# A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs

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## Introduction

- All matrices are real and symmetric.
- $\text{corank}(A) = \text{null}(A) = \dim(\ker(A)) = \text{mult}_A(0)$
- All graphs are simple.
- $H$ is a minor of $G$ if $H$ is obtained from $G$ by a sequence of edge deletions, isolated vertex deletions and edge contractions (in any order).

The *graph* $\mathcal{G}(A) = (V, E)$ of $n \times n$ symmetric matrix $A$ is

- $V = \{1, ..., n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of $A$ is ignored.

The family of symmetric matrices described by a graph is

$$\mathcal{S}(G) = \{A : A^T = A \text{ and } \mathcal{G}(A) = G\}.$$

## Minimum rank and maximum eigenvalue multiplicity

The minimum rank of graph $G$:

$$\mathrm{mr}(G) = \min_{A \in \mathcal{S}(G)} \operatorname{rank} A.$$

The maximum eigenvalue multiplicity:

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\mathrm{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$

**Problem** Determine the minimum rank (or maximum eigenvalue multiplicity) of a graph $G$.

## Properties of minimum rank and maximum multiplicity

- $M(G) = \max_{A \in \mathcal{S}(G)} \{\text{null}(A)\}$.
- $M(G) + \text{mr}(G) = |V(G)|$.
- If $G$ is the disjoint union of graphs $G_i$ then

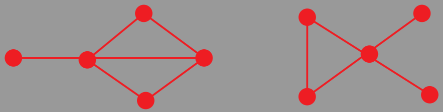
$$M(G) = \sum M(G_i)$$

$$\text{mr}(G) = \sum \text{mr}(G_i)$$

- If $G$ is connected,
  $\text{mr}(G) = 0$ iff $G$ is single vertex.
  $\text{mr}(G) = 1$ iff $G = K_n,\ n \geq 2$.

- $M(G) = 1$ if and only if $G$ is a path.
  [Fiedler 1969]

- For a connected graph $G$, $\text{mr}(G) \leq 2$ iff $G$ does not contain as an induced subgraph any of $P_4, K_{3,3,3}$ or

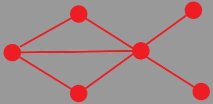


[Barrett, van der Holst, and Loewy]

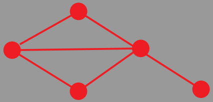
Minimum rank is monotone on induced subgraphs, i.e., if $H$ is an induced subgraph of $G$, then

$$\text{mr}(H) \leq \text{mr}(G).$$

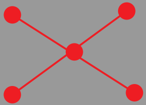
Unfortunately, $M$ is not monotone on induced subgraphs:



Graph $F$



Induced subgraph $D$



Induced subgraph $K_{1,4}$

$3 = \text{mr}(D) \leq \text{mr}(F)$, so $M(F) = 2$, but $M(K_{1,4}) = 3$.

- For a tree $T$,

$$M(T) = P(T) = \Delta(T)$$

where $P(T)$ is the path cover number, and

$$\Delta(T) = \max\{p - |Q| : T - Q \text{ is } p \text{ paths}\}.$$

[Johnson, Leal-Duarte]

- For unicyclic graph $G$, $M(G) \leq P(G)$.
[Barioli, Fallat, Hogben]

- There are good algorithms for computing $\Delta(T), P(T)$.

Colin de Verdière defined new graph parameters $\mu$ and $\nu$ with monotonicity properties that bound $M$ from below, using the Strong Arnold Property.

**Definition** $X$ *fully annihilates* $A$ if

1. $AX = 0$;
2. $A \circ X = 0$, i.e. $X$ has 0 where $A$ has nonzero;
3. $I_n \circ X = 0$, i.e., all diagonal elements of $X$ are 0.

**Definition**
The matrix $A$ has the *Strong Arnold Property* (SAP) if the zero matrix is the only symmetric matrix that fully annihilates $A$.

SAP is equivalent to transverse intersection of the constant rank manifold and the constant pattern manifold.

**Definition** $\mu(G) = \max\{\text{null}(L)\}$ such that

1. $L$ is a generalized Laplacian matrix ($L \in \mathcal{S}(G)$ and off-diagonal entries nonpositive).
2. $L$ has exactly one negative eigenvalue (with multiplicity one).
3. $L$ has SAP.

For a connected simple graph $G$,

- $\mu(G) \leq 1$ if and only if $G$ is a path.
- $\mu(G) \leq 2$ if and only if $G$ is outerplanar.
- $\mu(G) \leq 3$ if and only if $G$ is planar.

[Colin de Verdière]

- $\mu$ is minor monotone, i.e., if $H$ is a minor of $G$, then

$$\mu(H) \leq \mu(G)$$

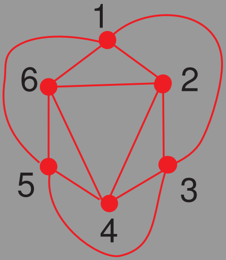
[Colin de Verdière 1990]

- For any graph,

$$\mu(G) \leq M(G).$$

- To study minimum rank, generalized Laplacians and number of negative eigenvalues are not relevant.

Example No generalized Laplacian of $K_{2,2,2}$ has rank 2 but $\mathrm{mr}(K_{2,2,2}) = 2$.




$$\operatorname{rank} A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & -3 & -2 & 1 & 0 & -1 \\ 1 & -2 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 \\ 1 & -1 & 0 & 1 & 2 & 1 \end{bmatrix} = 2$$

Another Colin de Verdière parameter:

Definition $\nu(G) = \max\{\text{null}(A)\}$ such that

1. $A \in \mathcal{S}(G)$.
2. $A$ is positive semi-definite.
3. $A$ has SAP.

- $\nu$ is minor monotone, i.e., if $H$ is a minor of $G$, then

$$\nu(H) \leq \nu(G)$$

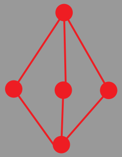
  [Colin de Verdière]

- For any graph,

$$\nu(G) \leq M(G).$$

To study minimum rank, positive semi-definite is not relevant.

Example No positive semi-definite matrix in $\mathcal{S}(K_{2,3})$ has rank 2 but $\mathrm{mr}(K_{2,3}) = 2$.



$$\operatorname{rank} A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2$$

## The new parameter $\xi$

- To study minimum rank, positive semi-definite, generalized Laplacians and number of negative eigenvalues are not relevant.
- Minor monotonicity is useful.

The new parameter:

$$\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has SAP}\}.$$

Example: $\xi(K_{2,2,2}) = 4 = M(K_{2,2,2})$.
Example: $\xi(K_{2,3}) = 3 = M(K_{2,3})$.

For any graph $G$,

- $\mu(G) \leq \xi(G)$.
- $\nu(G) \leq \xi(G)$.
- $\xi(G) \leq M(G)$.

- $\xi(P_n) = 1 = M(P_n)$
- $\xi(K_n) = n - 1 = M(K_n)$
- If $T$ is a non-path tree, $\xi(T) = 2$.

Theorem (Barioli, Fallat, Hogben, 2005)

*If $G$ is the disjoint union of graphs $G_i, i = 1, \ldots, k$ then*

$$\xi(G) = \max_{i=1,\ldots,k} \xi(G_i).$$

Example Here is why you can't sum the $\xi(G_i)$:

Let $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and let $x_i, i = 1, 2$ satisfy $A_i x_i = 0$.

Then $X = \begin{bmatrix} 0 & x_1 x_2^T \\ x_2 x_1^T & 0 \end{bmatrix}$ fully annihilates $A$, so $A$ does not have SAP.

- SAP comes from manifold theory.
- $\mathcal{R}_A = \{B : \operatorname{rank} B = \operatorname{rank} A\}$.
- $\mathcal{S}_A = \mathcal{S}(\mathcal{G}(A))$.
- $A$ has SAP if and only if manifolds $\mathcal{R}_A$ and $\mathcal{S}_A$ intersect transversally at $A$.
- Transversal intersection allows perturbation.

[van der Holst, Lovász, Schrijver]

Theorem (Barioli, Fallat, Hogben, 2005)

*If $H$ is a subgraph of $G$ then $\xi(H) \leq \xi(G)$.*

Proof.

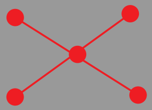
- Deletion of isolated vertices cannot increase $\xi$ by disjoint union theorem.
- Obtain $G'$ from $G$ by deleting edge $uv$ and show $\xi(G') \leq \xi(G)$:
- Choose $A' \in \mathcal{S}(G')$, $\text{null}(A') = \xi(G')$, and $A'$ has SAP, so $\mathcal{R}_{A'}$ and $\mathcal{S}_{A'}$ intersect transversally at $A'$.
- $\mathcal{S}(t)$ is the manifold of matrices obtained from matrices $B$ in $\mathcal{S}_{A'}$ by replacing the $u, v$- and $v, u$-entries of $B$ by $t$.
- $\mathcal{R}(t) = \mathcal{R}_{A'}$.
- For sufficiently small positive $t_0$, $\mathcal{R}(t_0)$ and $\mathcal{S}(t_0)$ intersect transversally at some $A$. So $A$ has SAP and $\mathcal{G}(A) = G$.
- $\xi(G') = \text{null}(A') = \text{null}(A) \leq \xi(G)$. □

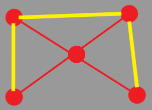
Corollary If $G$ has $q$ independent vertices then
$\xi(G) \leq |V(G)| - q + 1$.

Proof.
Add edges between independent vertices to obtain $\widetilde{G}$ having path $P_q$ as induced subgraph.

$q - 1 = \text{mr}(P_q) \leq \text{mr}(\widetilde{G})$.

$|V(G)| - (q-1) \geq |V(\widetilde{G})| - \text{mr}(\widetilde{G}) = M(\widetilde{G}) \geq \xi(\widetilde{G}) \geq \xi(G)$. □



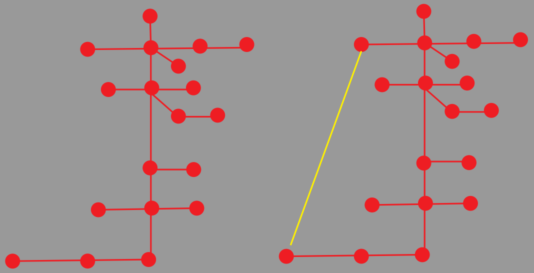


Corollary $\xi(K_{p,q}) = p + 1$ $(1 \leq p \leq q, 3 \leq q)$.
Proof. $p + 1 = \mu(K_{p,q}) \leq \xi(K_{p,q}) \leq p + q - (q-1)$.

Theorem (Barioli, Fallat, Hogben, 2005)

*If $T$ is a tree and $\xi(T) < M(T)$, then we can add an edge to $T$ to obtain graph $G$ such that $M(G) < M(T)$.*

Example



$M(T) = P(T) = 8 \qquad M(G) \leq P(G) = 7$

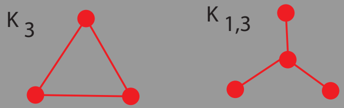
Theorem (Barioli, Fallat, Hogben, 2005)

*$\xi$ is minor monotone, i.e., if $H$ is a minor of $G$, then $\xi(H) \leq \xi(G)$.*

Forbidden minors

Since $\xi$ is minor monotone, the graphs $G$ such that $\xi(G) \leq k$ can be characterized by a finite set of forbidden minors.

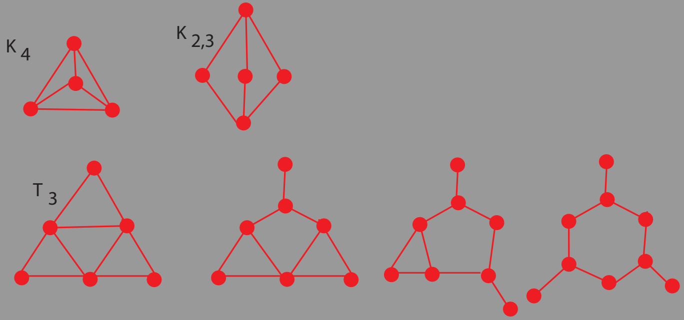
$\xi(G) \leq 1$ if and only if $G$ contains no $K_3$ or $K_{1,3}$ minor.



**Theorem (Hogben, van der Holst, 2006)**

*$\xi(G) \leq 2$ if and only if $G$ contains no minor in the $T_3$ family.*

$T_3$ family



Definition Let $G_i, i = 1, \ldots, k$ be subgraphs of $G$ such that

$$G = \bigcup_{i=1}^{k} G_i \text{ and for } i \neq j, V_i \cap V_j = \{v\}.$$

Then $G$ is called the 1-sum at $v$ of $G_i, i = 1, \ldots, k$.

Theorem (Barioli, Fallat, Hogben, 2004)

*Let $G$ be the 1-sum at $v$ of graphs $G_1, \ldots, G_k$. Then*

$$\sum_{i=1}^{k} M(G_i - v) - 1 \leq M(G) \leq \sum_{i=1}^{k} M(G_i - v) + 1.$$

*There is an exact formula for the computation of $M$ of a 1-sum in terms of the pieces.*

Theorem (Barioli, Fallat, Hogben, 2005)

*Let $G$ be 1-sum at $v$ of graphs $G_1, \ldots, G_k$. Then*

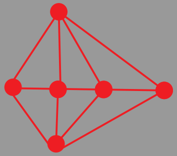
$$\max_{i=1}^{k} \xi(G_i) \leq \xi(G) \leq \max_{i=1}^{k} \xi(G_i) + 1.$$

Note: For a 1-sum,

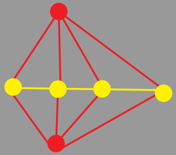
$$\mu(G) = \max_{i=1}^{k} \mu(G_i) \text{ and } \nu(G) = \max_{i=1}^{k} \nu(G_i).$$

## $\xi$ and minimum rank

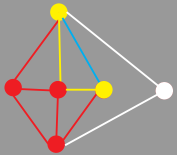
### Example



The graph $G$



$P_4$ is an induced subgraph



$K_4$ is a minor

$3 = \text{mr}(P_4) \leq \text{mr}(G)$.
$3 = \xi(K_4) \leq M(K_4) \leq M(G) = 6 - \text{mr}(G) \leq 6 - 3$.
So $\text{mr}(G) = 3$.

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