Combinatorial Matrix Theory and Spectral Graph Theory

Leslie Hogben

Iowa State University and American Institute of Mathematics

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Introduction

Inverse Eigenvalue Problem for a Graph (IEPG)

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  Basic properties of minimum rank
  Trees

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  Specific matrices $\mathcal{A}, \mathcal{L}, |\mathcal{L}|, \hat{\mathcal{A}}, \hat{\mathcal{L}}, |\hat{\mathcal{L}}|
  Relationships among $\mathcal{A}, \mathcal{L}, |\mathcal{L}|, \hat{\mathcal{A}}, \hat{\mathcal{L}}, |\hat{\mathcal{L}}|

Colin de Verdière Type Parameters
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Combinatorial Matrix Theory

- Studies patterns of entries in a matrix rather than values
- In some applications, only the sign of the entry (or whether it is nonzero) is known, not the numerical value
- Uses graphs or digraphs to describe patterns
- Uses graph theory and combinatorics to obtain results about matrices
- **Inverse Eigenvalue Problem of a Graph (IEPG)** associates a family of matrices to a graph and studies spectra
Algebraic Graph Theory

- Uses algebra and linear algebra to obtain results about graphs or digraphs.
- Groups used extensively in the study of graphs.
- **Spectral graph theory** uses matrices and their eigenvalues are used to obtain information about graphs.
- Specific matrices:
  - adjacency, Laplacian, signless Laplacian matrices
  - Normalized versions of these matrices
- Colin de Verdière type parameters associate families of matrices to a graph but still use the matrices to obtain information about the graph.
Connections between these two approaches have yielded results in both directions.

**Terminology and notation**

- All matrices are real and symmetric
- Matrix $B = [b_{ij}]$
- $\sigma(B)$ is the *ordered spectrum* (eigenvalues) of $B$, repeated according to multiplicity, in nondecreasing order
- All graphs are simple
- Graph $G = (V, E)$
- **Inverse Eigenvalue Problem**: What sets of real numbers \( \beta_1, \ldots, \beta_n \) are possible as the eigenvalues of a matrix satisfying given properties of a matrix?

- **Inverse Eigenvalue Problem of a Graph (IEPG)**: For a given graph \( G \), what eigenvalues are possible for a matrix \( B \) having nonzero off-diagonal entries determined by \( G \)?
The graph $\mathcal{G}(B) = (V, E)$ of $n \times n$ matrix $B$ is

- $V = \{1, \ldots, n\}$,
- $E = \{ij : b_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of $B$ is ignored.

**Example:**

$$B = \begin{bmatrix}
  2 & -1 & 3 & 5 \\
  -1 & 0 & 0 & 0 \\
  3 & 0 & -3 & 0 \\
  5 & 0 & 0 & 0
\end{bmatrix}$$

The **family of matrices described by $G$** is

$$\mathcal{S}(G) = \{B : B^T = B \text{ and } \mathcal{G}(B) = G\}.$$
Tools for IEPG

- $B$ is irreducible if and only if $G(B)$ is connected
- Any (symmetric) matrix is permutation similar to a block diagonal matrix and the spectrum of $B$ is the union of the spectra of these blocks
- The diagonal blocks correspond to the connected components of $G(B)$
- It is customary to assume a graph is connected (at least until we cut it up)
\( B(i) \) is the *principal submatrix* obtained from \( B \) by deleting the \( i^{th} \) row and column.

- Eigenvalue interlacing: If \( \beta_1 \leq \cdots \leq \beta_n \) are the eigenvalues of \( B \), and \( \gamma_1 \leq \cdots \leq \gamma_{n-1} \) are the eigenvalues of \( B(i) \), then

\[
\beta_1 \leq \gamma_1 \leq \beta_2 \leq \cdots \leq \gamma_{n-1} \leq \beta_n
\]

- [Parter 69], [Wiener 84] If \( G(B) \) is a tree and \( \text{mult}_B(\beta) \geq 2 \), then there is \( k \) such that

\[
\text{mult}_{B(k)}(\beta) = \text{mult}_B(\beta) + 1
\]
Ordered multiplicity lists

- If the distinct eigenvalues of $B$ are $\tilde{\beta}_1 < \cdots < \tilde{\beta}_r$ with multiplicities $m_1, \ldots, m_r$, then $(m_1, \ldots, m_r)$ is called the ordered multiplicity list of $B$.

- Determining the possible ordered multiplicity lists of matrices in $S(G)$ is the ordered multiplicity list problem for $G$.

- The IEPG of $G$ can be solved by
  - solving the ordered multiplicity list problem for $G$
  - proving that if ordered multiplicity list $(m_1, \ldots, m_r)$ is possible, then for any real numbers $\gamma_1 < \cdots < \gamma_r$, there is $B \in S(G)$ having eigenvalues $\gamma_1, \ldots, \gamma_r$ with multiplicities $m_1, \ldots, m_r$.
  - If this case, IEPG for $G$ is equivalent to the ordered multiplicity list problem for $G$
[Fiedler 69], [Johnson, Leal Duarte, Saiago 03],
[Barioli, Fallat 05]
The possible ordered multiplicity lists of matrices in \( S(T) \)
have been determined for the following families of trees \( T \):

- paths
- double paths
- stars
- generalized stars
- double generalized stars

For \( T \) in any of these families, IEPG for \( G \) is equivalent to
the ordered multiplicity list problem for \( G \)

It was widely believed that the ordered multiplicity list
problem was always equivalent to IEPG for trees.
Example

[Barioli, Fallat 03] For the tree $T_{BF}$

the ordered eigenvalue list for the adjacency matrix $A$ is

$$(-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5}),$$

so the ordered multiplicity list $(1, 2, 4, 2, 1)$ is possible.

But if $B \in S(T_{BF})$ has the five distinct eigenvalues

$\tilde{\beta}_1 < \tilde{\beta}_2 < \tilde{\beta}_3 < \tilde{\beta}_4 < \tilde{\beta}_5$ with ordered multiplicity list

$(1, 2, 4, 2, 1)$, then $\tilde{\beta}_1 + \tilde{\beta}_5 = \tilde{\beta}_2 + \tilde{\beta}_4$. 
First step in solving IEPG is determining maximum possibility multiplicity of an eigenvalue.

- **maximum multiplicity** $M(G) = \max_{B \in S(G)} \text{mult}_B(\beta)$
- **minimum rank** $mr(G) = \min_{B \in S(G)} \text{rank}(B)$
- $M(G)$ is the maximum nullity of a matrix in $S(G)$
- $M(G) + mr(G) = |G|$

The **Minimum Rank Problem for a Graph** is to determine $mr(G)$ for any graph $G$. 
Examples:
Path: $\text{mr}(P_n) = n - 1$. Complete graph: $\text{mr}(K_n) = 1$

\[
\begin{bmatrix}
? & * & 0 & \ldots & 0 & 0 \\
* & ? & * & \ldots & 0 & 0 \\
0 & * & ? & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & ? & * \\
0 & 0 & 0 & \ldots & * & ? \\
\end{bmatrix}

* is nonzero, ? is indefinite
Minimum rank

Let $G$ have $n$ vertices.

- It is easy to obtain a matrix $B \in S(G)$ with $\text{rank}(B) = n - 1$ (translate)
- It is easy to have full rank (use large diagonal)
- If $G$ is the disjoint union of graphs $G_i$ then
  \[
  \text{mr}(G) = \sum \text{mr}(G_i)
  \]
- Only connected graphs are studied
- If $G$ is connected, $\text{mr}(G) = 0$ iff $G$ is single vertex
  $\text{mr}(G) = 1$ iff $G = K_n$, $n \geq 2$
- $\text{mr}(G) = n - 1$ if and only if $G$ is a path [Fiedler 69]
Minimum Rank Problem for Trees

Let $T$ be a tree. $\Delta(T)$ is the maximum of $p - q$ such that there is a set of $q$ vertices whose deletion leaves $p$ paths.

**Theorem (Johnson, Leal Duarte 99)**

$$|T| - \text{mr}(T) = M(T) = \Delta(T)$$

A related method for computing $\text{mr}(T)$ directly appeared earlier in [Nylen 96].

Numerous algorithms compute $\Delta(T)$ by using high degree ($\geq 3$) vertices.

The following algorithm works from the outside in. $\nu$ is an outer high degree vertex if at most one component of $T - \nu$ contains high degree vertices.

Delete each outer high degree vertex. Repeat as needed.
Example

Compute $\text{mr}(T)$ by computing $\Delta(T) = M(T)$. 

![Graph Diagram]

Example

Compute $\text{mr}(T)$ by computing $\Delta(T) = M(T)$. 

![Graph Diagram]
Example

Compute $\text{mr}(T)$ by computing $\Delta(T) = M(T)$.
Example

*Compute* \( mr(T) \) *by computing* \( \Delta(T) = M(T) \).
Example

Compute \( \text{mr}(T) \) by computing \( \Delta(T) = M(T) \).
Example

Compute $\text{mr}(T)$ by computing $\Delta(T) = M(T)$:

- the six vertices $\{1, 2, 3, 5, 6, 7\}$ were deleted
- there are 18 paths
- $M(T) = \Delta(T) = 18 - 6 = 12$
- $\text{mr}(T) = 35 - 12 = 23$
Adjacency matrix

- $\mathcal{A}(G) = \mathcal{A} = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{if } i \not\sim j \end{cases}$
- $\sigma(\mathcal{A}) = (\alpha_1, \ldots, \alpha_n)$

Example $W_5$

$$
\mathcal{A} = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
$$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-2, 1 - \sqrt{5}, 0, 0, 1 + \sqrt{5})$
If two graphs have different spectra (equivalently, different characteristic polynomials) of the adjacency matrix, then they are not isomorphic.

However, non-isomorphic graphs can be cospectral.

Example

\[ p(x) = x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1 \]

Spectrally determined graphs:

- Complete graphs
- Graphs with one edge
- Regular of degree 2
- Regular of degree n-3
- \( K_n, n, ..., n \)
- Empty graphs
- Graphs missing only 1 edge
- \( mK_n \)
Laplacian matrix

\[ \mathcal{L}(G) = \mathcal{L} = D - \mathcal{A}(G) \text{ where} \]
\[ D = \text{diag}(\deg(1), \ldots, \deg(n)). \]
\[ \sigma(\mathcal{L}) = (\lambda_1, \ldots, \lambda_n) \]

Example \( W_5 \)

\[
\mathcal{L} = \begin{bmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 3 & -1 & 0 & -1 \\
-1 & -1 & 3 & -1 & 0 \\
-1 & 0 & -1 & 3 & -1 \\
-1 & -1 & 0 & -1 & 3
\end{bmatrix}
\]

\( (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0, 3, 3, 5, 5) \)
- isomorphic graphs must have the same Laplacian spectrum (i.e., Laplacian characteristic polynomial)
- non-isomorphic graphs can be Laplacian cospectral
- [Schwenk 73], [McKay 77] For almost all trees $T$ there is a non-isomorphic tree $T'$ that has both the same adjacency spectrum and the same Laplacian spectrum
- for any $G$, $\lambda_1(G) = 0$
- algebraic connectivity: $\lambda_2(G)$, second smallest eigenvalue of $\mathcal{L}$
- vertex connectivity: $\kappa_V(G)$, minimum number of vertices in a cutset ($G \neq K_n$)
- [Fiedler 73] $\lambda_2(G) \leq \kappa_V(G)$

Example $\lambda_2(W_5) = 3 = \kappa_V(W_5)$
Signless Laplacian matrix

- $|\mathcal{L}|(G) = |\mathcal{L}| = D + A(G)$ where $D = \text{diag}(\deg(1), \ldots, \deg(n))$
- $\sigma(|\mathcal{L}|) = (\mu_1, \ldots, \mu_n)$

Example $W_5$

$\begin{vmatrix}
2 & & & 4 \\
& 1 & & \\
& & 3 & \\
5 & & & 4
\end{vmatrix}$

$\begin{vmatrix}
4 & 1 & 1 & 1 \\
1 & 3 & 1 & 0 \\
1 & 0 & 1 & 3 \\
1 & 1 & 0 & 1
\end{vmatrix}$

$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = (1, \frac{9-\sqrt{17}}{2}, 3, 3, \frac{9+\sqrt{17}}{2})$
Normalized adjacency matrix

- \( \hat{A} = \sqrt{D^{-1}} A \sqrt{D^{-1}} \) where 
  \( D = \text{diag}(\deg(1), \ldots, \deg(n)) \)
- \( \sigma(\hat{A}) = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \)

Example \( W_5 \)

\[
\hat{A} = \begin{bmatrix}
0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
\end{bmatrix}
\]

\((\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5) = (-\frac{2}{3}, -\frac{1}{3}, 0, 0, 1)\)
Normalized Laplacian matrix

\[ \hat{\mathcal{L}} = \sqrt{D}^{-1} \mathcal{L} \sqrt{D}^{-1} \]

\[ \sigma(\hat{\mathcal{L}}) = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n) \]

Example \( W_5 \)

\[
\hat{\mathcal{L}} = \begin{bmatrix}
1 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\
-\frac{1}{2\sqrt{3}} & 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\
-\frac{1}{2\sqrt{3}} & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 \\
-\frac{1}{2\sqrt{3}} & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\
-\frac{1}{2\sqrt{3}} & -\frac{1}{3} & 0 & -\frac{1}{3} & 1
\end{bmatrix}
\]

\( (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\lambda}_5) = (0, 1, 1, \frac{4}{3}, \frac{5}{3}) \)
Normalized Signless Laplacian matrix

\[ \widehat{\mathcal{L}} = \sqrt{D^{-1}} |\mathcal{L}| \sqrt{D^{-1}} \]

\[ \sigma(\widehat{\mathcal{L}}) = (\hat{\mu}_1, \ldots, \hat{\mu}_n) \]

Example \( W_5 \)

\[
\begin{bmatrix}
1 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\
\frac{1}{2\sqrt{3}} & 1 & \frac{1}{3} & 0 & \frac{1}{3} \\
\frac{1}{2\sqrt{3}} & \frac{1}{3} & 1 & \frac{1}{3} & 0 \\
\frac{1}{2\sqrt{3}} & 0 & \frac{1}{3} & 1 & \frac{1}{3} \\
\frac{1}{2\sqrt{3}} & \frac{1}{3} & 0 & \frac{1}{3} & 1
\end{bmatrix}
\]

\[ (\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\mu}_4, \hat{\mu}_5) = \left( \frac{1}{3}, \frac{2}{3}, 1, 1, 2 \right) \]
Relationships between $\mathcal{A}$, $\mathcal{L}$, $|\mathcal{L}|$, $\hat{\mathcal{A}}$, $\hat{\mathcal{L}}$, $|\hat{\mathcal{L}}|$ and their spectra

Let $G$ be a graph of order $n$.  

- $\hat{\mathcal{A}} + \hat{\mathcal{L}} = I$
- So $\hat{\alpha}_{n-k+1} + \hat{\lambda}_k = 1$
- $|\hat{\mathcal{L}}| + \hat{\mathcal{L}} = 2I$
- So $\hat{\mu}_{n-k+1} + \hat{\lambda}_k = 2$

**Example** $W_5$ eigenvalue relationships

$(\hat{\alpha}_5, \hat{\alpha}_4, \hat{\alpha}_3, \hat{\alpha}_2, \hat{\alpha}_1) = (1, 0, 0, -\frac{1}{3}, -\frac{2}{3})$

$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4, \hat{\lambda}_5) = (0, 1, 1, \frac{4}{3}, \frac{5}{3})$

$(\hat{\mu}_5, \hat{\mu}_4, \hat{\mu}_3, \hat{\mu}_2, \hat{\mu}_1) = (2, 1, 1, \frac{2}{3}, \frac{1}{3})$
Let $G$ be an $r$-regular graph of order $n$.

- $\mathcal{L} + A = rl$, so $\lambda_k = r - \alpha_{n-k+1}$

- $|\mathcal{L}| - A = rl$, so $\mu_k = r + \alpha_k$

- $\hat{\mathcal{A}} = \frac{1}{r}A$, so $\hat{\alpha}_k = \frac{1}{r}\alpha_k$

- $|\hat{\mathcal{L}}| = \frac{1}{r}|\mathcal{L}|$, so $\hat{\mu}_k = \frac{1}{r}\mu_k$
\( \mathcal{A}, \mathcal{L}, |L|, \hat{\mathcal{A}}, \hat{\mathcal{L}}, \hat{|L|} \) and the incidence matrix

Let \( G \) be a graph with \( n \) vertices and \( m \) edges.

- **incidence matrix** \( \mathcal{P} = \mathcal{P}(G) \) is the \( n \times m \) 0,1-matrix with rows indexed by the vertices of \( G \) and columns indexed by the edges of \( G \)

\[
\mathcal{P} = [p_{ve}] \quad \text{where} \quad p_{ve} = \begin{cases} 1 & \text{if } e \text{ is incident with } v \\ 0 & \text{otherwise} \end{cases}
\]

- \( |L| = \mathcal{P}\mathcal{P}^T \)

- \( \hat{|L|} = \sqrt{D}^{-1}|L|\sqrt{D}^{-1} = (\sqrt{D}^{-1}\mathcal{P})(\sqrt{D}^{-1}\mathcal{P})^T \)

- \( \mathcal{L} = \mathcal{P}'\mathcal{P}'^T \) where \( \mathcal{P}' \) is an oriented incidence matrix.

- \( \hat{\mathcal{L}} = \sqrt{D}^{-1}\mathcal{L}\sqrt{D}^{-1} = (\sqrt{D}^{-1}\mathcal{P}')(\sqrt{D}^{-1}\mathcal{P}')^T \)

- \( \mathcal{L}, |L|, \hat{\mathcal{L}}, \hat{|L|} \) are all positive semidefinite

- all eigenvalues \( \lambda_k, \mu_k, \hat{\lambda}_k, \hat{\mu}_k \) are nonnegative.
The spectral radius of $B$ is $\rho(B) = \max_{\beta \in \sigma(B)} |\beta|$

$\mathcal{A}, |\mathcal{L}|, \widehat{\mathcal{A}}, |\widehat{\mathcal{L}}| \geq 0$ (nonnegative)

Theorem (Perron-Forbenius)
Let $P \geq 0$ be irreducible. Then

- $\rho(P) > 0$,
- $\rho(P)$ is an eigenvalue of $P$,
- eigenvalue $\rho(P)$ has a positive eigenvector, and
- $\rho(P)$ is a simple eigenvalue of $P$ (multiplicity 1).
Let $G \neq K_n$ be connected.

- $\sigma(|\hat{\mathcal{L}}|) \subseteq [0, 2]$ and $\hat{\mu}_n = 2 = \rho(|\hat{\mathcal{L}}|)$
- $\sigma(\hat{\mathcal{A}}) \subseteq [-1, 1]$ and $\hat{\alpha}_n = 1 = \rho(\hat{\mathcal{A}})$
- $\sigma(\hat{\mathcal{L}}) \subseteq [0, 2]$ and $\hat{\lambda}_1 = 0$
- $0 < \hat{\lambda}_2 \leq 2$
- $\frac{n}{n-1} \leq \hat{\lambda}_n \leq 2$ and $\hat{\lambda}_n = 2$ if and only if $G$ is bipartite
- $\sum_{i=1}^{n} \hat{\lambda}_i = n$
Colin de Verdière’s graph parameters

- Colin de Verdière defined new graph parameters $\mu(G)$ and $\nu(G)$
  - minor monotone
  - bound $M$ from below
  - use the Strong Arnold Property
- Unlike the specific matrices originally used in spectral graph theory, these parameters involve families of matrices
- Close connections with IEPG and minimum rank
A minor of $G$ is a graph obtained from $G$ by a sequence of edge deletions, vertex deletions, and edge contractions.

A graph parameter $\zeta$ is minor monotone if for every minor $H$ of $G$, $\zeta(H) \leq \zeta(G)$.

$X$ fully annihilates $B$ if

1. $BX = 0$
2. $X$ has 0 where $B$ has nonzero
3. all diagonal elements of $X$ are 0

The matrix $B$ has the Strong Arnold Property (SAP) if the zero matrix is the only symmetric matrix that fully annihilates $B$.
See [van der Holst, Lovász, Shrijver 99] for information about SAP

- SAP comes from manifold theory.
- $\mathcal{R}_B = \{C : \text{rank } C = \text{rank } B\}$.
- $S_B = S(\mathcal{G}(B))$.
- $B$ has SAP if and only if manifolds $\mathcal{R}_B$ and $S_B$ intersect transversally at $B$.
- Transversal intersection allows perturbation.
$$\mu(G) = \max\{\text{null}(L)\} \text{ such that}$$

1. $L$ is a generalized Laplacian matrix
   ($L \in S(G)$ and off-diagonal entries $\leq 0$)
2. $L$ has exactly one negative eigenvalue (with multiplicity one)
3. $L$ has SAP

**Theorem (Colin de Verdière 90)**

- $\mu$ is minor monotone
- $\mu(G) \leq 1$ if and only if $G$ is a path
- $\mu(G) \leq 2$ if and only if $G$ is outerplanar
- $\mu(G) \leq 3$ if and only if $G$ is planar
For any graph,
\[ \mu(G) \leq M(G). \]

If \( G \) is not planar then \( 3 < \mu(G) \leq M(G) \) is sometimes useful for small graphs.

To study minimum rank, generalized Laplacians and number of negative eigenvalues are not usually relevant.
Example

- $K_{2,2,2}$ is planar but not outer planar, so $\mu(K_{2,2,2}) = 3$ (no generalized Laplacian of $K_{2,2,2}$ has rank 2)
- $\text{mr}(K_{2,2,2}) = 2$ and $M(K_{2,2,2}) = 4$

\[
\text{rank } B = \begin{bmatrix}
0 & 1 & -1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
-1 & 1 & -2 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 2 \\
\end{bmatrix} = 2
\]
\[ \nu(G) = \max\{\null(B)\} \quad \text{such that} \]

1. \( B \in S(G) \)
2. \( B \) is positive semi-definite
3. \( B \) has SAP

- [Colin de Verdière] \( \nu \) is minor monotone
- For any graph, \( \nu(G) \leq M(G) \)
To study minimum rank, positive semi-definite is not usually relevant.

**Example**

- No positive semi-definite matrix in $S(K_{2,3})$ has rank 2 so $\nu(K_{2,3}) = 2$
- $\text{mr}(K_{2,3}) = 2$ and $M(K_{2,3}) = 3$

![Diagram of $K_{2,3}$](image)

$$\text{rank } B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2$$
The new parameter $\xi$

- To study minimum rank, positive semi-definite, generalized Laplacians and number of negative eigenvalues usually are not relevant.
- Minor monotonicity is useful.

Definition

$\xi(G) = \max \{ \text{null}(B) : B \in S(G), B \text{ has SAP} \}$

Example: $\xi(K_{2,2,2}) = 4 = M(K_{2,2,2})$ because the matrix $B$ has SAP

Example: $\xi(K_{2,3}) = 3 = M(K_{2,3})$ because the matrix $B$ has SAP
For any graph $G$,

- $\mu(G) \leq \xi(G)$
- $\nu(G) \leq \xi(G)$
- $\xi(G) \leq M(G)$
- $\xi(P_n) = 1 = M(P_n)$
- $\xi(K_n) = n - 1 = M(K_n)$
- If $T$ is a non-path tree, $\xi(T) = 2$. 
Theorem (Barioli, Fallat, Hogben 05)

\( \xi \) is minor monotone.

Forbidden minors

Since \( \xi \) is minor monotone, the graphs \( G \) such that \( \xi(G) \leq k \) can be characterized by a finite set of forbidden minors.

\( \xi(G) \leq 1 \) if and only if \( G \) contains no \( K_3 \) or \( K_{1,3} \) minor.
Theorem (Hogben, van der Holst 07)

\[ \xi(G) \leq 2 \text{ if and only if } G \text{ contains no minor in the } T_3 \text{ family.} \]

\[ \text{T}_3 \text{ family} \]

K_4

K_{2,3}

T_3
Minimum rank problem

Minimum rank is characterized for:

- trees [Nylen 96], [Johnson, Leal-Duarte 99]
- unicyclic graphs [Barioli, Fallat, Hogben 05]
- extreme minimum rank:
  \[ \text{mr}(G) = 0, 1, 2: \] [Barrett, van der Holst, Loewy 04]
  \[ |G| - 1, |G| - 2: \] [Fiedler 69],
  [Hogben, van der Holst 07], [Johnson, Loewy, Smith]

Reduction techniques for

- cut-set of order 1 [Barioli, Fallat, Hogben 04]
  and order 2 [van der Holst 08]
- joins [Barioli, Fallat 06]
Minimum rank graph catalogs

- Minimum rank of many families of graphs determined at the 06 AIM Workshop.
- On-line catalogs of minimum rank for small graphs and families developed.
- The ISU group determined the order of all graphs of order 7.

Minimum rank of families of graphs
http://aimath.org/pastworkshops/catalog2.html

Minimum rank of small of graphs
http://aimath.org/pastworkshops/catalog1.html
Thank You!