_n\} \]\nand \( \text{spec}_D(G') = \{\rho', \theta'_2, \ldots, \theta'_n\} \). Then\n\[ \text{spec}_D(G \square G') = \{n'\rho + n\rho'\} \cup \{n'\theta_2, \ldots, n'\theta_n\} \cup \{n\theta'_2, \ldots, n\theta'_n\} \cup \{0((n-1)(n'-1))\}. \]
Section II: Unimodality of the distance characteristic polynomial coefficients for trees

(A) Unimodality and Peak Conjectures of Graham and Lovász
(B) Collins’ counterexample to the Peak Conjecture
(C) Proof of the Graham/Lovász Unimodality Conjecture
(D) Collins/Shor Peak Conjecture
(E) Graphs that are not trees
(A) Unimodality and Peak Conjectures: Definitions

$G$ is a graph of order $n$.

- Graham and Lovász used the polynomial
  \[ \det(D(G) - xl) =: (-1)^n x^n + \delta_{n-1}(G) x^{n-1} + \cdots + \delta_1(G) x + \delta_0(G) \]

- $p_{D(G)}(x) = \det(xl - D(G)) = x^n + (-1)^n \delta_{n-1}(G) x^{n-1} + \cdots + (-1)^n \delta_1(G) x + (-1)^n \delta_0(G)$

- For $0 \leq k \leq n - 2$, the normalized coefficients are
  \[ d_k(G) := \left( \frac{1}{2^{n-2}} \right) 2^k |\delta_k(G)|. \]

- [Graham, Lovász; 1978] If $T$ is a tree, then the normalized coefficients represent counts of certain subforests of the tree.

- [Graham, Lovász; 1978] If $T$ is a tree, then
  \[ d_k(T) = (-1)^{n-1} \delta_k(T)/2^{n-k-2} \]
The real sequence $a_0, a_1, a_2, \ldots, a_n$ is unimodal if there is a $k$ such that $a_{i-1} \leq a_i$ for $i \leq k$ and $a_i \geq a_{i+1}$ for $i \geq k$.

**Conjecture (Graham, Lovász; 1978)**

*For a tree $T$ of order $n \geq 3$:*

- **Unimodality Conjecture** The sequence of normalized coefficients $d_0(T), \ldots, d_{n-2}(T)$ is unimodal.

- **Peak Conjecture** The peak of the unimodal sequence $d_0(T), \ldots, d_{n-2}(T)$ occurs at $\left\lfloor \frac{n}{2} \right\rfloor$. 
Theorem (Collins; 1989)

- **For both stars and paths the sequence** $d_0(T), \ldots, d_{n-2}(T)$ **is unimodal.**
- **For stars the peak is at** $\left\lfloor \frac{n}{2} \right\rfloor$.
- **For paths the peak is at approximately** $(1 - \frac{1}{\sqrt{5}})n \approx 0.553n$. 
Proof of the Unimodality Conjecture: Key definitions and results

- $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ is unimodal if there is a $k$ such that $a_i - 1 \leq a_i$ for $i \leq k$ and $a_i \geq a_{i+1}$ for $i \geq k$.
- $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ is log-concave if $a_j^2 \geq a_{j-1} a_{j+1}$ for all $j = 1, \ldots, n - 1$.
- $a_0, a_1, a_2, \ldots, a_n$ is the coefficient sequence of polynomial $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$.

The following results appears in various surveys about unimodality.

**Theorem**

(i) If $p(x)$ is a real-rooted polynomial, then the coefficient sequence of $p(x)$ is log-concave.

(ii) If $a_0, a_1, a_2, \ldots, a_n$ is positive and log-concave, then $a_0, a_1, a_2, \ldots, a_n$ is unimodal.
**Theorem (Aalipour, Abiad, Berikkyzy, Hogben, Kenter, Lin, Tait)**

*Let $T$ be a tree of order $n \geq 3$.  

(i) The coefficient sequence of the distance characteristic polynomial $p_{D(T)}(x)$ is log-concave.  

(ii) The sequence $|\delta_0(T)|, \ldots, |\delta_{n-2}(T)|$ of absolute values of coefficients of the distance characteristic polynomial is log-concave and unimodal.  

(iii) The sequence $d_0(T), \ldots, d_{n-2}(T)$ of normalized coefficients of the distance characteristic polynomial is log-concave and unimodal.*
Proof of the Unimodality Theorem: Distance coefficients

- $p_{D}(T)(x)$ is real-rooted
- The coefficient sequence $(-1)^n \delta_0(T), \ldots, (-1)^n \delta_{n-2}(T), 0, 1$ is log-concave.
- $(-1)^n \delta_0(T), \ldots, (-1)^n \delta_{n-2}(T)$ is log-concave.
- $(-1)^{n-1} \delta_k(T) > 0$ for $0 \leq k \leq n - 2$.
- $(-1)^n \delta_k(T) < 0$ for $0 \leq k \leq n - 2$.
- All of the terms $(-1)^n \delta_0(T), \ldots, (-1)^n \delta_{n-2}(T)$ are negative.
- $\{|\delta_k(T)|\}_{k=0}^{n-2}$ is log-concave and positive.
- $|\delta_0(T)|, \ldots, |\delta_{n-2}(T)|$ is unimodal.
Proof: Normalized distance coefficients

Observation

Let $a_0, a_1, a_2, \ldots, a_n$ be a sequence of real numbers, let $c$ and $s$ be nonzero real numbers, and define $b_k = sc^k a_k$. Then $a_0, a_1, a_2, \ldots, a_n$ is log-concave if and only if $b_0, b_1, b_2, \ldots, b_n$ is log-concave.

Proof continued

- $|\delta_0(T)|, \ldots, |\delta_{n-2}(T)|$ is log-concave.
- $d_k(T) = \left(\frac{1}{2^{n-2}}\right) 2^k |\delta_k(T)|$ for $k = 0, \ldots, n-2$.
- $d_0(T), \ldots, d_{n-2}(T)$ is log-concave and positive.
- $d_0(T), \ldots, d_{n-2}(T)$ is unimodal.
Conjecture (Collins; 1989; attributed to Shor)

For every tree $T$ of order at least 3, the location of the peak of the unimodal sequence of normalized distance coefficients $d_0(T), \ldots, d_{n-2}(T)$

is between

$$\left\lfloor \frac{n}{2} \right\rfloor \text{ and } \left\lceil n - \frac{n}{\sqrt{5}} \right\rceil.$$

Computations on Sage confirm this conjecture for all trees of order at most 20.
The Heawood graph and a plot of its normalized distance coefficients,

81, 924, 3794, 5460, 3801, 14728, 1848,
17928, 6545, 9492, 9114, 3164, 441.
Section III: On distance spectra

(A) Optimistic strongly regular graphs
(B) Distance regular graphs having one positive distance eigenvalue
(C) Distance determinants and inertias of barbells and lollipops
(D) Number of distinct distance eigenvalues
G is optimistic if $G$ has more positive than negative distance eigenvalues.


[Azarija; 2014] exhibited families of strongly regular optimistic graphs.

[AABCDGHHKLT, 2016] characterized which strongly regular graphs are optimistic in terms of their parameters.
Strongly regular graphs

- **G** is an \((n, k, \lambda, \mu)\) strongly regular graph (SRG) if
  - **G** has \(n\) vertices,
  - **G** is \(k\)-regular,
  - every two adjacent vertices have \(\lambda\) common neighbors, and
  - every two nonadjacent vertices have \(\mu\) common neighbors.

- At most 3 adjacency eigenvalues of a strongly regular graph:
  - \(\rho = k\) with multiplicity 1.
  - \(\theta > 0\) with multiplicity \(m_\theta \geq 0\) \((m_\theta = 0\) iff \(K_n\)).
  - \(\tau < 0\) with multiplicity \(m_\tau > 0\).

- Formulas for \(\theta, \tau, m_\theta, m_\tau\) in terms of parameters \((n, k, \lambda, \mu)\)
  are well known.
Suppose $G$ is a connected $(n, k, \lambda, \mu)$-SRG, so $\mu \geq 1$ (or $G = K_n$).

- The maximum distance between vertices is 2.
- $\mathcal{D}(G) = 2(J - I - \mathcal{A}(G)) + \mathcal{A}(G) = 2(J - I) - \mathcal{A}(G)$.
- Since $G$ is regular, $J$ commutes with $\mathcal{A}(G)$, and eigenvalue $n$ of $J$ and $k$ of $G$ have a common eigenvector $1$.

$$\text{spec}(\mathcal{D}(G)) = \{2n - 2 - \rho\} \cup \{\lambda : \lambda \in \text{spec}(\mathcal{A}(G)) \text{ and } \lambda \neq \rho\}$$

$$= \{2n - 2 - k, (-2 - \theta)^{(m_\theta)}, (-2 - \tau)^{(m_\tau)}\}$$

$$= \{\rho_D, \theta_D^{(m_\theta)}, \tau_D^{(m_\tau)}\}.$$
A strongly regular graph is a conference graph if
\((n, k, \lambda, \mu) = (n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})\), or equivalently, if \(m_\theta = m_\tau\)

**Theorem (Azarija; 2014)**

*A conference graph of order \(n\) is optimistic if and only if \(n \geq 13\).*
Optimistic strongly regular graphs

\[ \text{spec}(\mathcal{D}(G)) = \{2n - 2 - k, (-2 - \theta)^{m_\theta}, (-2 - \tau)^{m_\tau}\} \]
\[ = \{\rho_\mathcal{D}, \theta_\mathcal{D}^{(m_\theta)}, \tau_\mathcal{D}^{(m_\tau)}\}. \]

**Theorem (AABCDGHHKLT; 2016)**

Let \( G \) be a strongly regular graph with parameters \((n, k, \lambda, \mu)\). The graph \( G \) is optimistic if and only if \( \tau_{\mathcal{D}} > 0 \) and \( m_\tau \geq m_\theta \). That is, \( G \) is optimistic if and only if

\[ \mu - \frac{2k}{n - 1} \leq \lambda < \frac{-4 + \mu + k}{2}. \]
If $G$ is strongly regular, then $\overline{G}$ is strongly regular.

**Theorem (AABCDGHHKL; 2016)**

Let $G$ be a strongly regular graph. Both $G$ and $\overline{G}$ are optimistic if and only if $G$ is a conference graph and $n \geq 13$. 
Optimistic strongly regular graphs with $\lambda = \mu$

$G$ is optimistic if and only if $\tau_D > 0$ and $m_\tau \geq m_\theta$. That is, $G$ is optimistic if and only if

$$\mu - \frac{2k}{n-1} \leq \lambda < \frac{-4 + \mu + k}{2}.$$  

**Corollary (AABCDGHHKLT; 2016)**

A strongly regular graph with parameters $(n, k, \mu, \mu)$ is optimistic if and only if $k > \mu + 4$. 
Other families of optimistic strongly regular graphs

A strongly regular graph with parameters \((n, k, \mu, \mu)\) is optimistic if and only if \(k > \mu + 4\).

- The symplectic graph \(Sp(2m, q)\) is a strongly regular graph with parameters
  \[(n, k, \lambda, \mu) = \left(\frac{q^{2m-1}}{q-1}, q^{2m-1}, q^{2m-2}(q - 1), q^{2m-2}(q - 1)\right)\]

- For \(m \geq 2\) and \(e = \pm 1\), the graph \(O_{2m+1}(3)\) on one type of nonisotropic points is a strongly regular graph with parameters
  \[\left(\frac{3^m(e+3^m)}{2}, \frac{3^{m-1}(3^m-e)}{2}, \frac{3^{m-1}(3^{m-1}-e)}{2}, \frac{3^{m-1}(3^{m-1}-e)}{2}\right).\]

**Corollary (AABCDGHHKLT, 2016)**

\(Sp(2m, q)\) is optimistic except for \(q = 2\) and \(m = 2\).
\(O_{2m+1}(3)\) is optimistic.
Distance regular graphs with one positive distance eigenvalue

- \( G = (V, E) \) is distance regular if for \( u, v \in V \) with \( d(u, v) = k \), the number of vertices \( w \in V \) such that \( d(u, w) = i \) and \( d(v, w) = j \) is independent of the choice of \( u \) and \( v \).

- Distance regular graphs with exactly one positive distance eigenvalue (counted according to multiplicity) are of particular interest.

- [Koolen and Shpectorov, 1994] determined the distance regular graphs with exactly one positive distance eigenvalue (counted according to multiplicity).

- Distance spectra of specific graphs can be determined by computation.

- Distance spectra of families are determined theoretically.

- This determination is now complete.
Distance regular graphs with one positive distance eigenvalue: Specific graphs

<table>
<thead>
<tr>
<th>KS #</th>
<th>Graph $G$</th>
<th>$\text{spec}_D(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(II)</td>
<td>Gosset graph</td>
<td>${84, 0^{(48)}, (-12)^{(7)}}$</td>
</tr>
<tr>
<td>(III)</td>
<td>Schlafli graph</td>
<td>${36, 0^{(20)}, (-6)^{(6)}}$</td>
</tr>
<tr>
<td>(VI)</td>
<td>Chang graphs (3)</td>
<td>${42, 0^{(20)}, (-6)^{(7)}}$</td>
</tr>
<tr>
<td>(IX)</td>
<td>Icosahedral graph</td>
<td>${18, 0^{(5)}, (-3 + \sqrt{5})^{(3)}, (-3 - \sqrt{5})^{(3)}}$</td>
</tr>
<tr>
<td>(XII)</td>
<td>Petersen graph</td>
<td>${15, 0^{(4)}, (-3)^{(5)}}$</td>
</tr>
<tr>
<td>(XIII)</td>
<td>Dodecahedral graph</td>
<td>${50, 0^{(9)}, (-7 + 3\sqrt{5})^{(3)}, (-2)^{(4)}, (-7 - 3\sqrt{5})^{(3)}}$.</td>
</tr>
</tbody>
</table>
Distance regular graphs with one positive distance eigenvalue: Families of graphs

(I) Cocktail party graphs $CP(m) = K_{2,2,...,2}$, strongly regular with parameters $(2m, 2m - 2, 2m - 4, 2m - 2)$.

$\text{spec}_D(CP(m)) = \{2m, 0^{(m-1)}, -2^{(m)}\}$.

(X) Cycles: $\text{spec}_D(C_n) =$

- $\left\{ \frac{n^2 - 1}{4}, \left(\frac{1}{4} \sec^2\left(\frac{\pi j}{n}\right)\right)^{(2)}, j = 1, \ldots, p \right\}$, for $n = 2p + 1$;
- $\left\{ \frac{n^2}{4}, 0^{(p-1)}, \left(-\csc^2\left(\frac{\pi(2j-1)}{n}\right)\right)^{(2)}, j = 1, \ldots, \left\lfloor \frac{p}{2} \right\rfloor \right\} (\cup \{-1\})$ if $p$ is odd), for $n = 2p$.

[Fowler, Caporossi, Hansen; 2001], [Atik, Panigrahi; 2015]
Distance regular graphs with one positive distance eigenvalue: Families of graphs

- Hamming graphs $H(d, n) = K_n \square \cdots \square K_n$.
- Hypercube $Q_d = H(d, 2)$.
- Halved cube $\frac{1}{2} Q_d = Q_{d-1}^2$.

(VII) Hamming graphs: $\text{spec}_D(H(d, n)) = \left\{ dn^{d-1}(n - 1), 0^{(n^d - d(n-1) - 1)}, (-n^{d-1})(d(n - 1)) \right\}$. [Indulal; 2009]

(IV) Halved cube: $\text{spec}_D(\frac{1}{2} Q_d) = \left\{ d2^{d-3}, 0^{(2^{d-1} - (d+1))}, (-2^{d-3})(d) \right\}$. [Atik, Panigrahi; 2016]
Distance regular graphs with one positive distance eigenvalue: Families of graphs

(VIII) Doob graph $D(m, d)$:
\[
\{3(2m + d)4^{2m+d-1}, 0(4^{2m+d} - 6m - 3d - 1), (-4^{2m+d} - 1)(6m + 3d)\}.
\]
[AABCDGHHKLT; 2016]

(V) Johnson graphs
\[
\text{spec}_D((J(n, r))) = \left\{ s(n, r), 0\binom{n}{r} - n, \left( -\frac{s(n, r)}{n-1} \right)^{n-1} \right\}
\]
where $s(n, r) = \sum_{j=0}^{r} j(r) \binom{n-r}{j}$ [Atik, Panigrahi; 2015]

(XI) double odd graphs
\[
\text{spec}_D(\text{DO}(r)) = \left\{ (2r+1)\binom{2r+1}{r}, 0\binom{2^{r+1}}{r} - 2 - 2, \left( -\frac{2s(2r+1, r)}{r} \right)^{2r}, -(2r+1)\binom{2r+1}{r} + 4s(2r+1, r) \right\}
\]
[AABCDGHHKLT; 2016]
For $k \geq 2$, $\ell \geq 0$, a lollipop graph $L(k, \ell)$ has a $k$-clique attached by an edge to one end of a path on $\ell$ vertices.

For $k \geq 2$, $\ell \geq 0$, a barbell graph $B(k, \ell)$ has 2 $k$-cliques each attached by an edge to one of the end vertices of a path on $\ell$ vertices.

![Diagram of L(3, 2) and B(4, 1)]
For $k, m \geq 2$, $\ell \geq 0$, a generalized barbell graph, $B(k; m; \ell)$, has a $k$-clique attached by an edge to one end of a path on $\ell$ vertices, and an $m$-clique attached by an edge to the other end.

$L(k, \ell) = B(k; 2; \ell - 2)$ for $\ell \geq 2$.

$B(k, \ell) = B(k; k; \ell)$ for $k \geq 2$ and $\ell \geq 0$. 
**Theorem (AABCDGHHKLT; 2016)**

\[
det D(B(k; m; \ell)) = (-1)^{k+m+\ell-1}2^\ell(km(\ell + 5) - 2(k + m)).
\]

**Theorem (AABCDGHHKLT; 2016)**

*Inertia of* \( D(B(k; m; \ell)) *:

\[
(n_+, n_0, n_-) = (1, 0, k + m + \ell - 1).
\]
Distance determinants and inertias of generalized barbells: Examples

\[ L(3, 2) = B(3, 2, 0) \]

\[ B(4, 1) = B(4, 4, 1) \]

- \( \det D(B(3; 2; 0)) = 20 \).
- \( \text{spec}_D(B(3; 2; 0)) = \{-1, -4.26261, -1.17742, -0.568579, 7.00861\} \).
- \( \det D(B(4; 4; 1)) = 160 \).
- \( \text{spec}_D(B(4; 4; 1)) = \{-11.2915, -2, (-1)^4, -0.708497, -0.539392, 18.5394\} \).
(D) Number of distinct distance eigenvalues:
Few distinct distance eigenvalues and not distance regular

Question (Atik, Panigrahi; 2015; Problem 4.3)

Is there a graph $G$ with less than $\text{diam}(G) + 1$ distinct distance eigenvalues and $G$ is not distance regular?

Example (AABCDGHHKLT, 2016)

$Q_d^\ell$ is not regular. $\text{diam}(Q_d^\ell) = d + 1$ but $Q_d^\ell$ has at most 5 distinct distance eigenvalues since its distance eigenvalues interlace those of $Q_d$, which has 3 distinct distance eigenvalues.
Few distinct distance eigenvalues and many degrees

If $G$ is a transmission regular graph of order $n$ with $\text{spec}_D(G) = \{\rho, \theta_2, \ldots, \theta_n\}$, then

$$\text{spec}_D(G \Box G) = \{2n\rho\} \cup \{(n\theta_2)^{(2)}, \ldots, (n\theta_n)^{(2)}\} \cup \{0^{((n-1)^2)}\}.$$  

Example (AABCDGHHKL; 2016)

$$H^{2^k} := H^{2^{k-1}} \Box H^{2^{k-1}} \text{ has } k + 1 \text{ degrees and } 5 \text{ distinct distance eigenvalues.}$$
- $q(G)$ is the number of distinct (adjacency) eigenvalues of $G$.
- $q_D(G)$ is the number of distinct distance eigenvalues of $G$.
- It is well known that for a tree $T$,
  \[ \text{diam}(T) + 1 \leq q(T). \]

Conjecture (AABCDGHHKLT; 2016)

For a tree $T$,

\[ \text{diam}(T) + 1 \leq q_D(T). \]

Theorem (AABCDGHHKLT; 2016)

For a tree $T$,

\[ \left\lceil \frac{\text{diam}(T)}{2} \right\rceil \leq q_D(T). \]
[Merris, 1990] For $T$ a tree, $L(T)$ is the line graph of $T$, $A(L(T))$ is the adjacency matrix of $L(T)$, and $K := 2I + A(L(T))$. The eigenvalues of $-2K^{-1}$ interlace the eigenvalues of $D(T)$.

Proof of $\left\lceil \frac{\text{diam}(T)}{2} \right\rceil \leq q_D(T)$.

- $\text{diam}(L(T)) = \text{diam}(T) - 1$.
- $A(L(T)) \geq 0$ has at least $\text{diam}(L(T)) + 1 = \text{diam}(T)$ distinct eigenvalues.
- $-2K^{-1}$ also has at least $\text{diam}(T)$ distinct eigenvalues.
- The eigenvalues of $-2K^{-1}$ interlace the eigenvalues of $D$.
- Thus $D$ has at least $\left\lfloor \frac{\text{diam}(T)}{2} \right\rfloor$ distinct eigenvalues.
Still many interesting open questions

Conjecture (Collins; 1989; attributed to Shor)

For every tree $T$ of order at least 3, the location of the peak of the unimodal sequence of normalized distance coefficients $d_0(T), \ldots, s_{n-2}(T)$ is between $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil n - \frac{n}{\sqrt{5}} \right\rceil$.

Question

What is the distance spectrum of the generalized barbell $B(k; m; \ell)$?

Conjecture (AABCDGHHKLT; 2016)

For a tree $T$, $\text{diam}(T) + 1 \leq q_D(T)$. 

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References (Introduction)


References (On Distance Spectra)


