THE COPOSITIVE COMPLETION PROBLEM:
UNSPECIFIED DIAGONAL ENTRIES

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Abstract. In [1] it was shown that any partial (strictly) copositive matrix all of whose diagonal entries are specified can be completed to a (strictly) copositive matrix. In this note we show that every partial strictly copositive matrix (possibly with unspecified diagonal entries) can be completed to a strictly copositive matrix, but there is an example of a partial copositive matrix with an unspecified diagonal entry that cannot be completed to a copositive matrix.

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The real symmetric $n \times n$ matrix $A$ is a strictly copositive matrix if $v^T A v > 0$ for all $v \in \mathbb{R}^n$ such that $v \geq 0$ and $v \neq 0$; $A$ is a copositive matrix if $v^T A v \geq 0$ for all $v \geq 0$ ($v \geq 0$ means every entry of $v$ is nonnegative). Copositive matrices have been studied extensively; see [1] for more information.

A partial matrix is a square array in which some entries are specified and others are not. (A conventional matrix, with all entries specified, and a matrix with no specified entries are also considered partial matrices.) An unspecified entry is denoted by ? or by $x_{ij}$. A completion of a partial matrix is a choice of values for the unspecified entries.

The partial matrix $B$ is a partial strictly copositive matrix (respectively, partial copositive matrix) if every fully specified principal submatrix of $B$ is a strictly copositive matrix (respectively, copositive matrix) and whenever $b_{ij}$ is specified then so is $b_{ji}$ and $b_{ji} = b_{ij}$.

In [1] it was shown that any partial (strictly) copositive matrix all of whose diagonal entries are specified can be completed to a (strictly) copositive matrix. In this note we address the issue of partial (strictly) copositive matrices with unspecified diagonal entries. All matrices discussed are real and symmetric.

Algorithm 1. Let $B = \begin{bmatrix} x_{11} & b^T \\ b & B_1 \end{bmatrix}$ be a partial strictly copositive $n \times n$ matrix having all entries except the 1,1-entry specified. Let $\| \cdot \|$ be a vector norm. Complete
by choosing a value for \( x_{11} \) as follows:

\[
\begin{align*}
\beta &= \min_{\mathbf{y} \in \mathbb{R}^{n-1}, \mathbf{y} \geq 0, \|\mathbf{y}\| = 1} \mathbf{b}^T \mathbf{y}, \\
\gamma &= \min_{\mathbf{y} \in \mathbb{R}^{n-1}, \mathbf{y} \geq 0, \|\mathbf{y}\| = 1} \mathbf{y}^T \mathbf{B}_1 \mathbf{y}, \\
x_{11} &= \frac{\beta^2}{\gamma}.
\end{align*}
\]

**Theorem 2.** Let \( B \) be a partial strictly copositive matrix having all entries except the 1,1-entry specified. Any value of \( x_{11} \) chosen as in Algorithm 1 completes \( B \) to a strictly copositive matrix.

**Proof.** The values \( \beta \) and \( \gamma \) are attained because a continuous function attains its extreme values on a compact set. Thus \( \gamma > 0 \), since \( \mathbf{B}_1 \) is strictly copositive, and \( x_{11} > 0 \).

If \( \mathbf{y} \neq 0 \), we can scale \( \mathbf{v} \) so that \( \|\mathbf{y}\| = 1 \), without affecting whether \( \mathbf{v}^T \mathbf{Bv} > 0 \).

Then
\[
\mathbf{v}^T \mathbf{Bv} = x_{11} z^2 + 2\mathbf{b}^T \mathbf{y} z + \mathbf{y}^T \mathbf{B}_1 \mathbf{y} \geq x_{11} z^2 + 2\beta z + \gamma > 0,
\]
by the choice of \( x_{11} \).

**Corollary 3.** Every partial strictly copositive matrix can be completed to a strictly copositive matrix.

**Proof.** Let \( B \) be a partial strictly copositive matrix. If \( B \) contains one or more unspecified diagonal entries, select such an entry \( x_{ii} \) and use Algorithm 1 (applied to every principal submatrix completed by specification of \( x_{ii} \)) to choose a value for \( x_{ii} \) large enough to ensure that every such principal submatrix of \( B \) is strictly copositive.

Repeat until all diagonal entries are specified. Then complete \( B \) as in [1], by setting \( x_{ij} = x_{ji} = \sqrt{b_{ii} b_{jj}} \), to obtain a strictly copositive matrix. \( \square \)

Corollary 3 is false for copositive matrices.

**Example 4.** \( B = \begin{bmatrix} ? & -1 \\ -1 & 0 \end{bmatrix} \) is a partial copositive matrix that cannot be completed to a copositive matrix.

Example 4 is a special case of the next lemma (which establishes the assertion that the given matrix cannot be completed to a copositive matrix).

**Lemma 5.** If a partial copositive matrix \( B \) contains a principal submatrix of the form
\[
\begin{bmatrix} ? & \mathbf{b}^T \\ \mathbf{b} & \mathbf{B}_1 \end{bmatrix}
\]
(with all elements of \( \mathbf{b} \) and \( \mathbf{B}_1 \) specified) and there exists \( \mathbf{y} \geq 0 \) such that \( \mathbf{b}^T \mathbf{y} < 0 \) and \( \mathbf{y}^T \mathbf{B}_1 \mathbf{y} = 0 \), then \( B \) cannot be completed to a copositive matrix.

**Proof.** Let \( B = \begin{bmatrix} x_{11} & \mathbf{b}^T \\ \mathbf{b} & \mathbf{B}_1 \end{bmatrix} \) and suppose there exists a vector \( \mathbf{y} \geq 0 \) such that \( \mathbf{b}^T \mathbf{y} < 0 \) and \( \mathbf{y}^T \mathbf{B}_1 \mathbf{y} = 0 \). Choose a value for \( x_{11} \).

If \( x_{11} = 0 \), then with \( \mathbf{v} = \begin{bmatrix} 1 \\ \mathbf{y} \end{bmatrix} \), \( \mathbf{v}^T \mathbf{Bv} = 2\mathbf{b}^T \mathbf{y} < 0 \).
If $x_{11} > 0$, then for the vector $\mathbf{v} = \begin{bmatrix} -b^T y \\ x_{11} y \end{bmatrix}$, $\mathbf{v}^T B \mathbf{v} = -x_{11} (b^T y)^2 < 0$. The result follows since copositivity is inherited by principal submatrices.

However, if all diagonal entries are unspecified, completion is possible.

**Theorem 6.** Let $B$ be an $n \times n$ partial symmetric matrix having all diagonal entries unspecified. Let $\hat{b} = \max_{i,j} |b_{ij}|$. If $B'$ is the partial (or full) matrix obtained from $B$ by setting all diagonal entries of $B$ to $\hat{n}b$, then $B'$ is a partial copositive matrix.

**Proof.** Let $m \leq n$, let $C'$ be an $m \times m$ fully specified principal submatrix of $B'$ and let $\mathbf{v} = [v_1, \ldots, v_m]^T \geq 0$.

\[
\mathbf{v}^T C' \mathbf{v} = \sum_{i=1}^{m} \hat{b} v_i^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} b_{ij} v_i v_j \\
\geq \left( \sum_{i=1}^{m} \hat{b} v_i^2 - 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} |b_{ij}| v_i v_j \right) \\
\geq \hat{b} \left( \sum_{i=1}^{m} n v_i^2 - 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j \right) \\
> \hat{b} \left( \sum_{i=1}^{m} (m-1) v_i^2 - 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_i v_j \right) \\
\geq \hat{b} \left( \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (v_i - v_j)^2 \right) \\
\geq 0.
\]

□

**References**