Minimum Rank and Maximum Eigenvalue Multiplicity of Symmetric Tree Sign Patterns and Trees

Atoshi Chowdhury*, Luz DeAlba, Timothy Hardy, Irvin Hentzel, Leslie Hogben*, Jude Melancon*, Rana Mikkelson, Amy Wangsness

August 1, 2005

*Research supported by NSF grant DMS-0353880.

- A graph G allows loops but does not allow multiple edges.
- A simple graph does not allow loops or multiple edges.
- A tree T is a connected acyclic graph (ignoring loops).
- A sign pattern Z is a matrix with entries in $\{+, -, 0\}$.
- All sign patterns Z and matrices A discussed are symmetric, i.e., $z_{ij} = z_{ji}$.
- $\mathcal{G}(Z)$ (or $\mathcal{G}(A)$) is the simple graph with vertices $1, \ldots, n$ such that ij is an edge of $\mathcal{G}(Z)$ (or $\mathcal{G}(A)$) if and only if $i \neq j$ and $z_{ij} \neq 0$ ($a_{ij} \neq 0$).
- Z is a symmetric tree sign pattern if Z is symmetric and $\mathcal{G}(Z)$ is a tree.
- $\mathcal{S}(G) = \{A : A \text{ is a symmetric matrix and } \mathcal{G}(A) = G\}$.
- $\mathcal{S}(Z) =$ $\{A : A \text{ is a symmetric matrix and for all } i, j, \ \operatorname{sgn}(a_{ij}) = z_{ij}\}$.
- $S = \mathcal{S}(Z)$ or $\mathcal{S}(T)$.

- The minimum rank of G is defined by

$$\mathrm{mr}(G) = \min_{A \in \mathcal{S}(G)} \{\mathrm{rank}(A)\}.$$

- For A a symmetric matrix, $\mathrm{mult}_\lambda(A)$ denotes the multiplicity of eigenvalue λ of A.
- The maximum multiplicity of an eigenvalue λ in $\mathcal{S}(G)$ is defined by

$$M_\lambda(G) = \max_{A \in \mathcal{S}(G)} \{\mathrm{mult}_\lambda(A)\}.$$

- All nonzero eigenvalues have the same maximum multiplicity.
- $\mathrm{mr}(G) + M_0(G) =$ the number of vertices of G.

- The minimum rank of Z is defined by

$$\mathrm{mr}(Z) = \min_{A \in \mathcal{S}(Z)} \{\mathrm{rank}(A)\}.$$

- The maximum multiplicity of an eigenvalue λ in $\mathcal{S}(G)$ is defined by

$$M_\lambda(Z) = \max_{A \in \mathcal{S}(G)} \{\mathrm{mult}_\lambda(A)\}.$$

- All positive (respectively, negative) eigenvalues have the same maximum multiplicity.
- $\mathrm{mr}(Z) + M_0(Z) =$ the size of Z.

A family S of symmetric matrices allows eigenvalue λ if there is a matrix $A \in S$ having eigenvalue λ.

Lemma

- $\mathcal{S}(G)$ *allows a nonzero eigenvalue if and only if* G *has an edge (a loop is an edge).*
- $\mathcal{S}(Z)$ *allows a positive (respectively, negative) eigenvalue if and only if* Z *has a nonzero off-diagonal entry or has a positive (negative) diagonal entry.*
- *Let* T *be a tree.* $\mathcal{S}(T)$ *allows 0 eigenvalue if and only if* T *has no permutation digraphs or at least two permutation digraphs.*
- *Let* Z *be a symmetric tree sign pattern.* $\mathcal{S}(Z)$ *allows eigenvalue 0 if and only if* $\det Z$ *is identically 0 or has both positive and negative terms.*

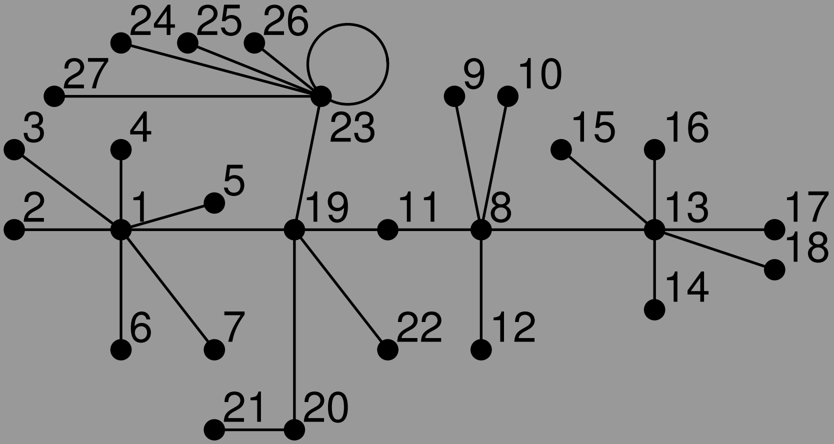
Algorithm (Computation of maximum multiplicity of λ)

Let T be a simple tree ($\mathcal{G}(Z)$ or a tree with loops suppressed).
Set $Q = \emptyset$ (set of vertices to be deleted).
Set $H = \{$vertices of T with degree $\geq 3\}$ (candidates for deletion).
While $H \neq \emptyset$:

1. *Set $\widetilde{T}$ = the unique component of $T - Q$ that contains an H-vertex.*
2. *Set $W = \{w \in H\colon$ at most 1 component of $\widetilde{T} - w$ is not H-free$\}$.*
3. *For each $w \in W$, if there are at least two H-free components of $\widetilde{T} - w$) that allow eigenvalue λ, then $Q = Q \cup \{w\}$.*
4. *$H = H - W$.*
5. *For each $v \in H$, if $\deg_{T-Q} v \leq 2$, remove v from H.*

$M_\lambda(Z)$=(number of components of $T - Q$ that allow λ) $- |Q|$.

- Example 1: Compute the maximum multiplicity of eigenvalue 1 for the tree:



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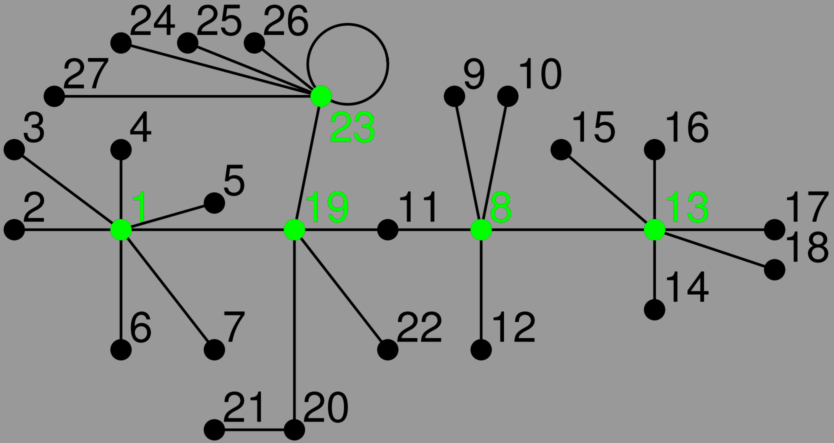

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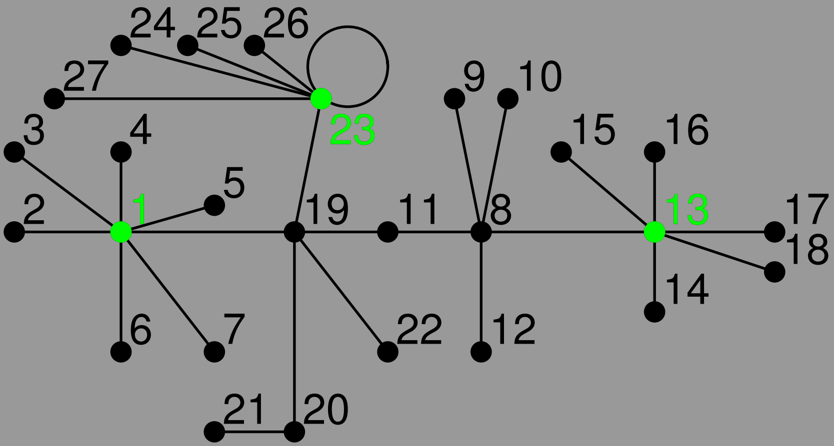

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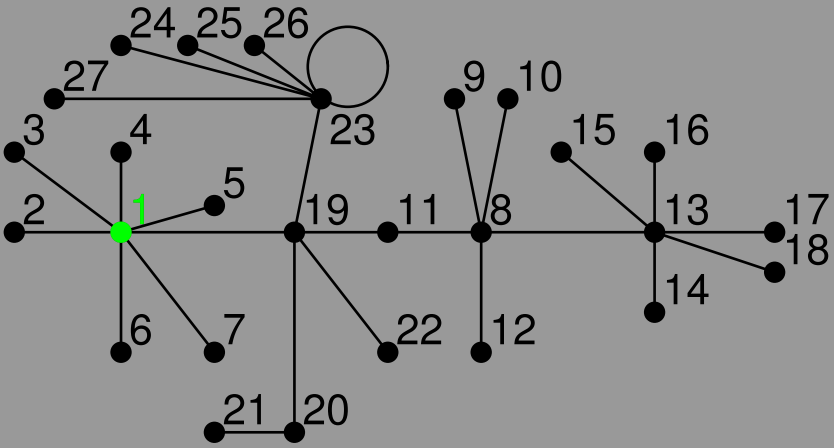

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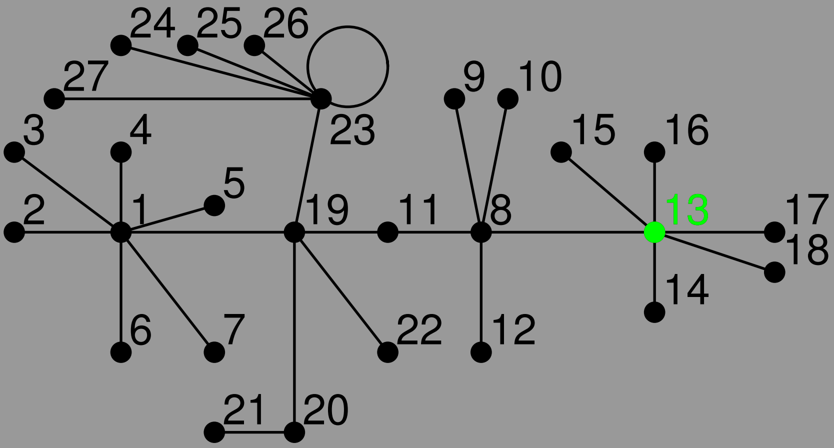

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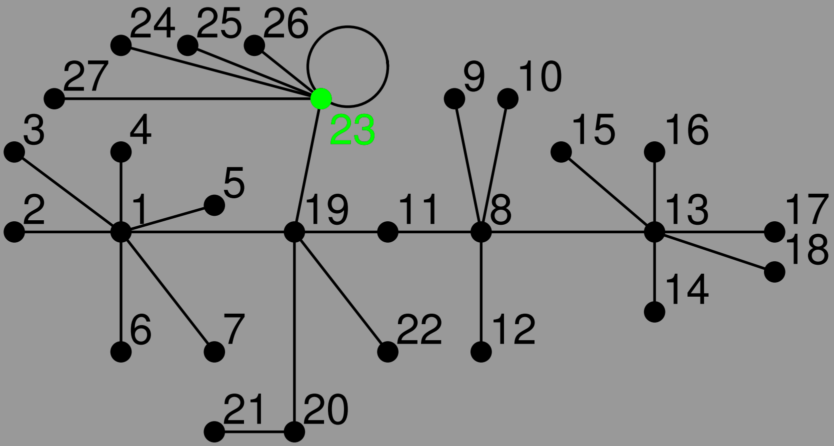

Algorithm (Computation of maximum multiplicity of λ)

Finish 1st iteration

4 $H = H - W$, *i.e.*, $H = \{8, 19\}$.

5 *For each* $v \in H$, *if* $\deg_{T-Q} v \leq 2$, *remove* v *from* H. *(no change)*

Start 2nd iteration

1. *Set* $\widetilde{T}$ = *the unique component of* $T - Q$ *that contains an H-vertex. (no change)*
2. *Set* $W = \{w \in H\colon$ *at most 1 component of* $\widetilde{T} - w$ *is not H-free*$\}$, *i.e.*, $W = \{8, 19\}$.
3. *For each* $w \in W$, *if there are at least two H-free components of* $\widetilde{T} - w)$ *that allow eigenvalue* λ, *then* $Q = Q \cup \{w\}$.

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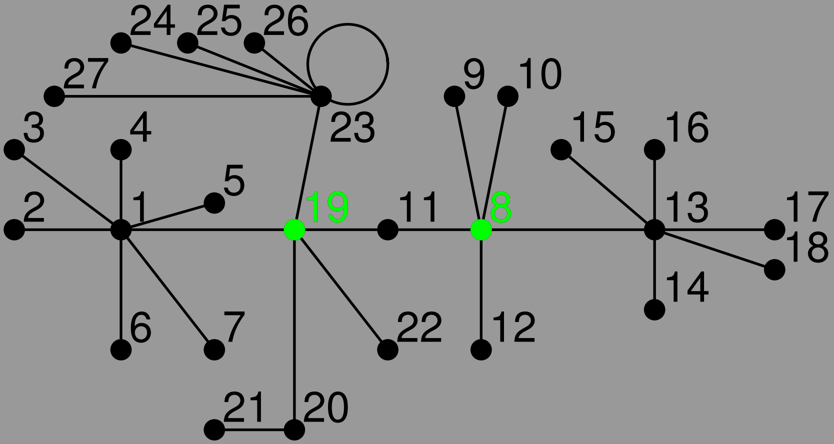

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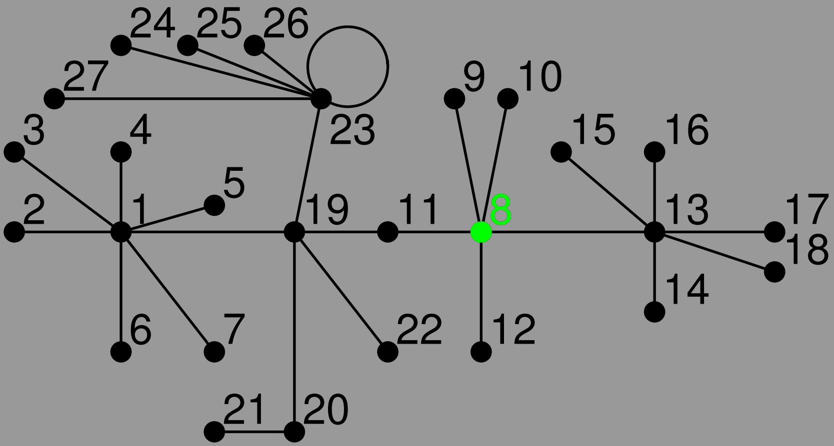

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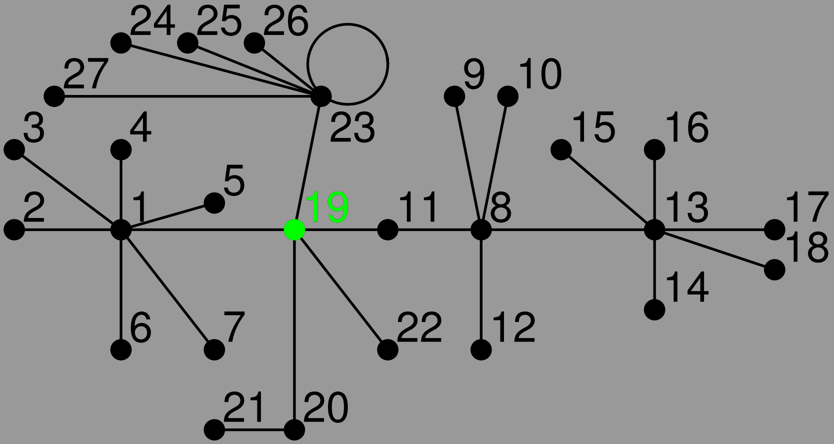

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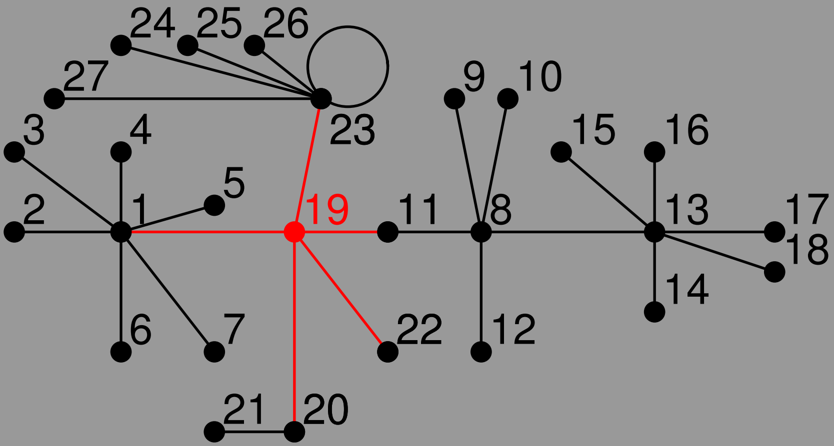

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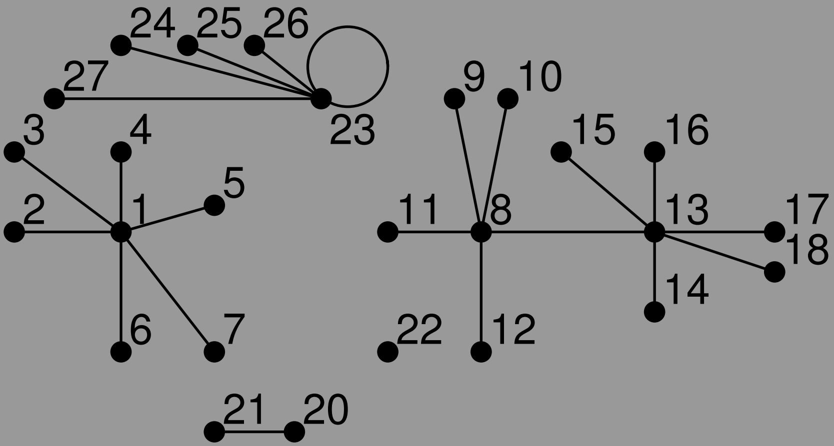

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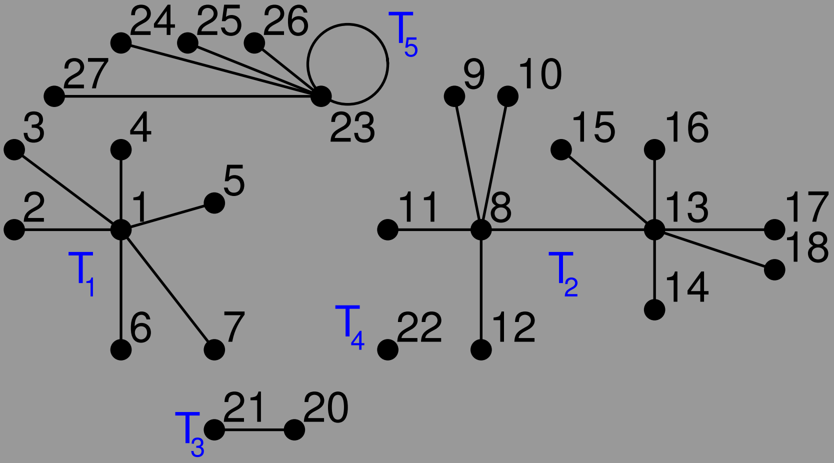
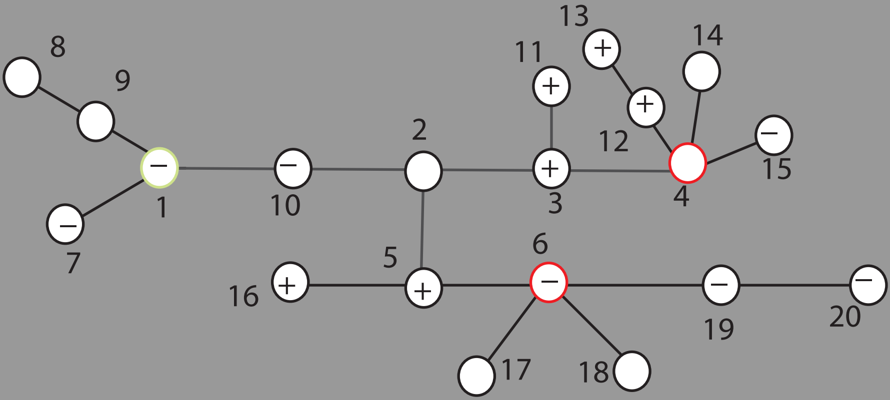


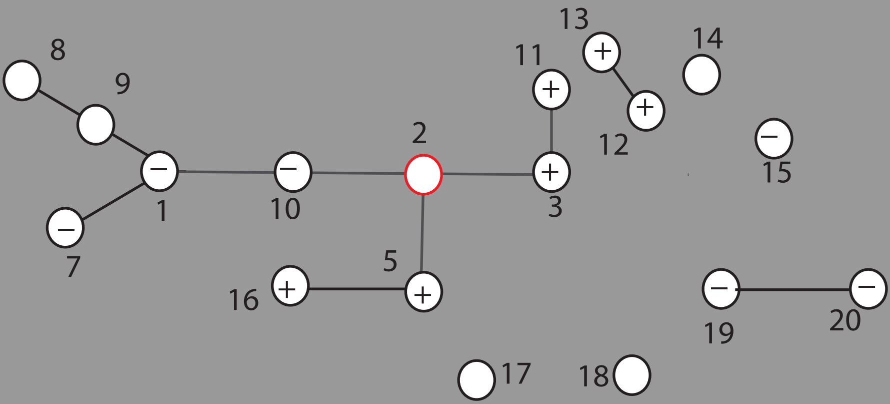
- Example 1: Compute the maximum multiplicity of eigenvalue 1 for the tree:

- $Q = \{19\}$, i.e., 1 vertex was deleted.
- There are 4 components that allow eigenvalue 1, T_1, T_2, T_3, T_5.
- $M_1(T) = 4 - 1 = 3$.

- Example 2: Compute the minimum rank of the sign pattern by computing maximum multiplicity of eigenvalue 0:



- Example 2: Compute the minimum rank of the sign pattern by computing maximum multiplicity of eigenvalue 0:



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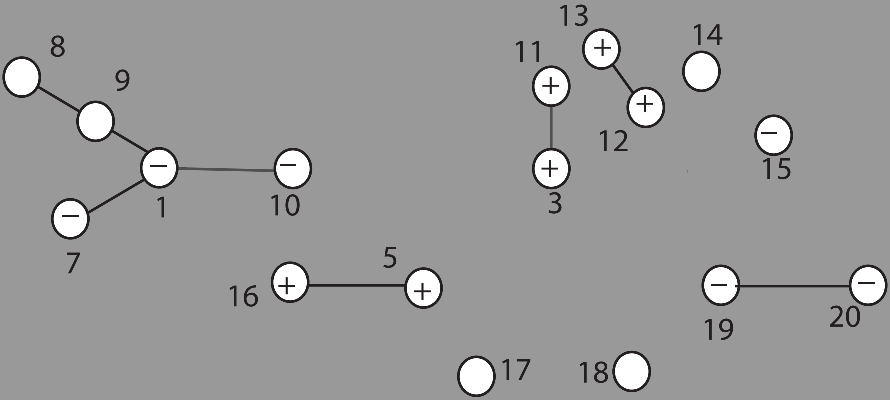

- Example 2: Compute the minimum rank of the sign pattern by computing maximum multiplicity of eigenvalue 0:

- $Q = \{2, 4, 6\}$, i.e., 3 vertices were deleted.
- There are 8 components that allow eigenvalue 0.
- $M_0(Z) = 8 - 3 = 5$.
- $\text{mr}(Z) = 20 - 5 = 15$.

The assertions about components allowing eigenvalue 0 rely on results about permutation digraphs.

Algorithm (Construction of a rational matrix realizing maximum multiplicity)

To find a symmetric rational matrix $A \in S = \mathcal{S}(T)$ or $S = \mathcal{S}(Z)$ having maximum multiplicity for rational eigenvalue λ:

1. *If $\lambda < 0$ and using a sign pattern, replace Z by $-Z$.*
2. *Apply the algorithm to obtain a set of components.*
3. *For each component, find a matrix having eigenvalue*
   - *0 if $\lambda = 0$.*
   - *1 if λ is nonzero.*
4. *Embed these matrices as principal submatrices in a large matrix $A \in S$*
5. *If $\lambda \neq 0$, multiply A by λ.*

We have algorithms to construct a symmetric rational matrix A satisfying:

- $A \in \mathcal{S}(T)$ and 0 is an eigenvalue of A.
- $A \in \mathcal{S}(Z)$ and 0 is an eigenvalue of A.
- $A \in \mathcal{S}(T)$ and 1 is an eigenvalue of A:
  - If T has a loop.
  - If T does not have a loop.
- $A \in \mathcal{S}(Z)$ and 1 is an eigenvalue of A:
  - If Z has a positive diagonal entry.
  - If Z does has a negative diagonal entry but no positive diagonal entry.
  - If all the diagonal entries of Z are 0.

- Example 1: A rational matrix having eigenvalue 1 with multiplicity 3 for the given tree:

$A(\{19\}) = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$

$$\begin{bmatrix}
 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{3}{5} &&&&&&&&&&& & 1 &&&&&&&& \\
\frac{1}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
\frac{1}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
\frac{1}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
\frac{2}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
\frac{3}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
\frac{3}{5} &&&&&&&&&&&&&&&&&&&&&&&&&& \\
&&&&&&&& \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} &&&&&&&&&&&&&& \\
&&&&&&& \frac{1}{4} &&&&&&&&&&&&&&&&&&& \\
&&&&&&& \frac{1}{4} &&&&&&&&&&&&&&&&&&& \\
&&&&&&& \frac{1}{4} &&&&&&&&&& & 1 &&&&&&&& \\
&&&&&&& \frac{1}{4} &&&&&&&&&&&&&&&&&&& \\
&&&&&&& \frac{3}{4} &&&&& & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} &&&&&&&&& \\
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0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Theorem

For any symmetric tree sign pattern Z and any rational number λ, there is a rational matrix $A \in \mathcal{S}(Z)$ such that $\operatorname{mult}_\lambda(A) = M_\lambda(Z)$, and such a matrix can be constructed from the algorithms. For $\lambda = 0$, $\operatorname{rank} A = \operatorname{mr}(Z)$.

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