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DETERMINING WHETHER A MATRIX IS STRONGLY EVENTUALLY NONNEGATIVE

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Abstract. A matrix A can be tested to determine whether it is eventually positive by examination of its Perron-Frobenius structure, i.e., by computing its eigenvalues and left and right eigenvectors for the spectral radius $\rho(A)$. No such "if and only if" test using Perron-Frobenius properties exists for eventually nonnegative matrices. The concept of a strongly eventually nonnegative matrix was was introduced in [2] to define a class of eventually nonnegative matrices with a stronger connection to Perron-Frobenius theory (and to exclude nilpotent matrices and related problems). This paper presents an algorithm that uses necessary and sufficient Perron-Frobenius-type conditions to determine whether a matrix is strongly eventually nonnegative. To establish the validity of the algorithm, eventually r-cyclic matrices are defined, and it is shown that a strongly eventually nonnegative matrix that is not eventually positive is eventually r-cyclic, and an eventually r-cyclic matrix A having $\operatorname{rank} A^2 = \operatorname{rank} A$ is r-cyclic.

Key words. Strongly eventually nonnegative matrix, eventually nonnegative matrix, eventually r-cyclic matrix, Perron-Frobenius.

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1. Introduction.

A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually nonnegative* (respectively, *eventually positive*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k \geq 0$ (respectively, $A^k > 0$), and the least such k_0 is called the *power index* of A. A matrix $A \in \mathbb{R}^{n\times n}$ is *strongly eventually nonnegative* if A is eventually nonnegative and there is a positive integer k such that $A^k \geq 0$ and A^k is irreducible [2].

For a fixed n, the power index of an eventually positive or eventually nonnegative $n\times n$ matrix may be arbitrarily large, so it is not possible to show a matrix is not eventually positive or eventually nonnegative by computing powers. Eventual positivity is characterized by Perron-Frobenius properties, which provide necessary and sufficient conditions to determine whether a matrix is eventually positive. Unfortunately, nilpotent matrices, which have no Perron-Frobenius structure, are eventually nonnegative, and there is no known "if and only if" test using Perron-Frobenius-type properties for eventual nonnegativity. The concept of strong eventual nonnegativity was introduced in [2] to define a subset of the eventually nonnegative matrices having connections with Perron-Frobenius theory similar to the connections between eventually positive

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matrices and Perron-Frobenius theory, and also to eliminate nilpotent matrices and related difficulties. Algorithm 4.1 below uses necessary and sufficient conditions developed in [2] and in this paper to determine whether a matrix is strongly eventually nonnegative, and thus provides a way to show a matrix is not strongly eventually nonnegative.

Throughout this paper all matrices are real, and we follow the notations and conventions of [2]. Some of the less standard definitions from that paper that we adopt include: An eigenvalue λ of A is a *dominant* eigenvalue if $|\lambda| = \rho(A)$, and is *strictly dominant* if it is the unique dominant eigenvalue of A (and is simple). A matrix A has the *strong Perron-Frobenius property* if A has a positive strictly dominant eigenvalue having a positive eigenvector. A matrix A has the *semi-strong Perron-Frobenius property* if A has a simple positive dominant eigenvalue having a positive eigenvector. A matrix is eventually positive if and only if A and A^T have the strong Perron-Frobenius property [5].

THEOREM 1.1. [2] *A matrix A is strongly eventually nonnegative if and only if A is eventually nonnegative and both A and A^T have the semi-strong Perron-Frobenius property.*

Unfortunately, the semi-strong Perron-Frobenius property for A and A^T is not enough to guarantee eventual nonnegativity (see, for example, [3, Example 2.5] or [2, Example 3]).

Just as digraphs are central to the Perron-Frobenius theory of nonnegative matrices, they are central to our analysis of strongly eventually nonnegative matrices, and we need additional notation and terminology. A *digraph* $\Gamma = (V, E)$ consists of a finite, nonempty set V of vertices, together with a set $E \subseteq V \times V$ of arcs. Note that a digraph allows loops (arcs of the form (v, v)) and may have both arcs (v, w) and (w, v) but not multiple copies of the same arc.

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. The *digraph of A*, denoted $\Gamma(A)$, has vertex set $\{1, \dots, n\}$ and arc set $\{(i, j) : a_{ij} \neq 0\}$. If $R, C \subseteq \{1, 2, \dots, n\}$, then $A[R|C]$ denotes the *submatrix* of A whose rows and columns are indexed by R and C, respectively. If $C = R$, then $A[R|R]$ can be abbreviated to $A[R]$. For a digraph $\Gamma = (V, E)$ and $W \subseteq V$, the *induced subdigraph* $\Gamma[W]$ is the digraph with vertex set W and arc set $\{(v, w) \in E : v, w \in W\}$. For a square matrix A, $\Gamma(A[W])$ is identified with $\Gamma(A)[W]$ by a slight abuse of notation.

A square matrix A is *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are nonempty square matrices and 0 is a (possibly rectangular) block consisting entirely of zero entries, or A is the 1×1 zero matrix. If A is not reducible, then A is called *irreducible.* A digraph Γ is *strongly connected* (or *strong*) if for any two distinct vertices v and w of Γ, there is a walk in Γ from v to w. It is well known that for $n \geq 2$, A is irreducible if and only if $\Gamma(A)$ is strongly connected. For a strong digraph Γ, the *index of imprimitivity* is the greatest common divisor of the the lengths of the closed walks in Γ. A strong digraph is *primitive* if its index of imprimitivity is one; otherwise it is *imprimitive.* The *strong components* of Γ are the maximal strongly connected subdigraphs of Γ.

For $r \geq 2$, a digraph $\Gamma = (V, E)$ is *cyclically r-partite* if there exists an ordered partition $(V_1, \ldots, V_r)$ of V into r nonempty sets such that for each arc $(i, j) \in E$, there exists $\ell \in \{1, \ldots, r\}$ with $i \in V_\ell$ and $j \in V_{\ell+1}$ (where we adopt the convention that index $r + 1$ is interpreted as 1). For $r \geq 2$, a strong digraph Γ is cyclically r-partite if and only if r divides the index of imprimitivity (see, for example, [1, p. 70]). For $r \geq 2$, a matrix $A \in \mathbb{R}^{n\times n}$ is called *r-cyclic* if $\Gamma(A)$ is cyclically r-partite. If $\Gamma(A)$ is cyclically r-partite with ordered partition Π, then we say A is *r-cyclic with partition* Π, or Π *describes* the r-cyclic structure of A. The ordered partition $\Pi = (V_1, \ldots, V_r)$ is *consecutive* if $V_1 = \{1, \ldots, i_1\}, V_2 = \{i_1 + 1, \ldots, i_2\}, \ldots, V_r = \{i_{r-1} + 1, \ldots, n\}$. If A is r-cyclic with consecutive ordered partition Π, then A has the block form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r} \\ A_{r1} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.1}$$

where $A_{i,i+1} = A[V_i|V_{i+1}]$. For any r-cyclic matrix A, there exists a permutation matrix P such that PAP^T is r-cyclic with consecutive ordered partition.

An irreducible nonnegative matrix B is *primitive* if $\Gamma(B)$ is primitive, and the *index of imprimitivity* of B is the index of imprimitivity of $\Gamma(B)$. It is well known that a nonnegative matrix is primitive if and only if it is eventually positive. Let $B \geq 0$ be irreducible with index of imprimitivity $r \geq 2$. Then $\Gamma(B)$ is cyclically r-partite with ordered partition $\Pi = (V_1, \ldots, V_r)$ and the sets V_i are uniquely determined (up to cyclic permutation of the V_i) (see, for example, [1, p. 70]). Furthermore, $\Gamma(B^r)$ is the disjoint union of r primitive digraphs on the sets of vertices $V_i, i = 1, \ldots, r$ (see, for example, [6, Fact 29.7.3]).

Section 2 introduces eventually r-cyclic matrices and establishes some of their properties, and in Section 3 it is shown that a strongly eventually nonnegative matrix is eventually r-cyclic or eventually positive. These results are used in Section 4 to establish the validity of Algorithm 4.1, which tests whether a matrix is strongly eventually nonnegative; examples illustrating the use of the algorithm are included.

2. Eventually r-cyclic matrices. In this section we examine matrices whose cyclic structure is eventually described by a single partition, and in the next section we show that strongly eventually nonnegative matrices have that property. First we introduce some terminology.

Definition 2.1. For an ordered partition $\Pi = (V_1, \ldots, V_r)$ of $\{1, \ldots, n\}$ into r nonempty sets, the *cyclic characteristic matrix* $C_\Pi = [c_{ij}]$ of Π is the $n \times n$ matrix such that $c_{ij} = 1$ if there exists $\ell \in \{1, \ldots, r\}$ such that $i \in V_\ell$ and $j \in V_{\ell+1}$, and $c_{ij} = 0$ otherwise.

Note that for any ordered partition $\Pi = (V_1, \ldots, V_r)$ of $\{1, \ldots, n\}$ into r nonempty sets, C_Π is r-cyclic, and $\Gamma(C_\Pi)$ contains every arc (v, w) for $v \in V_\ell$ and $w \in V_{\ell+1}$.

Definition 2.2. For matrices $A = [a_{ij}], C = [c_{ij}] \in \mathbb{R}^{n\times n}$, matrix A is *conformal* with C if for all $i, j = 1, \ldots, n$, $c_{ij} = 0$ implies $a_{ij} = 0$. Equivalently, A is conformal with C if $\Gamma(A)$ is a subdigraph of $\Gamma(C)$ (with the same set of vertices).

Let Π be an ordered partition into r nonempty sets. Then A is r-cyclic with partition Π if and only if A is conformal with C_Π.

Observation 2.3. *If $A, B, C, D \in \mathbb{R}^{n\times n}$, $C, D \geq 0$, A is conformal with C and B is conformal with D, then AB is conformal with CD. If A is an r-cyclic matrix with partition Π, then A^k is conformal with ${C_\Pi}^k$.*

Observation 2.4. *Let $B \geq 0$ be irreducible with index of imprimitivity $r \geq 2$ and let Π describe the r-cyclic structure of B. Then for d large enough, C_Π is conformal with B^{dr+1}, i.e., $\Gamma(B^{dr+1}) = \Gamma(C_\Pi)$.*

Definition 2.5. A matrix A is *eventually r-cyclic* if there exists an ordered partition Π of $\{1, \ldots, n\}$ into $r \geq 2$ nonempty sets, and a positive integer m such that for all $k \geq m$, A^k is conformal with ${C_\Pi}^k$. In this case, we say that Π *describes the eventually r-cyclic structure of A.*

It is common to establish an eventual property by establishing the property for two consecutive powers of a matrix, as in [5, Theorem 1] for eventually positive matrices, [2, Proposition 1.3] for eventually nonnegative matrices, and the following proposition for eventually r-cyclic matrices.

Proposition 2.6. *If A is a matrix and for some nonnegative integer d, A^{dr+1} is r-cyclic with partition Π and A^{dr} is conformal with C_Π^r, then A is eventually r-cyclic and Π describes the eventually r-cyclic structure of A.*

Proof. For every positive integer k sufficiently large, there exist $a, b \geq 0$ such that $k = a(dr) + b(dr + 1)$ (see e.g., [1, Lemma 3.5.5]). Fix $k = a(dr) + b(dr + 1)$. Then $A^k = A^{a(dr)+b(dr+1)} = (A^{dr})^a(A^{dr+1})^b$ is conformal with $({C_\Pi}^r)^a {C_\Pi}^b$, which is conformal with ${C_\Pi}^{adr}{C_\Pi}^{b(dr+1)} = {C_\Pi}^{k}$. □

Proposition 2.6 provides a convenient way to establish that a matrix is eventually r-cyclic, and will be used in Section 3.

For any square matrix A, $\operatorname{rank} A^2 = \operatorname{rank} A$ if and only if the degree of 0 as a root of the minimal polynomial of A is at most 1. A matrix with this property behaves very nicely in regard to being eventually r-cyclic, because this property eliminates issues caused by a nonzero nilpotent part. The following notation will be used in the next proof. The *nullspace* of a (possibly rectangular) $p \times q$ matrix M is $\mathrm{NS}(M) = \{\mathbf{v} \in \mathbb{R}^q : M\mathbf{v} = 0\}$, and the *left nullspace* of M is $\mathrm{LNS}(M) = \{\mathbf{w} \in \mathbb{R}^p : \mathbf{w}^T M = 0\}$.

THEOREM 2.7. *If $A \in \mathbb{R}^{n\times n}$, $\operatorname{rank} A^2 = \operatorname{rank} A$, and there is a positive integer m divisible by r such that A^{m+1} is r-cyclic with partition Π and A^m is conformal with ${C_\Pi}^r$, then A is r-cyclic with partition Π.*

Proof. Assume that A, m, r and $\Pi = (V_1, \dots, V_r)$ satisfy the hypotheses. Since $\operatorname{rank} A^2 = \operatorname{rank} A$, for every positive integer k, $\operatorname{rank} A^k = \operatorname{rank} A$. Thus $\mathrm{NS}(A^k) = \mathrm{NS}(A)$ and $\mathrm{LNS}(A^k) = \mathrm{LNS}(A)$.

Initially, we assume that Π is consecutive. Partition $A = [A_{ij}]$ where $A_{ij} = A[V_i|V_j]$. By hypothesis, $A^m = B_1 \oplus \dots \oplus B_r$ is a block diagonal matrix, and thus:

$$\mathrm{NS}(A^m) = \{[\mathbf{v}_1^T, \dots, \mathbf{v}_r^T]^T : \mathbf{v}_\ell \in \mathrm{NS}(B_\ell), i = 1, \dots, r\};$$
$$\mathrm{LNS}(A^m) = \{[\mathbf{w}_1^T, \dots, \mathbf{w}_r^T]^T : \mathbf{w}_\ell \in \mathrm{LNS}(B_\ell), i = 1, \dots, r\},$$

For $\mathbf{v}_\ell \in \mathrm{NS}(B_\ell)$, define $\hat{\mathbf{v}}_\ell = [0^T, \dots, 0^T, \mathbf{v}_\ell^T, 0^T, \dots, 0^T]^T$, so $A^m\hat{\mathbf{v}}_\ell = 0$. Since $\mathrm{NS}(A) = \mathrm{NS}(A^m)$,

$$0 = A\hat{\mathbf{v}}_\ell = \begin{bmatrix} A_{1\ell}\mathbf{v}_\ell \\ \vdots \\ A_{r\ell}\mathbf{v}_\ell \end{bmatrix},$$

and so $A_{i\ell}\mathbf{v}_\ell = 0, i = 1, \dots, r$. Similarly, $\mathbf{w}_\ell^T A_{\ell j} = 0^T, j = 1, \dots, r$ for $\mathbf{w}_\ell \in \mathrm{LNS}(B_\ell)$. That is, for all $i, j = 1, \dots, r$,

$$\mathrm{NS}(B_\ell) \subseteq \mathrm{NS}(A_{i\ell}) \qquad \text{and} \qquad \mathrm{LNS}(B_\ell) \subseteq \mathrm{NS}(A_{\ell j}). \tag{2.1}$$

Now consider

$$A^{m+1} = A^m A = \begin{bmatrix} B_1A_{11} & B_1A_{12} & \dots & B_1A_{1r} \\ B_2A_{21} & B_2A_{22} & \dots & B_2A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_rA_{r1} & B_rA_{r2} & \dots & B_rA_{rr} \end{bmatrix}.$$

Since A^{m+1} is conformal with C_Π,

$$B_\ell A_{\ell j} = 0 \text{ unless } j \equiv \ell + 1 \mod r.$$

Since $B_\ell \mathbf{v} = 0$ implies $A_{i\ell}\mathbf{v} = 0, i = 1, \ldots, r$,

$$A_{i\ell} A_{\ell j} = 0 \text{ unless } j \equiv \ell + 1 \mod r. \tag{2.2}$$

By considering $A^{m+1} = AA^m$ and the left null space,

$$A_{i\ell} A_{\ell j} = 0 \text{ unless } i \equiv \ell - 1 \mod r. \tag{2.3}$$

So the only product of the form $A_{i\ell}A_{\ell j}$ that is not required to be 0 is $A_{\ell-1,\ell}A_{\ell,\ell+1}$ (with indices mod r). Thus,

$$B_\ell = (A_{\ell,\ell+1} \cdots A_{r1} A_{12} \cdots A_{\ell-1,\ell})^{m/r},$$

so $\mathrm{NS}(A_{\ell-1,\ell}) \subseteq \mathrm{NS}(B_\ell)$ and $\mathrm{LNS}(A_{\ell,\ell+1}) \subseteq \mathrm{LNS}(B_\ell)$. Then by (2.1),

$$\mathrm{NS}(A_{\ell-1,\ell}) = \mathrm{NS}(B_\ell) \qquad \text{and} \qquad \mathrm{LNS}(A_{\ell,\ell+1}) = \mathrm{LNS}(B_\ell). \tag{2.4}$$

So by (2.1), $\mathrm{NS}(A_{\ell-1,\ell}) \subseteq \mathrm{NS}(A_{i,\ell})$ for $i = 1, \ldots, r$. This implies that for each i there exists a (possibly rectangular) matrix M_i such that

$$A_{i,\ell} = M_i A_{\ell-1,\ell}. \tag{2.5}$$

So for $i \not\equiv \ell - 1 \mod r$,

$$\begin{aligned}
0 &= \mathrm{rank}(A_{i\ell}A_{\ell,\ell+1}) \qquad \text{by (2.3)} \\
&= \mathrm{rank}(M_i A_{\ell-1,\ell} A_{\ell,\ell+1}) \qquad \text{by (2.5)} \\
&\geq \mathrm{rank}(M_i A_{\ell-1,\ell}) + \mathrm{rank}(A_{\ell-1,\ell}A_{\ell,\ell+1}) - \mathrm{rank}(A_{\ell-1,\ell}) \qquad \text{by [7, (2.7)]} \\
&= \mathrm{rank}(M_i A_{\ell-1,\ell}) \qquad \text{because } \mathrm{LNS}(A_{\ell-1,\ell}A_{\ell,\ell+1}) = \mathrm{LNS}(A_{\ell-1,\ell}) \text{ from (2.4)} \\
&= \mathrm{rank}(A_{i\ell}) \qquad \text{by (2.5).}
\end{aligned}$$

Thus $A_{i\ell} = 0$ for $i \not\equiv \ell - 1 \mod r$, and A is r-cyclic with partition Π.

Without the assumption that Π is consecutive, there exists a permutation matrix P such that $(PAP^T)^{m+1} = PA^{m+1}P^T$ is r-cyclic with consecutive partition Π' and $(PAP^T)^m = PA^mP^T$ is conformal with ${C_{\Pi'}}^r$. Since $\mathrm{rank}(PAP^T)^2 = \mathrm{rank}(PAP^T)$, $(PAP^T)_{ij} = 0$ unless $j \equiv i + 1 \mod r$ (using the block structure of $C_{\Pi'}$). Thus A is r-cyclic with partition Π. ☐

Corollary 2.8. *Let $A \in \mathbb{R}^{n\times n}$ have $\mathrm{rank}\, A^2 = \mathrm{rank}\, A$. Then A is eventually r-cyclic if and only if A is r-cyclic.*

3. Cyclic properties of strongly eventually nonnegative matrices.

We now return to strongly eventually nonnegative matrices. We need some preliminary results.

Theorem 3.1. [2] *Let A be strongly eventually nonnegative with spectral radius ρ, power index k_0, and the number of dominant eigenvalues of A denoted by r.*

1. *If $r = 1$ then A is eventually positive.*
2. *If $r \geq 2$ then the dominant eigenvalues of A are $\{\rho, \rho\omega, \dots, \rho\omega^{r-1}\}$ where $\omega = e^{2\pi i/r}$. For $k \geq k_0$, the following are equivalent.*
    - *(a)* $\gcd(r, k) = 1$.
    - *(b) A^k is irreducible.*
    - *(c) A^k is r-cyclic with index of imprimitivity r.*

Lemma 3.2. *Let A be a strongly eventually nonnegative matrix with power index k_0, and $r \geq 2$ dominant eigenvalues. Then for any $k \geq k_0$ such that $k \equiv 0 \mod r$, $\Gamma(A^k)$ has at least r strong components.*

Proof. Assume $k \equiv 0 \mod r$ and $k \geq k_0$. Then $A^k \geq 0$, and by Theorem 3.1, the dominant eigenvalues of A^k are r copies of $\rho(A)$. If $\Gamma(A^k)[W]$ is a strong component of $\Gamma(A^k)$, then $A^k[W]$ is an irreducible nonnegative matrix, so the multiplicity of $\rho(A^k[W])$ is 1. Since $\sigma(A^k)$ is the (multiset) union of $\sigma(A^k[W])$ taken over the strong components $\Gamma(A^k)[W]$ of $\Gamma(A^k)$, $\Gamma(A^k)$ must have at least r strong components. □

Lemma 3.3. *If A and B are $n \times n$ nonnegative matrices having all diagonal entries positive, then $\Gamma(A) \cup \Gamma(B) \subseteq \Gamma(AB)$.*

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. If $(u, v) \in \Gamma(A)$, then

$$(AB)_{uv} = \sum_{i=1}^{n} a_{ui} b_{iv} \geq a_{uv} b_{vv} > 0,$$

so $(u, v) \in \Gamma(AB)$. Thus $\Gamma(A) \subseteq \Gamma(AB)$. The case $\Gamma(B) \subseteq \Gamma(AB)$ is similar. □

If A is a strongly eventually nonnegative matrix with power index k_0 that has r dominant eigenvalues, then by Theorem 3.1, A^k is r-cyclic for every $k \geq k_0$ such that $\gcd(k, r) = 1$. However, the definition of eventually r-cyclic requires more, namely a single partition for all such powers (beyond a certain point). This is established in next two results.

Theorem 3.4. *Let A be strongly eventually nonnegative matrix A having $r \geq 2$ dominant eigenvalues and power index k_0. Then there exists a positive integer $m \geq k_0$ divisible by r such that A^{m+1} is r-cyclic with partition Π and A^m is conformal with ${C_\Pi}^r$.*

Proof. Let d be a positive integer such that $dr + 1 \geq k_0$. Then $A^{dr+1} \geq 0$ has index of imprimitivity r by Theorem 3.1. We let $\Pi = (V_1, \ldots, V_r)$ denote an ordered partition that describes the r-cyclic structure of A^{dr+1}, and let $m = (dr+1)r$. By Theorem 3.1, A^{m+1} has index of imprimitivity r. Let $\Psi = (W_1, \ldots, W_r)$ be an ordered partition that describes the r-cyclic structure of A^{m+1}. It suffices to show that A^m is conformal with ${C_\Psi}^m$. Note that for an r-cyclic matrix, in any power that is a multiple of r, the order of the sets in the partition is irrelevant, since all arcs are within partition sets. Thus it suffices to show that the unordered sets $\{V_1, \ldots, V_r\}$ and $\{W_1, \ldots, W_r\}$ are equal.

By Observation 2.4, we can choose s large enough so that the diagonal blocks $A^{msr}[V_i]$ and $A^{(m+1)sr}[W_i]$ are positive for $i = 1, \ldots, r$. By Lemma 3.2, $\Gamma(A^{msr}A^{(m+1)sr}) = \Gamma(A^{(2ms+s)r})$ has at least r strong components. Since all diagonal entries of $\Gamma(A^{msr})$ and $\Gamma(A^{(m+1)sr})$ are positive, by Lemma 3.3,

$$\Gamma(A^{msr}) \cup \Gamma(A^{(m+1)sr}) \subseteq \Gamma(A^{msr}A^{(m+1)sr}).$$

But $\Gamma(A^{msr}) \cup \Gamma(A^{(m+1)sr})$ contains the complete digraphs on $V_i, i = 1, \ldots, r$ and $W_i, i = 1, \ldots, r$, so the only way for $\Gamma(A^{msr}A^{(m+1)sr})$ to have r strong components is to have $\{V_1, \ldots, V_r\} = \{W_1, \ldots, W_r\}$. □

Corollary 3.5. *If $A \in \mathbb{R}^{n\times n}$ is strongly eventually nonnegative with $r \geq 2$ dominant eigenvalues, then A is eventually r-cyclic.*

Corollary 3.6. *If $A \in \mathbb{R}^{n\times n}$ is strongly eventually nonnegative with $r \geq 2$ dominant eigenvalues and* $\operatorname{rank} A^2 = \operatorname{rank} A$*, then A is r-cyclic.*

4. Testing for strong eventual nonnegativity.

In this section we provide an algorithm to test whether a matrix is strongly eventually nonnegative and prove that it works, illustrate the algorithm with examples, and discuss computational issues related to the algorithm.

4.1. Algorithm and proof.

Algorithm 4.1. Test a matrix for strong eventual nonnegativity.
Let A be an $n \times n$ real matrix.

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1. Compute $\sigma(A)$, set $r$ equal to the number of dominant eigenvalues, and set $\omega = e^{2\pi i/r}$.
2. If the multiset of dominant eigenvalues is not $\{\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}\}$,
      then $A$ is not strongly eventually nonnegative,
      else continue.
3. Compute eigenvectors $\mathbf{v}$ and $\mathbf{w}$ for $\rho(A)$ for $A$ and $A^T$.
4. If $\mathbf{v}$ or $\mathbf{w}$ is not a multiple of a positive eigenvector,
      then $A$ is not strongly eventually nonnegative,
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else continue.

5. *If $r = 1$,*
   then A is eventually positive (and thus is strongly eventually nonnegative),
   else continue.
6. *Set $B = \frac{1}{\rho(A)} A$ and compute a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that*
   $$B = S(\operatorname{diag}(1, \omega, \ldots, \omega^{r-1}) \oplus M)S^{-1}.$$
7. *Set $B_1 = S(\operatorname{diag}(1, \omega, \ldots, \omega^{r-1}) \oplus 0)S^{-1}$.*
8. *If B_1 is not nonnegative or B_1 is not r-cyclic,*
   then A is not strongly eventually nonnegative,
   else continue.
9. *Set $q = \lceil \frac{n}{r} \rceil r$. Then A is strongly eventually nonnegative if and only if B^q and B^{q+1} are conformal with ${B_1}^r$ and B_1, respectively.*

THEOREM 4.2. *Algorithm 4.1 is correct.*

Proof. The first three assertions that A is or is not strongly eventually nonnegative are justified by the following theorems.

2. Theorem 3.1
4. Theorem 1.1
5. Theorem 3.1

There are two remaining assertions, in Steps 8 and 9. There exists a nonsingular matrix $T \in \mathbb{R}^{(n-r)\times(n-r)}$ such that $M = T(G \oplus N)T^{-1}$ where N is nilpotent and G is nonsingular. Define $B_0 = S(0 \oplus T(G \oplus 0)T^{-1})S^{-1}$. From the definitions of B_1 and B_0,

$$\begin{gathered} {B_1}^{dr+1} = B_1 \text{ for } d \geq 0, \qquad \rho(B_1) = 1, \qquad \rho(B_0) < 1, \\ B^k = {B_1}^k + {B_0}^k \text{ for } k \geq n, \qquad \text{and } \operatorname{rank}(B_1 + B_0)^2 = \operatorname{rank}(B_1 + B_0). \end{gathered}$$

Thus $\lim_{k\to\infty} {B_0}^k = 0$, and

$$\lim_{d\to\infty} B^{dr+1} = B_1. \tag{4.1}$$

Thus if B_1 has an negative entry or is not r-cyclic, B^{dr+1} retains this property for arbitrarily large d and so B and thus A are not eventually nonnegative. This establishes the validity of Step 8.

For Step 9, we may assume that $B_1 \geq 0$ is r-cyclic with partition Π. By (4.1), for k large enough, $({B_1}^k)_{ij} > 0$ implies $(B^k)_{ij} > 0$. By the construction of B_1 from S, B_1 and B_1^T have the semi-strong Perron-Frobenius property, so by Theorem 1.1, B_1 is strongly eventually nonnegative, and so irreducible. Then by Observation 2.4 and the fact that ${B_1}^{dr+1} = B_1$, C_Π is conformal with B_1.

First assume B^q and B^{q+1} are conformal with ${B_1}^r$ and B_1, respectively. By Lemma 2.6, B is eventually r-cyclic and Π describes the eventually r-cyclic structure of B. So for k large enough, (4.1) implies $B^k \geq 0$ and if $\gcd(r, k) = 1$, then B^k is irreducible. Thus B and hence A are strongly eventually nonnegative.

For the converse, assume that A is strongly eventually nonnegative, so $B_1 + B_0$ is strongly eventually nonnegative. By Theorem 3.4, there exists a positive integer $m \geq k_0$ divisible by r such that $(B_1 + B_0)^{m+1}$ is r-cyclic with partition Π and $(B_1 + B_0)^m$ is conformal with ${C_\Pi}^r$. Since $\operatorname{rank}(B_1 + B_0)^2 = \operatorname{rank}(B_1 + B_0)$, by Theorem 2.7, $B_1 + B_0$ is conformal with C_Π. As a consequence of (4.1), B_1 must be r-cyclic with the same partition Π. Since $B_1 \geq 0$ and C_Π is conformal with B_1, a matrix is conformal with ${C_\Pi}^k$ if and only if it is conformal with ${B_1}^k$. Thus $(B_1 + B_0)^q$ and $(B_1 + B_0)^{q+1}$ are conformal with ${B_1}^r$ and B_1, respectively. Since $q \geq n$, $B^q = (B_1 + B_0)^q$ and $B^{q+1} = (B_1 + B_0)^{q+1}$. □

4.2. Examples.

We illustrate the algorithm with examples.

Example 4.3. Let

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 2 & -1 & 2 & 0 & 1 & 0 \end{bmatrix}.$$

Step 1: $\sigma(A) = \left\{4, -2 + 2i\sqrt{3}, -2 - 2i\sqrt{3}, 0, 0, 0\right\}$, so $r = 3$ and $\rho(A) = 4$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 4$ are both $[1, 1, 1, 1, 1, 1, 1, 1]^T$. Set $B = \frac{1}{4}A$. For Step 6, a possible S is

$$S = \begin{bmatrix} 1 & \frac{1}{2}\left(-1 - i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & 0 & 0 & -1 \\ 1 & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & \frac{1}{2}\left(-1 - i\sqrt{3}\right) & 0 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2}\left(-1 - i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & \frac{1}{2}\left(-1 - i\sqrt{3}\right) & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

With this S, in Step 7,

$$B_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $B_1 \geq 0$. By examining $\Gamma(B_1)$ we see that B_1 is 3-cyclic with partition $(\{1,3\},\{2,5\},\{4,6\})$. Computations then verify that B^6 and B^7 are conformal with ${B_1}^6$ and B_1, respectively, so B is strongly eventually nonnegative.

Example 4.4. Let

$$A = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} & -\frac{3}{4} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \end{bmatrix}.$$

Step 1: $\sigma(A) = \{2,-2,-1,-1,1,1,0,0\}$, so $r = 2$ and $\rho(A) = 2$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 2$ are both $[1,1,1,1,1,1,1,1]^T$. Set $B = \frac{1}{2}A$. For Steps 6 and 7, a possible S and the resulting B_1 are

$$S = \begin{bmatrix} 1 & -1 & -1 & 8 & 0 & 0 & 0 & -4 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -8 & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & -1 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \end{bmatrix},$$

and B_1 is clearly nonnegative and 2-cyclic. Step 9: Since

$$B^9 = \begin{bmatrix} \frac{1}{2048} & \frac{5}{2048} & \frac{7}{1024} & -\frac{5}{512} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2048} & -\frac{3}{2048} & -\frac{1}{256} & \frac{5}{1024} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2048} & -\frac{1}{2048} & -\frac{5}{2048} & \frac{7}{2048} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2048} & -\frac{1}{2048} & -\frac{1}{2048} & \frac{3}{2048} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{521}{2048} & \frac{521}{2048} & \frac{521}{2048} & \frac{485}{2048} & \frac{1}{2048} & -\frac{3}{2048} & \frac{1}{2048} & \frac{1}{2048} \\ \frac{503}{2048} & \frac{503}{2048} & \frac{503}{2048} & \frac{539}{2048} & -\frac{1}{2048} & \frac{3}{2048} & -\frac{1}{2048} & -\frac{1}{2048} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \end{bmatrix},$$

is not conformal with B_1, A is not strongly eventually nonnegative.

Example 4.5. Let

$$A = \begin{bmatrix} 0 & 0 & 45 & 1155 \\ 0 & 0 & 2097 & -897 \\ 871 & 329 & 0 & 0 \\ 187 & 1013 & 0 & 0 \end{bmatrix}.$$

Step 1: $\sigma(A) = \{1200, -1200, 684i\sqrt{3}, -684i\sqrt{3}\}$, so $r = 2$ and $\rho(A) = 1200$. The eigenvectors of A and A^T for eigenvalue $\rho(A) = 1200$ are $[1, 1, 1, 1]^T$ and $[7, 5, 9, 3]^T$, respectively. Set $B = \frac{1}{1200}A$. For Steps 6 and 7, a possible S and the resulting B_1 are

$$S = \begin{bmatrix} 1 & -1 & -\frac{5i}{3\sqrt{3}} & \frac{5i}{3\sqrt{3}} \\ 1 & -1 & \frac{7i}{3\sqrt{3}} & -\frac{7i}{3\sqrt{3}} \\ 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{7}{12} & \frac{5}{12} & 0 & 0 \\ \frac{7}{12} & \frac{5}{12} & 0 & 0 \end{bmatrix},$$

so $B_1 \geq 0$ and 2-cyclic. Since B is conformal with B_1, B^4 and B^5 are conformal with ${B_1}^4$ and B_1, respectively, and A is strongly eventually nonnegative.

In this particular case (because the spectrum consists entirely of real multiples of roots of unity), we can extend the spectral analysis in the algorithm to estimate the power index of A. Let $\alpha = \rho(B - B_1)$ and define

$$\hat{B}_0 = \frac{1}{\alpha}(B - B_1) = \begin{bmatrix} 0 & 0 & -\frac{5}{4\sqrt{3}} & \frac{5}{4\sqrt{3}} \\ 0 & 0 & \frac{7}{4\sqrt{3}} & -\frac{7}{4\sqrt{3}} \\ \frac{1}{4\sqrt{3}} & -\frac{1}{4\sqrt{3}} & 0 & 0 \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 \end{bmatrix}.$$

Since $\sigma(\hat{B}_0) = \{i, -i, 0, 0\}$, ${\hat{B}_0}^{4k+1} = \hat{B}_0$. Solving $\alpha^k |(\hat{B}_0)_{24}| = (B_1)_{24}$ yields $k = 109.001$, and in fact $A^{109} \ngeq 0$, but A is nonnegative thereafter.

4.3. Computational issues. The computations in Examples 4.3, 4.4, and 4.5 were all done in exact arithmetic, so there was no issue of roundoff error. However, eigenvalues will generally need to be computed as decimal approximations, and roundoff error is an issue. Fortunately, to implement Algorithm 4.1 it is not necessary to compute Jordan forms (or eigenvectors for repeated eigenvalues), which are difficult to do in decimal arithmetic. If the matrix A is eventually nonnegative, then the dominant eigenvalues are simple and well spread out. The accuracy of the computations will depend on the condition number of each dominant eigenvalue, which in turn depends on the angle between the eigenvectors of A and A^T (see, for example, [4, p. 323]). Step 6 of Algorithm 4.1 requires computing a matrix $S = [\mathbf{s}_1, \ldots, \mathbf{s}_n]$ such that

$$S^{-1}BS = \operatorname{diag}(1, \omega, \ldots, \omega^{r-1}) \oplus M.$$

This can be done as follows.

- Compute eigenvectors $\mathbf{s}_1, \ldots, \mathbf{s}_r$ for the dominant eigenvalues $\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}$.
- Extend $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ to a basis $\{\mathbf{s}_1, \ldots, \mathbf{s}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$.

- Set $U = [\mathbf{s}_1, \ldots, \mathbf{s}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_n]$. Then

$$U^{-1}BU = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix} \quad \text{where } H_{11} = \operatorname{diag}(1, \omega, \ldots, \omega^{r-1}).$$

- Since $\sigma(H_{11}) \cap \sigma(H_{22}) = \emptyset$, by [4, Lemma 7.1.5], we can solve a system of linear equations to find a matrix $Z \in \mathbb{R}^{r\times(n-r)}$ such that $H_{11}Z - ZH_{22} = -H_{12}$.
- Then for $Y = \begin{bmatrix} I_r & Z \\ 0 & I_{n-r} \end{bmatrix}$, $Y^{-1}U^{-1}BUY = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix}$, and $S = UY$ is a satisfactory matrix for Step 6.

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