A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs

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Introduction

- All matrices are real and symmetric.

- $\text{corank}(A) = \text{null}(A) = \dim(\ker(A)) = \text{mult}_A(0)$

- All graphs are simple.

- $H$ is a minor of $G$ if $H$ is obtained from $G$ by a sequence of edge deletions, isolated vertex deletions and edge contractions (in any order).
The graph $G(A) = (V, E)$ of $n \times n$ symmetric matrix $A$ is

- $V = \{1, \ldots, n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of $A$ is ignored.

The family of symmetric matrices described by a graph is

$$S(G) = \{A : A^T = A \text{ and } G(A) = G\}.$$
Minimum rank and maximum eigenvalue multiplicity

The minimum rank of graph $G$:

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank } A.$$  

The maximum eigenvalue multiplicity:

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{mult}_A(\lambda) : \lambda \in \sigma(A)\}.$$  

**Problem** Determine the minimum rank (or maximum eigenvalue multiplicity) of a graph $G$.  

Properties of minimum rank and maximum multiplicity

- \( M(G) = \max_{A \in \mathcal{S}(G)} \{ \text{null}(A) \} \).
- \( M(G) + \text{mr}(G) = |V(G)| \).
- If \( G \) is the disjoint union of graphs \( G_i \) then
  \[
  M(G) = \sum M(G_i) \\
  \text{mr}(G) = \sum \text{mr}(G_i)
  \]
- If $G$ is connected, 
  $\text{mr}(G) = 0$ iff $G$ is single vertex.
  $\text{mr}(G) = 1$ iff $G = K_n$, $n \geq 2$. 

- $M(G) = 1$ if and only if $G$ is a path. 
  [Fiedler 1969]

- For a connected graph $G$, $\text{mr}(G) \leq 2$ iff $G$ does not contain as an induced subgraph any of $P_4$, $K_{3,3,3}$ or

  ![Diagram](image_url)

  [Barrett, van der Holst, and Loewy]
Minimum rank is monotone on induced subgraphs, i.e., if $H$ is an induced subgraph of $G$, then

$$\text{mr}(H) \leq \text{mr}(G).$$

Unfortunately, $M$ is not monotone on induced subgraphs:

Graph $F$ \quad Induced subgraph $D$ \quad Induced subgraph $K_{1,4}$

$$3 = \text{mr}(D) \leq \text{mr}(F), \text{ so } M(F) = 2, \text{ but } M(K_{1,4}) = 3.$$
- For a tree $T$,

$$M(T) = P(T) = \Delta(T)$$

where $P(T)$ is the path cover number, and

$$\Delta(T) = \max\{p - |Q| : T - Q \text{ is } p \text{ paths}\}.$$

[Johnson, Leal-Duarte]

- For unicyclic graph $G$, $M(G) \leq P(G)$.

[Barioli, Fallat, Hogben]

- There are good algorithms for computing $\Delta(T), P(T)$. 
Colin de Verdière defined new graph parameters $\mu$ and $\nu$ with monotonicity properties that bound $M$ from below, using the Strong Arnold Property.

**Definition** $X$ fully annihilates $A$ if

1. $AX = 0$;
2. $A \circ X = 0$, i.e. $X$ has 0 where $A$ has nonzero;
3. $I_n \circ X = 0$, i.e., all diagonal elements of $X$ are 0.

**Definition**

The matrix $A$ has the *Strong Arnold Property* (SAP) if the zero matrix is the only symmetric matrix that fully annihilates $A$.

SAP is equivalent to transverse intersection of the constant rank manifold and the constant pattern manifold.
Definition $\mu(G) = \max\{\text{null}(L)\}$ such that

1. $L$ is a generalized Laplacian matrix ($L \in \mathcal{S}(G)$ and off-diagonal entries nonpositive).
2. $L$ has exactly one negative eigenvalue (with multiplicity one).
3. $L$ has SAP.

For a connected simple graph $G$,

- $\mu(G) \leq 1$ if and only if $G$ is a path.
- $\mu(G) \leq 2$ if and only if $G$ is outerplanar.
- $\mu(G) \leq 3$ if and only if $G$ is planar.

[Colin de Verdière]
- $\mu$ is minor monotone, i.e., if $H$ is a minor of $G$, then

$$\mu(H) \leq \mu(G)$$

[Colin de Verdière 1990]

- For any graph,

$$\mu(G) \leq M(G).$$

- To study minimum rank, generalized Laplacians and number of negative eigenvalues are not relevant.
Example: No generalized Laplacian of $K_{2,2,2}$ has rank 2 but $\text{mr}(K_{2,2,2}) = 2$.

\[
\text{rank } A = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & -3 & -2 & 1 & 0 & -1 \\
1 & -2 & -1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 3 & 2 \\
1 & -1 & 0 & 1 & 2 & 1
\end{bmatrix} = 2
\]
Another Colin de Verdière parameter:

Definition \( \nu(G) = \max\{\text{null}(A)\} \) such that

1. \( A \in S(G) \).
2. \( A \) is positive semi-definite.
3. \( A \) has SAP.

- \( \nu \) is minor monotone, i.e., if \( H \) is a minor of \( G \), then

\[
\nu(H) \leq \nu(G)
\]

[Colin de Verdière]

- For any graph,

\[
\nu(G) \leq M(G).
\]
To study minimum rank, positive semi-definite is not relevant.

Example No positive semi-definite matrix in $S(K_{2,3})$ has rank 2 but $\text{mr}(K_{2,3}) = 2$. 

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2$$
The new parameter $\xi$

- To study minimum rank, positive semi-definite, generalized Laplacians and number of negative eigenvalues are not relevant.
- Minor monotonicity is useful.

The new parameter:

$$\xi(G) = \max\{\text{null}(A) : A \in S(G), A \text{ has SAP}\}.$$ 

Example: $\xi(K_{2,2,2}) = 4 = M(K_{2,2,2})$. 
Example: $\xi(K_{2,3}) = 3 = M(K_{2,3})$. 
For any graph $G$,

- $\mu(G) \leq \xi(G)$.
- $\nu(G) \leq \xi(G)$.
- $\xi(G) \leq M(G)$.

- $\xi(P_n) = 1 = M(P_n)$
- $\xi(K_n) = n - 1 = M(K_n)$
- If $T$ is a non-path tree, $\xi(T) = 2$. 
Theorem (Barioli, Fallat, Hogben, 2005)

If $G$ is the disjoint union of graphs $G_i$, $i = 1, \ldots, k$ then

$$\xi(G) = \max_{i=1,\ldots,k} \xi(G_i).$$

Example Here is why you can't sum the $\xi(G_i)$:

Let $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and let $x_i, i = 1, 2$ satisfy $A_ix_i = 0$.

Then $X = \begin{bmatrix} 0 & x_1x_2^T \\ x_2x_1^T & 0 \end{bmatrix}$ fully annihilates $A$, so $A$ does not have SAP.
- SAP comes from manifold theory.
- $\mathcal{R}_A = \{B : \text{rank } B = \text{rank } A\}$.
- $S_A = S(\mathcal{G}(A))$.
- $A$ has SAP if and only if manifolds $\mathcal{R}_A$ and $S_A$ intersect transversally at $A$.
- Transversal intersection allows perturbation.

[van der Holst, Lovász, Schrijver]
Theorem (Barioli, Fallat, Hogben, 2005)

If $H$ is a subgraph of $G$ then $\xi(H) \leq \xi(G)$. 

Proof.

• Deletion of isolated vertices cannot increase $\xi$ by disjoint union theorem.
• Obtain $G'$ from $G$ by deleting edge $uv$ and show $\xi(G') \leq \xi(G)$:
• Choose $A' \in \mathcal{S}(G')$, $\text{null}(A') = \xi(G')$, and $A'$ has SAP, so $\mathcal{R}_{A'}$ and $\mathcal{S}_{A'}$ intersect transversally at $A'$.
• $\mathcal{S}(t)$ is the manifold of matrices obtained from matrices $B$ in $\mathcal{S}_{A'}$ by replacing the $u, v$- and $v, u$-entries of $B$ by $t$.
• $\mathcal{R}(t) = \mathcal{R}_{A'}$.
• For sufficiently small positive $t_0$, $\mathcal{R}(t_0)$ and $\mathcal{S}(t_0)$ intersect transversally at some $A$. So $A$ has SAP and $\mathcal{G}(A) = G$.
• $\xi(G') = \text{null}(A') = \text{null}(A) \leq \xi(G)$. 
\qed
Corollary If \( G \) has \( q \) independent vertices then
\[
\xi(G) \leq |V(G)| - q + 1.
\]

Proof.
Add edges between independent vertices to obtain \( \tilde{G} \) having path \( P_q \) as induced subgraph.

\[
q - 1 = \text{mr}(P_q) \leq \text{mr}(\tilde{G}).
\]

\[
|V(G)| - (q - 1) \geq |V(\tilde{G})| - \text{mr}(\tilde{G}) = M(\tilde{G}) \geq \xi(\tilde{G}) \geq \xi(G). \quad \square
\]

Corollary \( \xi(K_{p,q}) = p + 1 \) (\( 1 \leq p \leq q \), \( 3 \leq q \)).
Proof. \( p + 1 = \mu(K_{p,q}) \leq \xi(K_{p,q}) \leq p + q - (q - 1) \).
Theorem (Barioli, Fallat, Hogben, 2005)

If $T$ is a tree and $\xi(T) < M(T)$, then we can add an edge to $T$ to obtain graph $G$ such that $M(G) < M(T)$.

Example

$M(T) = P(T) = 8 \quad M(G) \leq P(G) = 7$
Theorem (Barioli, Fallat, Hogben, 2005)

\[ \xi \text{ is minor monotone, i.e., if } H \text{ is a minor of } G, \text{ then } \xi(H) \leq \xi(G). \]

Forbidden minors

Since \( \xi \) is minor monotone, the graphs \( G \) such that \( \xi(G) \leq k \) can be characterized by a finite set of forbidden minors.

\[ \xi(G) \leq 1 \text{ if and only if } G \text{ contains no } K_3 \text{ or } K_{1,3} \text{ minor.} \]
Theorem (Hogben, van der Holst, 2006)

\[ \xi(G) \leq 2 \text{ if and only if } G \text{ contains no minor in the } T_3 \text{ family.} \]

\[ T_3 \text{ family} \]
Definition Let $G_i, i = 1, \ldots, k$ be subgraphs of $G$ such that

$$G = \bigcup_{i=1}^{k} G_i$$

and for $i \neq j$, $V_i \cap V_j = \{v\}$.

Then $G$ is called the 1-sum at $v$ of $G_i, i = 1, \ldots, k$.

Theorem (Barioli, Fallat, Hogben, 2004)

Let $G$ be the 1-sum at $v$ of graphs $G_1, \ldots, G_k$. Then

$$\sum_{i=1}^{k} M(G_i - v) - 1 \leq M(G) \leq \sum_{i=1}^{k} M(G_i - v) + 1.$$  

There is an exact formula for the computation of $M$ of a 1-sum in terms of the pieces.
Theorem (Barioli, Fallat, Hogben, 2005)

Let $G$ be 1-sum at $v$ of graphs $G_1, \ldots, G_k$. Then

$$\max_{i=1}^k \xi(G_i) \leq \xi(G) \leq \max_{i=1}^k \xi(G_i) + 1.$$ 

Note: For a 1-sum,

$$\mu(G) = \max_{i=1}^k \mu(G_i) \text{ and } \nu(G) = \max_{i=1}^k \nu(G_i).$$
ξ and minimum rank

Example

The graph $G$  
$P_4$ is an induced subgraph  
$K_4$ is a minor

$3 = \text{mr}(P_4) \leq \text{mr}(G)$.  
$3 = \xi(K_4) \leq M(K_4) \leq M(G) = 6 - \text{mr}(G) \leq 6 - 3$.  
So $\text{mr}(G) = 3$. 


