

FORBIDDEN MINORS FOR THE CLASS OF GRAPHS G WITH $\xi(G) \leq 2$

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Abstract. For a given simple graph G , $\mathcal{S}(G)$ is defined to be the set of real symmetric matrices A whose (i, j) th entry is nonzero whenever $i \neq j$ and ij is an edge in G . In [2], $\xi(G)$ is defined to be the maximum corank (i.e., nullity) among $A \in \mathcal{S}(G)$ having the Strong Arnold Property; ξ is used to study the minimum rank/maximum eigenvalue multiplicity problem for G . Since ξ is minor monotone, the graphs G such that $\xi(G) \leq k$ can be described by a finite set of forbidden minors. We determine the forbidden minors for $\xi(G) \leq 2$ and present an application of this characterization to computation of minimum rank among matrices in $\mathcal{S}(G)$.

Key words. minimum rank, graph minor, corank, strong Arnold property, symmetric matrix.

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1. Introduction. Recently there has been considerable interest in the *minimum rank/maximum multiplicity problem* for a graph, that is, the problem of determining the minimum rank, or, equivalently, the maximum multiplicity of an eigenvalue, among the real symmetric matrices whose zero-nonzero pattern of entries is described by the graph (see, for example, the references in [1], [2] or [3]). This problem has been solved for trees, but only limited progress has been made toward determining minimum rank of graphs that are not trees. It is well-known that a graph has minimum rank one less than the order of the graph if and only if the graph is a path, and a connected graph has rank one if and only if it is a complete graph. Characterizations of graphs having minimum rank less than three are given in [3], and a method to compute the minimum rank of a graph with a cut-vertex from the minimum ranks of smaller subgraphs is given in [1].

Barioli, Fallat, and Hogben [2] introduced the Colin de Verdière-type parameter ξ for use in the study of the minimum rank/maximum multiplicity problem. The parameter ξ , like Colin de Verdière's parameters μ and ν , is minor monotone [2], so as noted in [6], the Robertson-Seymour graph minor theory applies to ξ , implying that the graphs G that have the property $\xi(G) \leq k$ can be characterized by a finite set of forbidden minors. Forbidden minors for low values of minor monotone graph parameters are often studied to obtain insight into the parameter or to facilitate application of the parameter. The main purpose of this note is to describe the forbidden minors for $\xi(G) \leq 2$ and to apply that result to characterize the 2-connected graphs of order n having minimum rank $n - 2$.

All matrices discussed in this paper are real and all graphs are simple, undirected, finite and of order at least 1. The following standard graph notation will be used: K_n ,

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$K_{p,q}, P_n$ denote the complete graph on n vertices, the complete bipartite graph on p, q vertices, and the path on n vertices respectively. The complement of a graph $G = (V, E)$ is $\overline{G} = (V, \overline{E})$, where \overline{E} is the set of edges that are not in E (between vertices in V). A *cut-vertex* is a vertex whose deletion increases the number of connected components. A graph is *2-connected* if its order is at least 3 and it has no cut-vertex. A *block* of a graph is a maximal connected subgraph that does not have a cut-vertex, so a block that is not 2-connected consists of a bridge and its endpoints or an isolated vertex. Let G be a graph and let v be a cut-vertex in G . Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs of G such that $G = G_1 \cup G_2$ and $V_1 \cap V_2 = \{v\}$. Then G is called the *1-sum* of G_1 and G_2 at v .

To facilitate the connection between a matrix A and a graph, we associate with A sets of row and column indices $\iota_r(A), \iota_c(A)$ by which the entries of A are indexed, i.e., $A = [a_{ij}]$ with $i \in \iota_r(A), j \in \iota_c(A)$. An ordinary (unindexed) $n \times n$ matrix A implicitly has index sets $\iota_r(A) = \iota_c(A) = \{1, \dots, n\}$. The *transpose* of $A = [a_{ij}]$, denoted A^T , is the matrix with index sets $\iota_r(A^T) = \iota_c(A)$ and $\iota_c(A^T) = \iota_r(A)$, and $(A^T)_{ij} = a_{ji}$. As usual, the matrix A is *symmetric* if $A = A^T$ (note that this imposes the condition that $\iota_r(A) = \iota_c(A)$). Most of the matrices of interest here will be square (in fact, symmetric), and for a square matrix A with $\iota_r(A) = \iota_c(A)$, we denote this common index set by $\iota(A)$. Most matrix functions, such as the determinant, can be computed as for unindexed matrices, but when computing the matrix product AB of indexed matrices A, B , it is required that $\iota_c(A) = \iota_r(B)$ (and $\iota_r(AB) = \iota_r(A), \iota_c(AB) = \iota_c(B)$). A *family* of matrices is a set of matrices all having the same sets of row and column indices.

A *vector* is a matrix with only one column; the column index is often ignored in working with vectors (e.g., when adding them). The *range* of matrix A , i.e., the span of its columns viewed as vectors, will be denoted by $R(A)$.

If A is a matrix, $R \subseteq \iota_r(A)$ and $C \subseteq \iota_c(A)$, then $A[R, C]$ denotes the submatrix of A lying in rows indexed by R and columns indexed by C , together with the row and column index sets R and C . Several abbreviations are also used: $A[R, R]$ can be denoted by $A[R]$, $A[\{v\}, C]$ can be denoted by $A[v, C]$, etc. Also, $A(R) = A[\overline{R}]$ where $\overline{R} = \iota(A) - R$.

If $S \subseteq \iota(B)$ such that $B[S]$ is nonsingular, we define the *Schur complement* of $B[S]$ to be the matrix

$$B/B[S] = B(S) - B[\overline{S}, S]B[S]^{-1}B[S, \overline{S}]$$

having $\iota(B/B[S]) = \overline{S}$. If $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ is a block matrix, then

$$\begin{bmatrix} I & 0 \\ -B_{21}B_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I & -B_{11}^{-1}B_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} B_{11} & 0 \\ 0 & B/B_{11} \end{bmatrix}. \quad (1.1)$$

If A is a fixed symmetric matrix, the *graph of A* , denoted by $G(A)$, has $\iota(A)$ as vertices, and as edges the unordered pairs ij such that $i \neq j$ and $a_{ij} \neq 0$. Graphs G of the form $G = G(A)$ do not have loops or multiple edges, and the diagonal of A

is ignored in the determination of $G(A)$. Similarly, for a given graph G , the set of symmetric matrices described by G is

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n \times n} : A \text{ is symmetric and } G(A) = G\}.$$

For a graph G , the *minimum rank of G* is defined by

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank}(A),$$

and the *maximum (eigenvalue) multiplicity of G* is defined by

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$

where $\sigma(A)$ denotes the spectrum of A . It is well-known (and easy to verify) that

$$M(G) = \max\{\text{corank}(A) : A \in \mathcal{S}(G)\},$$

(where the corank of A is the nullity of A) and

$$\text{mr}(G) = |V_G| - M(G).$$

The following definitions are taken from [2], in which the Colin de Verdière-type parameter ξ was introduced. Two $m \times n$ matrices having the same sets of row and column indices are *orthogonal* if, when viewed as mn -tuples in \mathbb{R}^{mn} they are orthogonal under the ordinary dot product. The matrix B is *orthogonal to the family \mathcal{F} of matrices* if B is orthogonal to every matrix $C \in \mathcal{F}$. Thus X orthogonal to $\mathcal{S}(G)$ requires that every diagonal entry of X is 0 and for every edge of G , the corresponding off-diagonal entry of X is 0. Let A, X be symmetric matrices with $\iota(X) = \iota(A)$. We say that X *fully annihilates A* if X is orthogonal to $\mathcal{S}(G(A))$ and $AX = 0$. The matrix A has the *Strong Arnold Property (SAP)* if the zero matrix is the only symmetric matrix that fully annihilates A .

For a given graph G , $\xi(G)$ is defined to be the maximum corank among matrices A that satisfy:

1. $A \in \mathcal{S}(G)$;
2. A has the SAP.

If $A \in \mathcal{S}(G)$ has $\text{corank}(A) = \xi(G)$ and A has the SAP, then we say A is ξ -*optimal* for G . The maximum multiplicity M is well-known for the standard graphs $K_n, K_{p,q}, P_n$, and the value of ξ was established for these graphs in [2]: $M(K_n) = n - 1 = \xi(K_n)$, $M(P_n) = 1 = \xi(P_n)$, $M(K_{p,q}) = p + q - 2$, and if $p \leq q$ and $3 \leq q$, then $\xi(K_{p,q}) = p + 1$.

The parameter ξ is called a ‘‘Colin de Verdière-type’’ parameter because Colin de Verdière defined two related parameters μ, ν [4, 5]. The parameter μ is discussed thoroughly from an algebraic perspective in [6]. For a graph G , $\mu(G)$ is defined to be the maximum corank among matrices L that satisfy:

1. L is a generalized Laplacian matrix (i.e., $L \in \mathcal{S}(G)$ and all off-diagonal entries are nonpositive);

2. L has exactly one negative eigenvalue (with multiplicity one);
3. L has the SAP.

For a graph G , $\nu(G)$ is defined to be the maximum corank among matrices A that satisfy:

1. $A \in \mathcal{S}(G)$;
2. A is positive semidefinite;
3. A has the SAP.

Recall that for a given edge $e = uv$ of a graph G , to *contract e in G* means to delete e from G and identify its ends u, v in such a way that the resulting vertex is adjacent to exactly the vertices that were originally adjacent to at least one of u, v . A *contraction of G* is then defined as any graph obtained from G by contracting an edge. For a given graph G , we call H a *minor of G* if H is obtained from G by a sequence of deletions of edges, deletions of isolated vertices, and contractions of edges. We say that G has an H -minor if G has a minor isomorphic to H . The parameter ξ , like Colin de Verdière's parameters μ and ν , is *minor monotone*, i.e., if H is a minor of G , then $\xi(H) \leq \xi(G)$ [2]. This is a powerful property that the maximum multiplicity parameter M lacks. In fact, M is not even monotone on induced subgraphs [1]. However, minimum rank is monotone on induced subgraphs, i.e., if H is an induced subgraph of G , then $\text{mr}(H) \leq \text{mr}(G)$ [3].

Furthermore, by the Robertson-Seymour theory of graph minors, the graphs G that have the property $\xi(G) \leq k$ can be characterized by a finite set of forbidden minors. For any graph G , $\xi(G) \geq 1$, and $\xi(G) \leq 1$ if and only if G is a disjoint union of paths [2]. The forbidden minors for $\xi(G) \leq 1$ are K_3 and $K_{1,3}$, because $\xi(K_3) = \xi(K_{1,3}) = 2$. Furthermore, K_3 is a minor of any cycle. If G has no cycles (i.e., G is a forest), then G is a disjoint union of paths if and only if G does not contain $K_{1,3}$ as a subgraph. The rest of this note is devoted to establishing the forbidden minors for $\xi(G) \leq 2$.

2. ΔY -transformations. Let $G = (V, E)$ be a graph. We say that G' is obtained from G by a ΔY -transformation if G' is obtained from G by deleting the edges of a triangle, adding a new vertex v and connecting v to the vertices of the triangle whose edges were deleted. For example, $K_{1,3}$ is obtained from K_3 by a ΔY -transformation.

We denote by T_3 the graph obtained from $K_{2,2,2}$ by deleting a triangle that includes a vertex from each of the partition sets (T_3 is the middle left graph in Figure 2.1). We denote by $T_3\Delta Y$ the graph obtained from T_3 by applying one ΔY -transformation on one of triangle containing a vertex of degree 2. We denote by $T_3(\Delta Y)^i$, for $i = 2, 3$, the graphs obtained from $T_3\Delta Y$ by applying $i - 1$ additional ΔY -transformations. The T_3 -family is the collection of all graphs that can be obtained from K_4 and T_3 by a number of ΔY -transformations, see Figure 2.1.

LEMMA 2.1. *Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be obtained from G by applying a ΔY -transformation. Then $\xi(G') \geq \xi(G)$.*

Proof. Let $\{v_1, v_2, v_3\}$ be the vertices of the triangle and let v_0 be the new vertex on which we apply the ΔY -transformation, and let $S = V - \{v_1, v_2, v_3\}$. Let $A = [a_{i,j}]$ be ξ -optimal for G . We distinguish two cases.

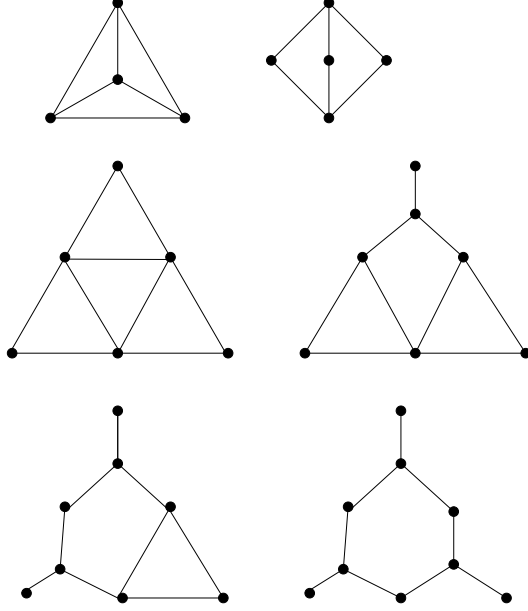


FIG. 2.1. *The T_3 -family.*

Suppose first that $a_{v_1,v_2}a_{v_2,v_3}a_{v_3,v_1} > 0$. Let

$$\begin{aligned}
 b_{v_0,v_1} &= \operatorname{sgn}(a_{v_2,v_3}) \sqrt{\frac{a_{v_1,v_3}a_{v_1,v_2}}{a_{v_2,v_3}}} \\
 b_{v_0,v_2} &= \operatorname{sgn}(a_{v_1,v_3}) \sqrt{\frac{a_{v_2,v_3}a_{v_1,v_2}}{a_{v_1,v_3}}} \\
 b_{v_0,v_3} &= \operatorname{sgn}(a_{v_1,v_2}) \sqrt{\frac{a_{v_1,v_3}a_{v_2,v_3}}{a_{v_1,v_2}}} \\
 b_{v_1,v_1} &= a_{v_1,v_1} - b_{v_0,v_1}^2 \\
 b_{v_2,v_2} &= a_{v_2,v_2} - b_{v_0,v_2}^2 \\
 b_{v_3,v_3} &= a_{v_3,v_3} - b_{v_0,v_3}^2
 \end{aligned}$$

where $\operatorname{sgn}(a) = 1, 0, -1$ according as $a > 0, a = 0, a < 0$, and let

$$B = [b_{i,j}] = \begin{bmatrix} -1 & b_{v_0,v_1} & b_{v_0,v_2} & b_{v_0,v_3} & 0 \\ b_{v_0,v_1} & b_{v_1,v_1} & 0 & 0 & A[v_1, S] \\ b_{v_0,v_2} & 0 & b_{v_2,v_2} & 0 & A[v_2, S] \\ b_{v_0,v_3} & 0 & 0 & b_{v_3,v_3} & A[v_3, S] \\ 0 & A[S, v_1] & A[S, v_2] & A[S, v_3] & A[S] \end{bmatrix}$$

with $\iota(B) = V'$. Applying the Schur complement, we see that $A = B/B[\{v_0\}]$ and

$LBL^\top = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b_{v_0, v_1} & 1 & 0 & 0 & 0 \\ b_{v_0, v_2} & 0 & 1 & 0 & 0 \\ b_{v_0, v_3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Thus $\text{corank}(B) = \text{corank}(A)$. Suppose that B does not have the SAP. Then there is a symmetric matrix $X = [x_{i,j}]$ with $\iota(X) = V'$ such that $x_{i,i} = 0$ for all $i \in V'$, $x_{i,j} = 0$ for all $ij \in E'$, and $BX = 0$. So

$$\begin{aligned} b_{v_0, v_2} x_{v_1, v_2} + b_{v_0, v_3} x_{v_1, v_3} &= 0 \\ b_{v_0, v_1} x_{v_1, v_2} + b_{v_0, v_3} x_{v_2, v_3} &= 0 \\ b_{v_0, v_1} x_{v_1, v_3} + b_{v_0, v_2} x_{v_2, v_3} &= 0, \end{aligned}$$

from which it follows that $x_{v_1, v_2} = x_{v_1, v_3} = x_{v_2, v_3} = 0$. So X can be partitioned (using $\{v_0\}, \{v_1, v_2, v_3\}, S$) into the block matrix

$$X = \begin{bmatrix} 0 & 0 & z^\top \\ 0 & 0 & Z^\top \\ z & Z & W \end{bmatrix}.$$

Then $BX = 0$ implies $-z^\top + b^\top Z^\top = 0$ (where $b = [b_{v_0, v_1}, b_{v_0, v_2}, b_{v_0, v_3}]^\top$), so X nonzero implies $X[V] = \begin{bmatrix} 0 & Z^\top \\ Z & W \end{bmatrix} \neq 0$. Since $BX = 0$, $LBL^\top(L^\top)^{-1}XL^{-1} = 0$. Note that $((L^\top)^{-1}XL^{-1})[V] = X[V]$. Thus $X[V]$ is a nonzero symmetric matrix that fully annihilates A . This contradiction shows that B has the SAP.

The case where $a_{v_1, v_2} a_{v_2, v_3} a_{v_3, v_1} < 0$ can be done similarly, using $b_{v_0, v_0} = 1$ and choosing $b = [b_{v_0, v_1}, b_{v_0, v_2}, b_{v_0, v_3}]^\top$ so that $A[\{v_1, v_2, v_3\}] + bb^\top$ is diagonal (cf. [6, Theorem 2.13]). \square

LEMMA 2.2. $\xi(K_4) = 3$ and $\xi(T_3) = 3$.

Proof. As noted earlier, $\xi(K_4) = 3$. Since (as shown in [5]), $\nu(T_3) = 3$, $\xi(T_3) \geq 3$. Since P_4 is an induced subgraph of T_3 , $\text{mr}(T_3) \geq 3$, so $\xi(T_3) \leq M(T_3) \leq 3$. \square

COROLLARY 2.3. *Each graph G in the T_3 -family has $\xi(G) > 2$.*

3. 1-sums of graphs. In order to establish the characterization of forbidden minors of $\xi(G) \leq 2$, we extend results from [2] that describe the behavior of ξ on 1-sums.

LEMMA 3.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs and let G be a 1-sum of G_1 and G_2 at v . Let $S_1 = V_1 \setminus \{v\}$ and $S_2 = V_2 \setminus \{v\}$. Let $A \in S(G)$ have the SAP. If $A[S_2]$ is nonsingular, then there exists a matrix $B \in S(G_1)$ that agrees with $A[V_1]$ at every entry except possibly the v, v -entry, such that $\text{corank}(B) = \text{corank}(A)$,*

and B has the SAP. In particular, if A is ξ -optimal for G and $A[S_2]$ is nonsingular, then $\xi(G) = \xi(G_1)$.

Proof. If $A[S_1, v] \notin R(A[S_1])$, let $B = A[V_1]$. Then by Lemma 3.5 (ii) of [2], B has the SAP, and $\text{corank}(B) = \text{corank}(A[S_1]) - 1 = \text{corank}(A)$.

If $A[S_1, v] \in R(A[S_1])$, let $B = A[V_1]$ except choose the v, v -entry so that $\text{corank}(B) = \text{corank}(A)$ ($= \text{corank}(A[V_1])$ or $\text{corank}(A[V_1]) + 1$, depending on a_{vv}). Then by [2, Lemma 3.5 (ii), (iii)], B has the SAP. \square

LEMMA 3.2. *Let G be a connected graph and let G be a 1-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ at v . Let $S_1 = V_1 \setminus \{v\}$ and $S_2 = V_2 \setminus \{v\}$. Let $G_1' = (V_1', E_1')$ be the graph obtained from K_2 and G_1 by identifying a vertex of K_2 with v . If A is ξ -optimal for G , $A[S_2]$ is singular and there is no nonzero vector y with $A[V_2, S_2]y = 0$, then $\xi(G) = \xi(G_1')$.*

Proof. By minor-monotonicity, $\xi(G_1') \leq \xi(G)$.

We now prove the converse inequality. Since there is no nonzero y with $A[S_2]y = 0$ and $A[v, S_2]y = 0$, $\text{corank}(A[S_2]) = 1$. Let $x \in \ker(A[S_2])$ be nonzero. Let $w \in S_2$ with $x_w \neq 0$. Let $Q = V_2 \setminus \{v, w\}$. Then $A[Q]$ is nonsingular, because if $A[Q]$ were singular, another (independent) vector could be constructed in $\ker(A[S_2])$. We may write

$$A = \begin{bmatrix} A[S_1] & A[S_1, v] & 0 & 0 \\ A[v, S_1] & a_{vv} & a_{vw} & A[v, Q] \\ 0 & a_{vw} & a_{ww} & A[w, Q] \\ 0 & A[Q, v] & A[Q, w] & A[Q] \end{bmatrix}.$$

Applying the Schur complement on $A[Q]$ yields the matrix

$$A/A[Q] = B = [b_{ij}] = \begin{bmatrix} A[S_1] & A[S_1, v] & 0 \\ A[v, S_1] & b_{vv} & b_{vw} \\ 0 & b_{vw} & b_{ww} \end{bmatrix},$$

where

$$\begin{bmatrix} b_{vv} & b_{vw} \\ b_{vw} & b_{ww} \end{bmatrix} = A[\{v, w\}] - A[\{v, w\}, Q]A[Q]^{-1}A[Q, \{v, w\}].$$

The corank of B is equal to the corank of A . Since $A[S_2]$ is singular, we know that $a_{ww} - A[w, Q]A[Q]^{-1}A[Q, w] = 0$. Suppose $0 = b_{vw} = a_{vw} - A[v, Q]A[Q]^{-1}A[Q, w]$. Then the vector

$$z = \begin{bmatrix} 1 \\ -A[Q]^{-1}A[Q, w] \end{bmatrix}$$

belongs to $\ker(A[V_2, S_2])$, contradicting the assumption. Therefore $b_{vw} \neq 0$, that is, $B \in \mathcal{S}(G_1')$. To show that $\xi(G_1') \geq \xi(G)$, it remains to show that B has the SAP.

Suppose for a contradiction that B does not have the SAP. Then there is a nonzero symmetric matrix $X = [x_{i,j}]$ with $\iota(X) = V_1'$ such that $x_{i,i} = 0$ for all $i \in V_1'$, $x_{i,j} = 0$ for all $ij \in E_1'$, and $BX = 0$. So

$$B[S_1]X[S_1] + B[S_1, \{v, w\}]X[\{v, w\}, S_1] = 0, \text{ i.e.,}$$

$$A[S_1]X[S_1] + A[S_1, \{v, w\}]X[\{v, w\}, S_1] = 0$$

and

$$B[\{v, w\}, S_1]X[S_1] + B[\{v, w\}]X[\{v, w\}, S_1] = 0, \text{ i.e.,}$$

$$A[\{v, w\}, S_1]X[S_1] + (A[\{v, w\}] - A[\{v, w\}, Q]A[Q]^{-1}A[Q, \{v, w\}])X[\{v, w\}, S_1] = 0.$$

Let

$$Z = -A[Q]^{-1}A[Q, \{v, w\}]X[\{v, w\}, S_1]$$

and

$$Y = \begin{bmatrix} X[S_1] & X[S_1, \{v, w\}] & Z^T \\ X[\{v, w\}, S_1] & 0 & 0 \\ Z & 0 & 0 \end{bmatrix}.$$

Then Y is a nonzero symmetric matrix with $\iota(Y) = V$ that fully annihilates A . Hence A would not have the SAP if B did not. \square

THEOREM 3.3. *Let G be a connected graph and let G be a 1-sum of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ at v . Let $G_i' = (V_i', E_i')$, for $i = 1, 2$, be the graph obtained from K_2 and G_i by identifying a vertex of K_2 with v . Then $\xi(G) = \max\{\xi(G_1'), \xi(G_2')\}$.*

Proof. Since G_1' and G_2' are isomorphic to minors of G , $\xi(G_1') \leq \xi(G)$ and $\xi(G_2') \leq \xi(G)$. Let A be ξ -optimal for G , let $S_1 = V_1 \setminus \{v\}$ and $S_2 = V_2 \setminus \{v\}$. If $A[S_1]$ or $A[S_2]$ is nonsingular, then, by Lemma 3.1, $\xi(G_1) = \xi(G)$ or $\xi(G_2) = \xi(G)$. Since G_i is a minor of G_i' for $i = 1, 2$, the theorem follows for this case. So we may assume that both $A[S_1]$ and $A[S_2]$ are singular. Since A has the SAP, it is not possible that there are nonzero vectors y and z with $A[V_1, S_1]y = 0$ and $A[V_2, S_2]z = 0$; say there is no nonzero vector z with $A[V_2, S_2]z = 0$. By Lemma 3.2, $\xi(G_1') = \xi(G)$. \square

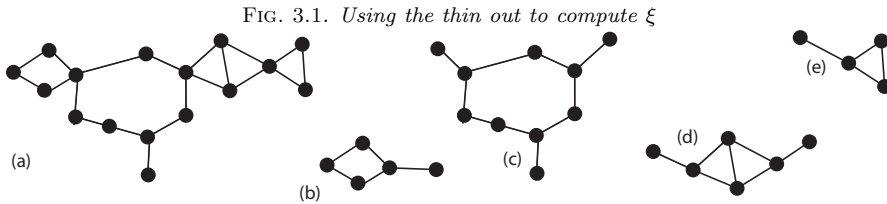
Let G be a graph and let C be a block of G . The *thin out* of C in G is the graph obtained from C by adding a pendant edge to each cut vertex v of G contained in C . So the thin out of C in G is isomorphic to a subgraph of G .

PROPOSITION 3.4. *Let G be a graph with $\xi(G) \geq 3$. Then there exists a 2-connected block C of G such that the thin out H of C satisfies $\xi(H) = \xi(G)$.*

Proof. If G has more than one component, by [2, Theorem 3.1], $\xi(G)$ is the maximum of ξ on the components of G , so we may assume G is connected. If G has no 2-connected blocks, then G is a tree. This contradicts the assumption that $\xi(G) \geq 3$, since for any tree T , $\xi(T) \leq 2$ [2]. We argue by induction on the number of 2-connected blocks; the result is clear when G has only one 2-connected block. Assume that for all graphs G having fewer than m 2-connected blocks and $\xi(G) \geq 3$, there exists a 2-connected block C of G such that the thin out H of C satisfies $\xi(H) = \xi(G)$.

Let G be a graph with $\xi(G) \geq 3$ having $m > 1$ 2-connected blocks. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs of G such that $G = G_1 \cup G_2$, $|V_1 \cap V_2| = 1$, and both G_1 and G_2 contain a 2-connected block. Let v be the vertex of $V_1 \cap V_2$, and

let G_i' , for $i = 1, 2$, be obtained from G_i and K_2 by identifying a vertex of K_2 with v . By Theorem 3.3, $\xi(G) = \max\{\xi(G_1'), \xi(G_2')\}$; we may assume that $\xi(G_1') = \xi(G)$. Since G_1' has fewer 2-connected blocks, G_1' has a 2-connected block C such that the thin out H of C in G_1' satisfies $\xi(H) = \xi(G)$. Since the thin out of C in G_1' is the same as the thin out of C in G , we have proven the proposition. \square



EXAMPLE 3.5. We use Proposition 3.4 to compute $\xi(G)$ for the graph G shown in Figure 3.1(a). The thin outs of the various blocks (in left to right order for the diagram of G in Figure 3.1(a), omitting the block of order 2) are shown in Figures 3.1(b)-(e). The thin out H of the 7-cycle in G is shown in Figure 3.1(c), and clearly has a $T_3(\Delta Y)^3$ -minor, so $3 = \xi(H) = \xi(G)$ (since it is also clear that the thin outs of the other blocks all have ξ equal to 2, by using an induced path to bound minimum rank in each).

LEMMA 3.6. *Let $G = (V, E)$ be a connected graph that has a vertex v of degree 1. Let w be the vertex adjacent to v . Let $A \in \mathcal{S}(G)$. If $a_{vv} = 0$, then $\text{corank}(A[V - \{v, w\}]) = \text{corank}(A)$.*

Proof. Let $S = V - \{v, w\}$. By reordering the indices if necessary, we can partition A into the block matrix

$$A = \begin{bmatrix} 0 & a_{vw} & 0 \\ a_{vw} & a_{ww} & A[w, S] \\ 0 & A[S, v] & A[S] \end{bmatrix},$$

which is equivalent (by elementary row and column operations) to

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A[S] \end{bmatrix}.$$

\square

4. Linear two-trees. A *linear 2-tree* is a 2-connected graph G that can be embedded in the plane such that the graph obtained from the dual of G after deleting the vertex corresponding to the infinite face is a path.

LEMMA 4.1. *Let $G = (V, E)$ be a 2-connected graph. If G has no K_4 -, $K_{2,3}$ -, and no T_3 -minor, then G is a linear 2-tree.*

Proof. Since G has no K_4 - and no $K_{2,3}$ -minor, G is outerplanar. Hence G can be embedded in the plane such that all its vertices are incident to the infinite face.

Construct the following tree T . The vertices of T are all finite faces of the plane embedding. Connect two vertices of the tree if the corresponding face have an edge in common. Then T is a path. For if not, there would be a face that has edges in common with at least three other faces. Such a graph has a T_3 -minor. \square

LEMMA 4.2. *Let $G = (V, E)$ be a linear 2-tree. If $A \in \mathcal{S}(G)$, then $\text{corank}(A) \leq 2$.*

Proof. Embed G in the plane such that the graph obtained from the dual of G after deleting the vertex corresponding to the infinite face is a path P . Let p be an end of P and let F be the face corresponding to p . Choose an edge e in the intersection of the infinite face and F , and let u, v be the ends of e . Suppose to the contrary that $\text{corank}(A) > 2$. Then there is a nonzero vector $x \in \ker(A)$ with $x_u = x_v = 0$. Then for each vertex w on the cycle bounding the face of p , $x_w = 0$. For otherwise we can find a vertex z of degree 2 with $x_z = 0$ that is adjacent to exactly one vertex w with $x_w \neq 0$. Repeating the same procedure on the cycle bounding the face of G corresponding to the vertex of P adjacent to p , and so on, shows that $x = 0$. This contradiction shows that $\text{corank}(A) \leq 2$. \square

COROLLARY 4.3. *If G is a minor of a linear 2-tree, then $\xi(G) \leq 2$.*

Since a linear 2-tree is 2-connected and so is not a path, it follows from Lemma 4.2 that if G is a linear 2-tree, then $\xi(G) = 2$.

5. Main result. We now present the description of the forbidden minors for $\xi(G) \leq 2$.

THEOREM 5.1. *Let $G = (V, E)$ be a graph. Then $\xi(G) \leq 2$ if and only if G has no minor isomorphic to a graph in T_3 -family.*

Proof. Since each graph H in the T_3 -family has $\xi(H) > 2$, G has no minor isomorphic to a graph in the T_3 -family if $\xi(G) \leq 2$.

To see the other implication, let G be a graph with no minor isomorphic to a graph in the T_3 -family. Then each 2-connected block of G is a linear 2-tree, by Lemma 4.1. If $\xi(G) \leq 2$ there is nothing to prove, so suppose for contradiction that $\xi(G) \geq 3$. By Proposition 3.4, there is a 2-connected block C such that the thin out, H' , of C in G has $\xi(H') = \xi(G)$. Notice that H' is a minor of G . Let $A' = [a'_{i,j}] \in \mathcal{S}(H')$ be ξ -optimal for H' . Let H be obtained from H' by deleting all vertices v of degree 1 such that $a'_{vv} \neq 0$. By Lemma 3.1, $\xi(H) = \xi(H')$. Let A be a ξ -optimal matrix for H .

Embed C in the plane such that each vertex is incident to the infinite face. Let \mathcal{B} be the collection of cycles bounding the finite faces, and let P be the path whose vertices are in correspondence with all finite faces and where pq is an edge if the faces corresponding to p and q share a common edge. Let p_1, p_2 be the ends of P . Let S be the collection of vertices of C to which a pendant edge is attached in H .

Since G has no $T_3(\Delta Y)$ -minor, no vertex $s \in S$ belongs to

$\bigcup_{q \neq p_1, p_2} B_q \setminus (B_{p_1} \cup B_{p_2})$. Hence each vertex of S belongs to B_{p_1} or B_{p_2} .

Suppose now that there is vertex $s \in S$ such that $s \in V(B)$ for each $B \in \mathcal{B}$. Let v be the other end of the pendant edge at s . Let $H_1 = H - \{s, v\}$. Then H_1 is a path with some pendant edges attached to it. By Lemma 3.6, $\text{corank}(A(\{s, v\})) = \text{corank}(A)$. If $H_1 - S$ has at least three components, then H has a $T_3(\Delta Y)^3$ -minor. Hence $H_1 - S$

has at most two components, each of which is a path. Applying Lemma 3.6 shows that $\text{corank}(A) = \text{corank}(A(S)) \leq 2$.

We may therefore assume that there is no vertex $s \in S$ such that $s \in V(B)$ for each $B \in \mathcal{B}$. Hence, if $p_1 = p_2$, then $S = \emptyset$. In this case it is clear that $\xi(H) \leq 2$. So we may assume that $p_1 \neq p_2$. If there are two vertices of $S \cap V(B_{p_1})$ at distance at least two on C , then H has a $T_3(\Delta Y)^2$ -minor. A similar statement holds for $S \cap V(B_{p_2})$. Hence, for $i = 1, 2$, there is an edge f_i such that each vertex in $S \cap V(B_{p_i})$ is an end of f_i . Append two 4-cycles C_1 and C_2 to C by identifying one edge of C_i with f_i for $i = 1, 2$. The resulting graph is a linear 2-tree and has H as a minor. By Corollary 4.3, $\xi(H) \leq 2$. \square

COROLLARY 5.2. *Let G be a 2-connected graph of order n . The following are equivalent:*

1. $\xi(G) = 2$.
2. $M(G) = 2$.
3. $\text{mr}(G) = n - 2$.
4. G has no K_4 -, $K_{2,3}$ -, or T_3 -minor.
5. G is a linear 2-tree.

Note that by [1, Theorem 2.3], the computation of the minimum rank of graph can be reduced to computation of the minimum rank of 2-connected graphs, so Corollary 5.2 in conjunction with this result renders straightforward the determination of whether an order n graph has minimum rank $n - 2$.

REFERENCES

- [1] F. Barioli, S. Fallat, and L. Hogben. Computation of minimal rank and path cover number for graphs. *Linear Algebra and Its Applications*, 392: 289–303, 2004.
- [2] F. Barioli, S. Fallat, and L. Hogben. A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs. *Electronic Journal of Linear Algebra*, 13: 387–404, 2005.
- [3] W. W. Barrett, H. van der Holst, and R. Loewy. Graphs whose minimal rank is two. *The Electronic Journal of Linear Algebra*, 11:258-280, 2004.
- [4] Y. Colin de Verdière. On a new graph invariant and a criterion for planarity. In *Graph Structure Theory*. Contemporary Mathematics 147, American Mathematical Society, Providence, 1993:137–147.
- [5] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width graphs. *Journal of Combinatorial Theory, Series B*, 74:121–146, 1998.
- [6] H. van der Holst, L. Lovász, and A. Schrijver. The Colin de Verdière graph parameter. In *Graph Theory and Computational Biology (Balatonlelle, 1996)*. Mathematical Studies, Janos Bolyai Math. Soc., Budapest, 1999:29–85.