THE COPOSITIVE COMPLETION PROBLEM: UNSPECIFIED DIAGONAL ENTRIES

LESLIE HOGBEN

Department of Mathematics
Iowa State University, Ames, IA 50011, USA
lhogben@iastate.edu

June 26, 2006

ABSTRACT. In [1] it was shown that any partial (strictly) copositive matrix all of whose diagonal entries are specified can be completed to a (strictly) copositive matrix. In this note we show that every partial strictly copositive matrix (possibly with unspecified diagonal entries) can be completed to a strictly copositive matrix, but there is an example of a partial copositive matrix with an unspecified diagonal entry that cannot be completed to a copositive matrix.

AMS Subject Classification: 15A18, 15A48, 15A63, 05C50
Keywords: Copositive; Strictly copositive; Matrix completion; Partial matrix

The real symmetric $n \times n$ matrix A is a *strictly copositive matrix* if $\mathbf{v}^T A\mathbf{v} > 0$ for all $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq 0$ and $\mathbf{v} \neq 0$; A is a *copositive matrix* if $\mathbf{v}^T A\mathbf{v} \geq 0$ for all $\mathbf{v} \geq 0$ ($\mathbf{v} \geq 0$ means every entry of $\mathbf{v}$ is nonnegative). Copositive matrices have been studied extensively; see [1] for more information.

A *partial matrix* is a square array in which some entries are specified and others are not. (A conventional matrix, with all entries specified, and a matrix with no specified entries are also considered partial matrices.) An unspecified entry is denoted by ? or by x_{ij}. A *completion* of a partial matrix is a choice of values for the unspecified entries.

The partial matrix B is a *partial strictly copositive matrix* (respectively, *partial copositive matrix*) if every fully specified principal submatrix of B is a strictly copositive matrix (respectively, copositive matrix) and whenever b_{ij} is specified then so is b_{ji} and $b_{ji} = b_{ij}$.

In [1] it was shown that any partial (strictly) copositive matrix all of whose diagonal entries are specified can be completed to a (strictly) copositive matrix. In this note we address the issue of partial (strictly) copositive matrices with unspecified diagonal entries. All matrices discussed are real and symmetric.

Algorithm 1. *Let* $B = \begin{bmatrix} x_{11} & \mathbf{b}^T \\ \mathbf{b} & B_1 \end{bmatrix}$ *be a partial strictly copositive* $n \times n$ *matrix having all entries except the 1,1-entry specified. Let* $\|\cdot\|$ *be a vector norm. Complete*

B by choosing a value for x_{11} as follows:

$$\beta = \min_{\mathbf{y}\in\mathbb{R}^{n-1},\mathbf{y}\geq 0,\|\mathbf{y}\|=1} \mathbf{b}^T\mathbf{y}. \tag{1}$$

$$\gamma = \min_{\mathbf{y}\in\mathbb{R}^{n-1},\mathbf{y}\geq 0,\|\mathbf{y}\|=1} \mathbf{y}^T B_1 \mathbf{y}. \tag{2}$$

$$x_{11} > \frac{\beta^2}{\gamma}. \tag{3}$$

Theorem 2. *Let B be a partial strictly copositive matrix having all entries except the 1,1-entry specified. Any value of x_{11} chosen as in Algorithm 1 completes B to a strictly copositive matrix.*

Proof. The values β and γ are attained because a continuous function attains its extreme values on a compact set. Thus $\gamma > 0$, since B_1 is strictly copositive, and $x_{11} > 0$. Let $\mathbf{v}^T = [z, \mathbf{y}^T] \geq 0$ and $\mathbf{v} \neq 0$. If $\mathbf{y} = 0$ then $z > 0$ and $\mathbf{v}^T B\mathbf{v} = x_{11}z^2 > 0$.

If $\mathbf{y} \neq 0$, we can scale $\mathbf{v}$ so that $\|\mathbf{y}\| = 1$, without affecting whether $\mathbf{v}^T B\mathbf{v} > 0$. Then

$$\mathbf{v}^T B\mathbf{v} = x_{11}z^2 + 2\mathbf{b}^T\mathbf{y}z + \mathbf{y}^T B_1\mathbf{y} \geq x_{11}z^2 + 2\beta z + \gamma > 0,$$

by the choice of x_{11}. □

Corollary 3. *Every partial strictly copositive matrix can be completed to a strictly copositive matrix.*

Proof. Let B be a partial strictly copositive matrix. If B contains one or more unspecified diagonal entries, select such an entry x_{ii} and use Algorithm 1 (applied to every principal submatrix completed by specification of x_{ii}) to choose a value for x_{ii} large enough to ensure that every such principal submatrix of B is strictly copositive. Repeat until all diagonal entries are specified. Then complete B as in [1], by setting $x_{ij} = x_{ji} = \sqrt{b_{ii}b_{jj}}$, to obtain a strictly copositive matrix. □

Corollary 3 is false for copositive matrices.

Example 4. $B = \begin{bmatrix} ? & -1 \\ -1 & 0 \end{bmatrix}$ *is a partial copositive matrix that cannot be completed to a copositive matrix.*

Example 4 is a special case of the next lemma (which establishes the assertion that the given matrix cannot be completed to a copositive matrix).

Lemma 5. *If a partial copositive matrix B contains a principal submatrix of the form $\begin{bmatrix} ? & \mathbf{b}^T \\ \mathbf{b} & B_1 \end{bmatrix}$ (with all elements of $\mathbf{b}$ and B_1 specified) and there exists $\mathbf{y} \geq 0$ such that $\mathbf{b}^T\mathbf{y} < 0$ and $\mathbf{y}^T B_1\mathbf{y} = 0$, then B cannot be completed to a copositive matrix.*

Proof. Let $B = \begin{bmatrix} x_{11} & \mathbf{b}^T \\ \mathbf{b} & B_1 \end{bmatrix}$ and suppose there exists a vector $\mathbf{y} \geq 0$ such that $\mathbf{b}^T\mathbf{y} < 0$ and $\mathbf{y}^T B_1\mathbf{y} = 0$. Choose a value for x_{11}.

If $x_{11} = 0$, then with $\mathbf{v} = \begin{bmatrix} 1 \\ \mathbf{y} \end{bmatrix}$, $\mathbf{v}^T B\mathbf{v} = 2\mathbf{b}^T\mathbf{y} < 0$.

If $x_{11} > 0$, then for the vector $\mathbf{v} = \begin{bmatrix} -\mathbf{b}^T\mathbf{y} \\ x_{11}\mathbf{y} \end{bmatrix}$, $\mathbf{v}^T B\mathbf{v} = -x_{11}(\mathbf{b}^T\mathbf{y})^2 < 0$. The result follows since copositivity is inherited by principal submatrices. □

However, if all diagonal entries are unspecified, completion is possible.

Theorem 6. *Let B be an $n \times n$ partial symmetric matrix having all diagonal entries unspecified. Let $\hat{b} = \max_{i,j} |b_{ij}|$. If B' is the partial (or full) matrix obtained from B by setting all diagonal entries of B to $n\hat{b}$, then B' is a partial copositive matrix.*

Proof. Let $m \leq n$, let C' by an $m \times m$ fully specified principal submatrix of B' and let $\mathbf{v} = [v_1, \ldots, v_m]^T \geq 0$.

$$\begin{aligned}
\mathbf{v}^T C'\mathbf{v} &= \sum_{i=1}^{m} n\hat{b}v_i^2 + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} b_{ij}v_iv_j \\
&\geq \left(\sum_{i=1}^{m} n\hat{b}v_i^2 - 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} |b_{ij}|v_iv_j\right) \\
&\geq \hat{b}\left(\sum_{i=1}^{m} nv_i^2 - 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} v_iv_j\right) \\
&> \hat{b}\left(\sum_{i=1}^{m}(m-1)v_i^2 - 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} v_iv_j\right) \\
&\geq \hat{b}\left(\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(v_i - v_j)^2\right) \\
&\geq 0.
\end{aligned}$$

□

References

[1] L. Hogben, C. R. Johnson and R. Reams. The Copositive Matrix Completion Problem. *Linear Algebra and Its Applications*, 408:207-211, 2005.