Minimum Rank Problems: Recent Developments

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General type of problem

Determine the minimum rank among a family of matrices described by a given zero-nonzero pattern.

Example

Let $Y_1 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ *and* $Y_2 = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}$.

Minimum rank of matrices described by Y_1 *is 1.*
Minimum rank of matrices described by Y_2 *is 2.*

Some topics with connections to minimum rank problems:

- Jordan Canonical Form for matrices described by a digraph
- Inverse Eigenvalue Problem for matrices described by a simple graph
- spectral graph theory, singular graphs, nullity of the adjacency matrix
- biclique decompositions and the bicliquecover number (Graham-Pollack Theorem)
- eigensharp graphs
- Lovász ϑ function
- communication complexity and minimum rank of ± 1 sign patterns

Graphs and digraphs

A square pattern can be described by a graph or digraph.

- a simple graph $G = (V_G, E_G)$ does not allow loops or multiple edges between the same pair of vertices
- a graph $\mathbb{G} = (V_{\mathbb{G}}, E_{\mathbb{G}})$ allows loops but not multiple edges between the same pair of vertices
- a digraph $\mathbb{D} = (V_{\mathbb{D}}, E_{\mathbb{D}})$ allows loops but not multiple arcs between the same ordered pair of vertices.

A graph or digraph describes a family of matrices.

- $\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} : A^T = A$ and
  for $i \neq j, \{i,j\} \in E_G \Leftrightarrow a_{i,j} \neq 0\}$
- $\mathcal{S}(\mathbb{G}) = \{A \in \mathbb{R}^{n\times n} : A^T = A$ and
  multiset $\{i,j\} \in E_{\mathbb{G}} \Leftrightarrow a_{i,j} \neq 0\}$
- $\mathcal{Q}(\mathbb{D}) = \{A \in \mathbb{R}^{n\times n} : (i,j) \in E_{\mathbb{D}} \Leftrightarrow a_{i,j} \neq 0\}$

Minimum rank and maximum nullity: digraphs

- $\text{mr}(\mathbb{D}) = \min\{\text{rank}(A) : A \in \mathcal{Q}(\mathbb{D})\}$
- $M(\mathbb{D}) = \max\{\text{null}(A) : A \in \mathcal{Q}(\mathbb{D})\}$
- $\text{mr}(\mathbb{D}) + M(\mathbb{D}) = |\mathbb{D}|$

Asymmetric minimum rank problem

Determine $\text{mr}(\mathbb{D})$ for any digraph $\mathbb{D}$.

Early work

Series of papers by Hershkowitz and Schneider in 1993 and 1994:

- Possible Jordan Canonical Forms for eigenvalue 0 of matrices in $\mathcal{Q}(\mathbb{D})$
- Complete solution for the generic case.

Minimum rank and maximum nullity: simple graphs

- $\operatorname{mr}(G) = \min\{\operatorname{rank}(A) : A \in \mathcal{S}(G)\}$
- $M(G) = \max\{\operatorname{null}(A) : A \in \mathcal{S}(G)\}$
- $\operatorname{mr}(G) + M(G) = |G|$

Symmetric minimum rank problem

Determine $\operatorname{mr}(G)$ for any simple graph G.

Early work

- Arose from the inverse eigenvalue problem for $\mathcal{S}(G)$.
- Complete solution for minimum rank of trees [Nylen, 96], [Johnson, Leal-Duarte, 99].
- Many results by more than 30 people from 2004 on.

Minimum rank and maximum nullity: graphs

- $\mathrm{mr}(\mathbb{G}) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(\mathbb{G})\}$
- $M(\mathbb{G}) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(\mathbb{G})\}$
- $\mathrm{mr}(\mathbb{G}) + M(\mathbb{G}) = |\mathbb{G}|$

Symmetric minimum rank problem for graphs with loops

Determine $\mathrm{mr}(\mathbb{G})$ for any graph $\mathbb{G}$.

Early work

- Complete solution for minimum rank of loop trees [DeAlba, Hardy, Hentzel, Hogben, Wangsness, 06]

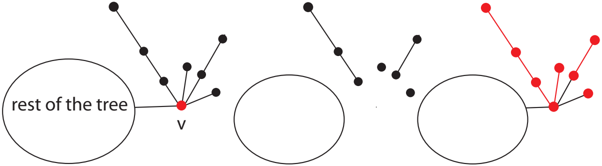
Simple trees

In [Johnson, Leal-Duarte, 99]:

- $\Delta(T)$ is the maximum of $p - q$ such that there is a set of q vertices whose deletion leaves p paths
- the *path cover number* $\mathcal{P}(T)$ is the minimum number of vertex disjoint paths that cover all the vertices of T
- proved that $\mathcal{P}(T) = M(T) = \Delta(T)$

Numerous algorithms compute $\Delta(T)$ and $\mathcal{P}(T)$ by using high degree (≥ 3) vertices.

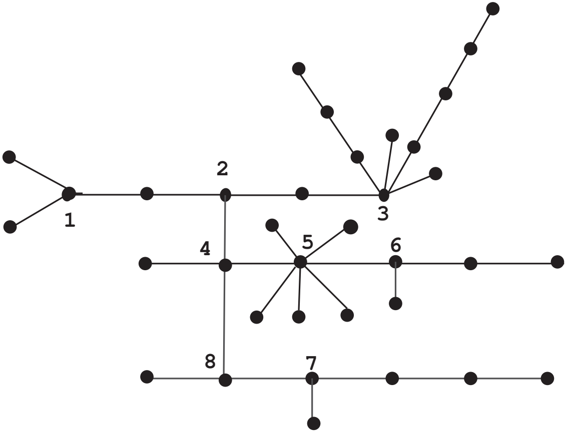
The following algorithms work from the outside in (start with a pendent generalized star).



- $\Delta(T)$: Delete each outer high degree vertex v. Repeat as needed.
- $\mathcal{P}(T)$: At each outer high degree vertex v, form a path of v and two pendent paths to make one path, and use additional pendent paths as needed. Repeat as needed.

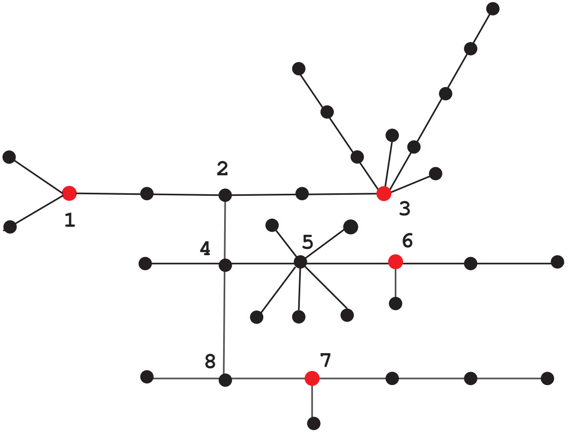
Example

Compute $\operatorname{mr}(T)$ *by computing* $\Delta(T) = M(T)$.



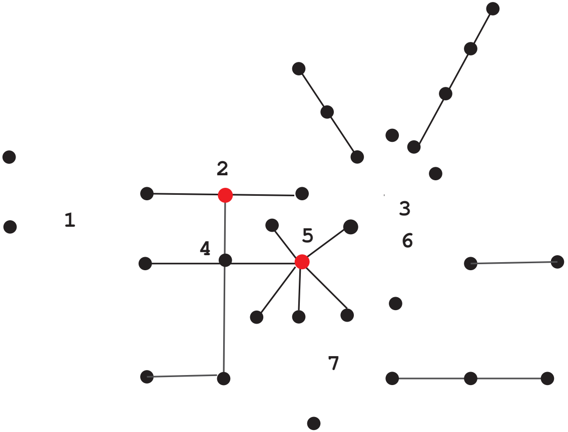
Example

Compute $\operatorname{mr}(T)$ *by computing* $\Delta(T) = M(T)$.



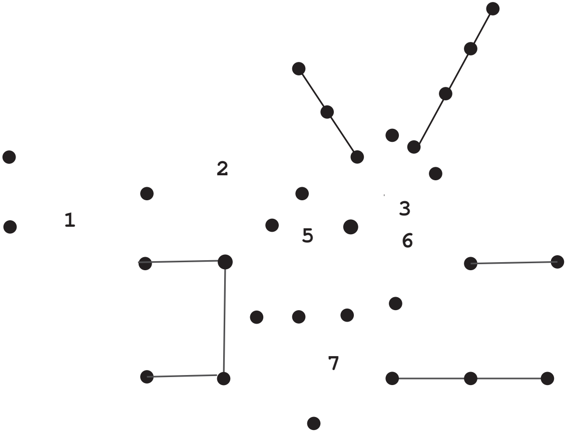
Example

Compute mr(T) *by computing* $\Delta(T) = M(T)$.



Example

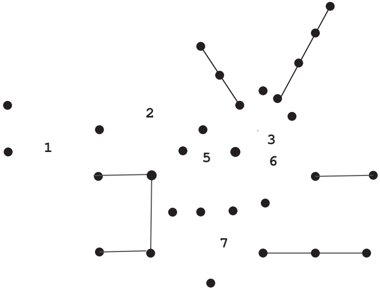
Compute $\operatorname{mr}(T)$ *by computing* $\Delta(T) = M(T)$.



Example

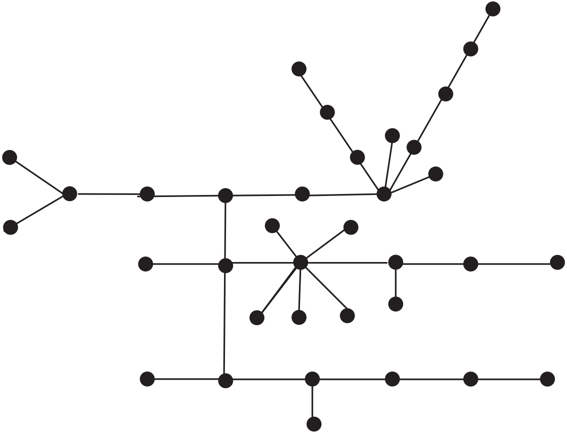
Compute $\mathrm{mr}(T)$ *by computing* $\Delta(T) = M(T)$*:*

- the six vertices $\{1, 2, 3, 5, 6, 7\}$ were deleted
- there are 18 paths
- $M(T) = \Delta(T) = 18 - 6 = 12$
- $\mathrm{mr}(T) = 35 - 12 = 23$



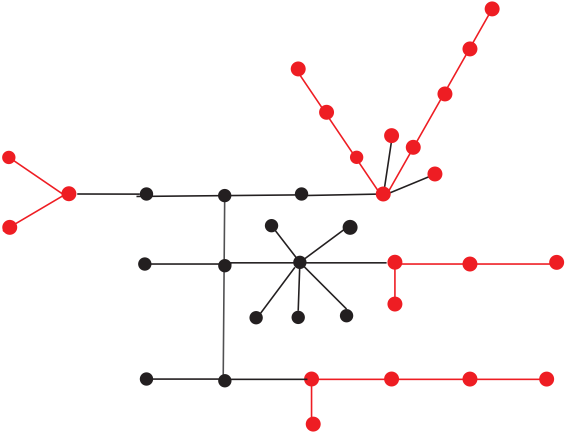
Example

Compute $\mathrm{mr}(T)$ *by computing* $\mathcal{P}(T) = M(T)$.



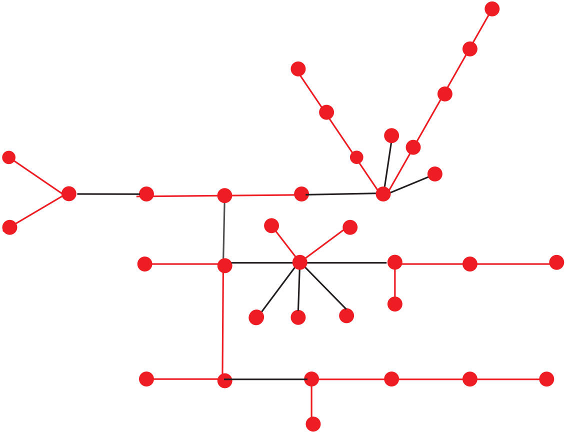
Example

Compute $\mathrm{mr}(T)$ *by computing* $\mathcal{P}(T) = M(T)$.



Example

$\mathcal{P}(T) = 12 = M(T)$ *and* $\operatorname{mr}(T) = 35 - 12 = 23.$



Loop trees

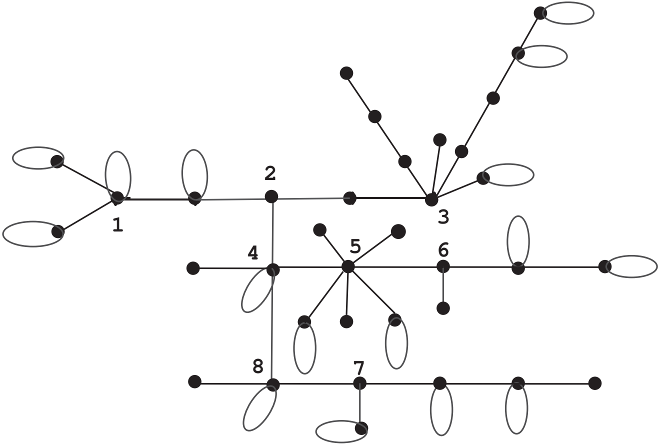
A loop tree $\mathbb{T}$ is a graph that allows loops but not multiple edges that is connected and has no cycles of length greater than 1.

In [DeAlba, Hardy, Hentzel, Hogben, Wangsness, 06]:

- $\mathcal{C}_0(\mathbb{T}) = \max\{c_0(Q) - |Q|\}$ where $c_0(Q)$ is the number of singular components of $\mathbb{T} - Q$
- algorithm to compute $\mathcal{C}_0(\mathbb{T})$ that generalizes the algorithm for computing $\Delta(T)$ working from the outside in.
- proved that $\mathcal{C}_0(\mathbb{T}) = M(\mathbb{T})$

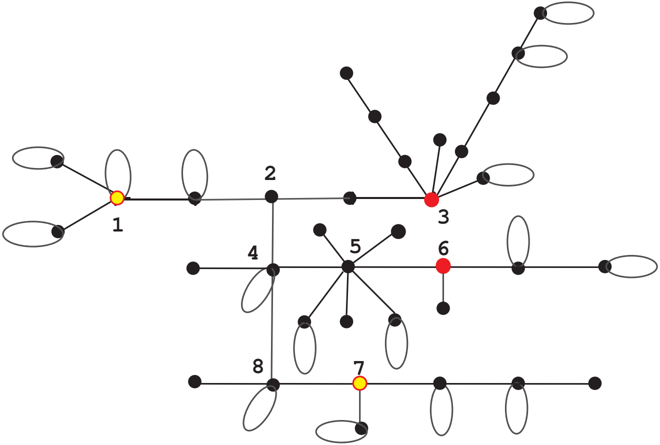
Example

Compute $\mathrm{mr}(\mathbb{T})$ *by computing* $\mathcal{C}_0(\mathbb{T}) = M(\mathbb{T})$.



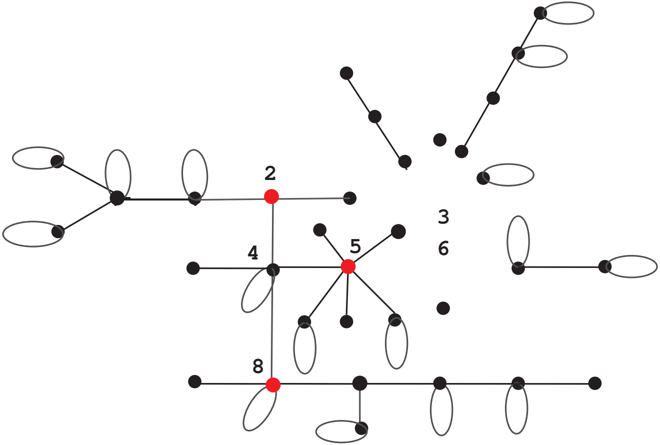
Example

Compute $\mathrm{mr}(\mathbb{T})$ *by computing* $\mathcal{C}_0(\mathbb{T}) = M(\mathbb{T})$.



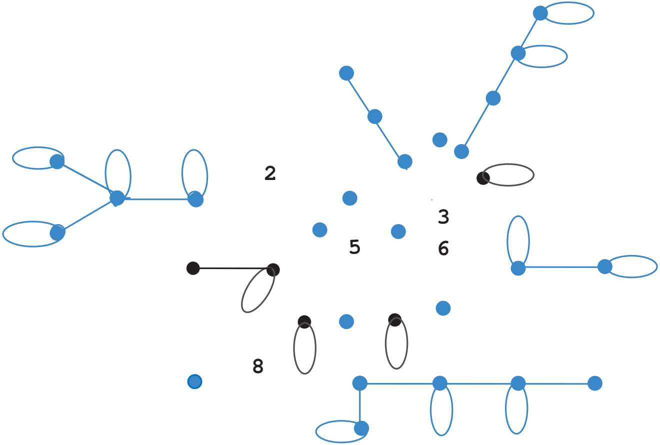
Example

Compute $\mathrm{mr}(\mathbb{T})$ *by computing* $\mathcal{C}_0(\mathbb{T}) = M(\mathbb{T})$.



Example

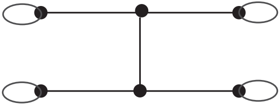
$M(\mathbb{T}) = \mathcal{C}_0(\mathbb{T}) = 12 - 5 = 7$ *and* $\operatorname{mr}(\mathbb{T}) = 35 - 7 = 28.$



The obvious generalization of path cover number fails to equal maximum nullity:

Example

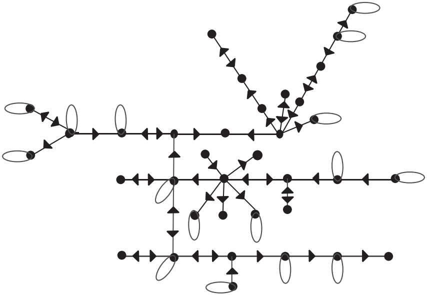
The minimum number of paths needed to cover the double path $\mathbb{T}$ *is 2 but* $M(\mathbb{T}) = 1$



Ditrees

The underlying simple graph of $\mathbb{D}$ is the simple graph obtained by deleting loops and then replacing every arc (v, w) or pair of arcs $(v, w), (w, v)$ by the edge $\{v, w\}$.

A ditree $\mathbb{T}$ is a digraph whose underlying simple graph is a tree.



The associated loop tree of a symmetric ditree is the loop tree obtained by replacing every pair of arcs $(v, w), (w, v)$ by the edge $\{v, w\}$ (and arc (v, v) by edge $\{v, v\}$).

Observation

If $\mathbb{T}$ *is a symmetric ditree and* $\mathbb{T}'$ *is the associated loop tree then* $\operatorname{mr}(\mathbb{T}) = \operatorname{mr}(\mathbb{T}')$.

Note that $\mathbb{T}'$ describes only symmetric matrices.

Results of the AIM Minimum Rank Square in February 08 [AIM Square] (Barioli, Fallat, Hall, Hershkowitz, Hogben, van der Holst, Shader):

Definition

The path cover number $\mathcal{P}(\mathbb{D})$ *of* $\mathbb{D}$ *is the minimum number of vertex disjoint paths whose deletion from* $\mathbb{D}$ *leaves a digraph that requires nonsingularity (or the empty set).*

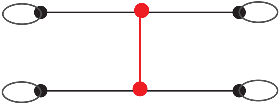
Theorem

If $\mathbb{T}$ *is a ditree then* $\mathcal{P}(\mathbb{T}) = M(\mathbb{T})$.

This definition of path cover number works for the double path:

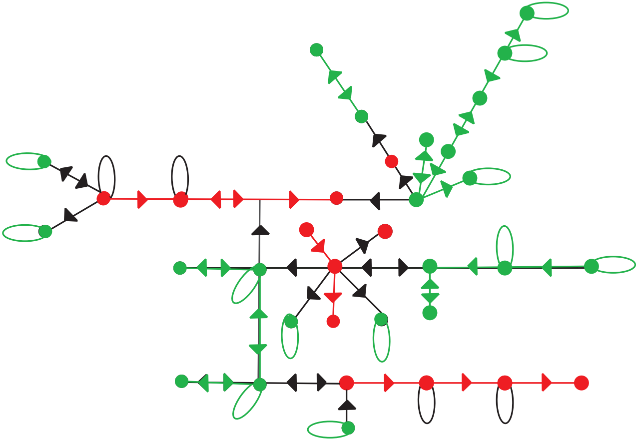
Example

The deletion of one path from $\mathbb{T}$ *leaves a nonsingular graph so* $\mathcal{P}(\mathbb{T}) = 1 = M(\mathbb{T})$.



Example

$M(\mathbb{T}) = \mathcal{P}(\mathbb{T}) = 5.$



How is $\mathcal{P}(\mathbb{T})$ found?

Computing minimum rank of ditrees

The zero forcing number was introduced for simple graphs in [AIM 08] (18 authors, based on the AIM 2006 workshop). We adapted it to digraphs. $\mathbb{D}$ is a digraph with each vertex colored either white or black.

- out color change rule: If u is a vertex of $\mathbb{D}$, and exactly one out-neighbor w of u is white, then change the color of w to black (u forces w).
- out derived coloring: result of applying the out color change rule until no more changes are possible.
- out zero forcing set: $Z \subset V_{\mathbb{D}}$ such that if Z is colored black, the out derived coloring is all black.
- zero forcing number: $Z_o(\mathbb{D})$: minimum of $|Z|$ over all out zero forcing sets $Z \subseteq V_{\mathbb{D}}$.

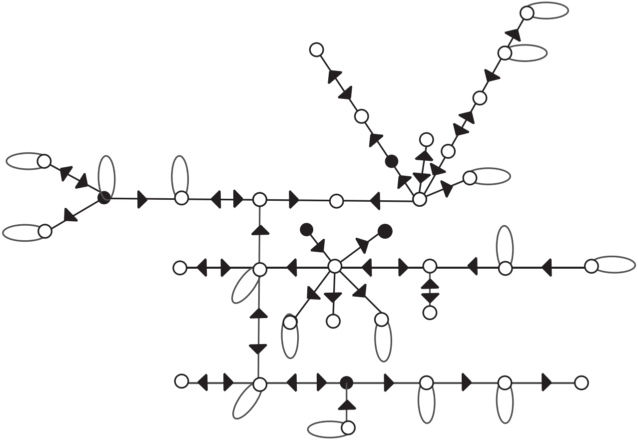
Theorem

[AIM Square] *If* $\mathbb{T}$ *is a ditree then* $\mathcal{P}(\mathbb{T}) = \mathrm{Z}_o(\mathbb{T})$.

- $\mathrm{Z}_o(\mathbb{D})$ can be computed by brute force.
- ISU group (DeLoss, Grout, McKay, Smith, Tims) have a program to compute $\mathrm{Z}_o(\mathbb{D})$ in Sage.
- It produced the following zero forcing set for the previous example.

Example

Verify $Z_o(\mathbb{T}) \le 5$ *by finding the out derived set.*



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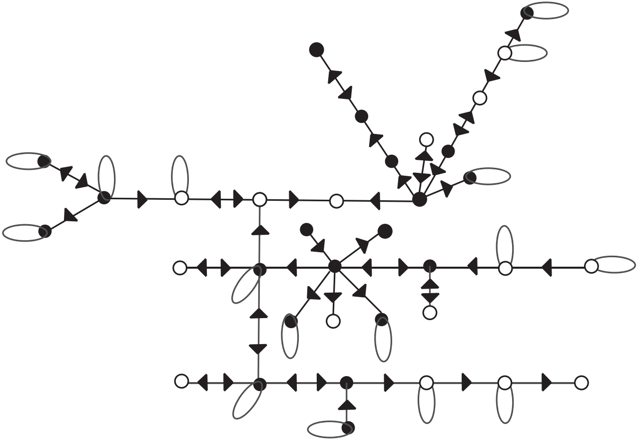
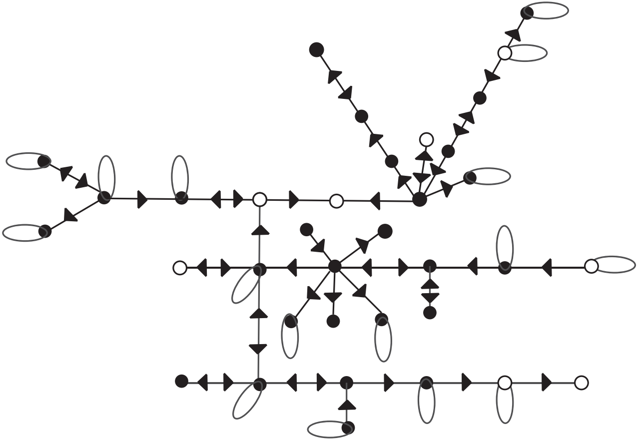


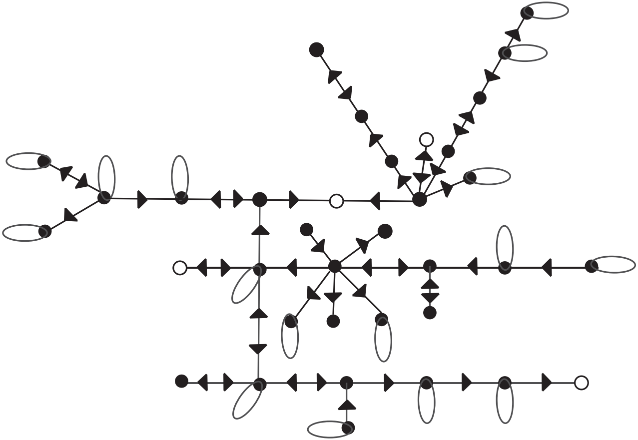
Example

Verify $Z_o(\mathbb{T}) \leq 5$ *by finding the out derived set.*



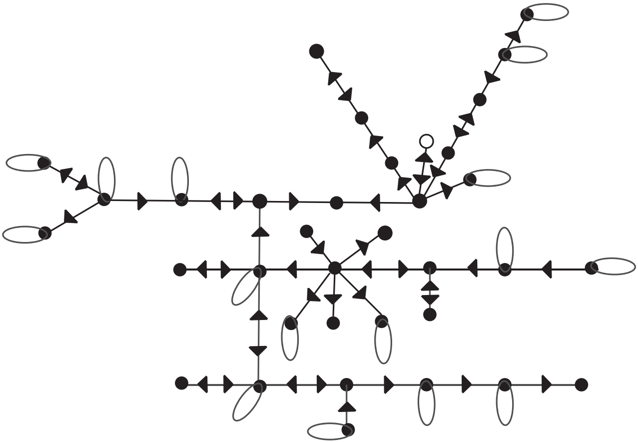
Example

Verify $Z_o(\mathbb{T}) \leq 5$ *by finding the out derived set.*



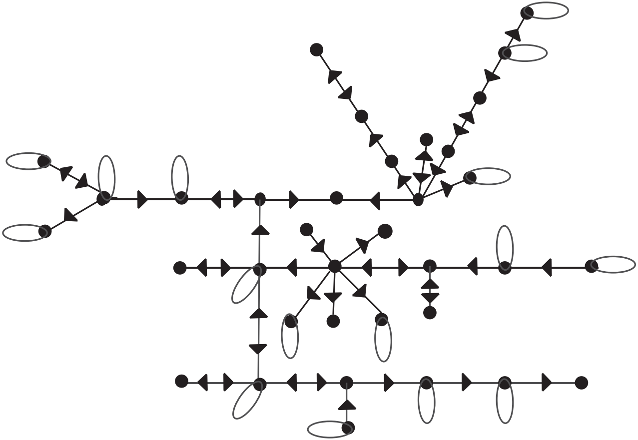
Example

Verify $Z_o(\mathbb{T}) \leq 5$ *by finding the out derived set.*



Example

Verify $Z_o(\mathbb{T}) \leq 5$ *by finding the out derived set.*



Outline of the proof of the ditree theorem

$\mathrm{M}(\mathbb{T}) = \mathcal{P}(\mathbb{T}) = \mathrm{Z}_o(\mathbb{T})$ [AIM Square]

Let Z be an out zero forcing set of a digraph $\mathbb{D}$. Construct the out derived set, recording the forces.

- forcing chain: a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, v_i forces v_{i+1}.
- forcing chain digraph (of the forcing chain $(v_1, v_2, \ldots, v_k)$): the digraph $\mathbb{H} = (V_{\mathbb{H}}, E_{\mathbb{H}})$ where $V_{\mathbb{H}} = \{v_1, v_2, \ldots, v_k\}$ and $E_{\mathbb{H}} = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$.

Lemma

Any forcing chain digraph is a path or a cycle.

Theorem (AIM Square)

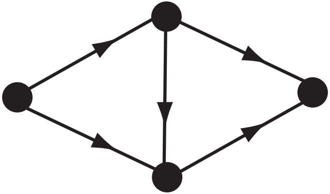
For any digraph $\mathbb{D}$, $\mathcal{P}(\mathbb{D}) \leq \mathrm{Z}_o(\mathbb{D})$.

Proof:

- Choose an out zero forcing set of order $\mathrm{Z}_o(\mathbb{D})$.
- P is the set of all maximal forcing chain digraphs that are paths.
- $\mathbb{D} - P$ can force itself.
- So $\mathbb{D} - P$ is nonsingular.
- $\mathcal{P}(\mathbb{D}) \leq |P| \leq |Z| = \mathrm{Z}_o(\mathbb{D})$.

We need to define path cover number without induced to obtain $\mathcal{P}(\mathbb{D}) \leq \mathrm{Z}_o(\mathbb{D})$.

Example



The triangle number has been used to bound minimum rank of a pattern from below ([Canto, 11th ILAS], [Johnson, 12th ILAS]).

- t-triangle of an $m \times n$ pattern Y: a $t \times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero.
- triangle number: $\operatorname{tri}(Y) =$ maximum size of a triangle in Y.
- For a digraph $\mathbb{D}$, $\operatorname{tri}(\mathbb{D}) = \operatorname{tri}(\mathcal{Y}(\mathbb{D}))$.
- $\operatorname{tri}(\mathbb{D}) \leq \operatorname{mr}(\mathbb{D})$

Theorem (AIM Square)

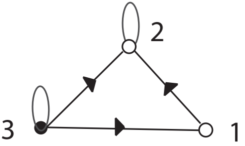
$\text{tri}(\mathbb{D}) + Z_o(\mathbb{D}) = |\mathbb{D}|$.

Proof: $Z_o(\mathbb{D}) \leq |\mathbb{D}| - \text{tri}(\mathbb{D})$

- Suppose $\mathbb{D}$ has a t-triangle.
- The columns not in the t-triangle constitute a zero forcing set.
- So $Z_o(\mathbb{D}) \leq |\mathbb{D}| - \text{tri}(\mathbb{D})$.

Example

$$\begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix}$$



3 is black, 1 forces 2 and then 3 forces 1, so $Z_o(\mathbb{D}) \leq 1$.

Theorem

$\mathrm{tri}(\mathbb{D}) + \mathrm{Z}_o(\mathbb{D}) = |\mathbb{D}|$.

Proof: $\mathrm{tri}(\mathbb{D}) \geq |\mathbb{D}| - \mathrm{Z}_o(\mathbb{D})$

- Let Z be an out zero forcing set.
- In $Y = \mathcal{Y}(\mathbb{D})$, delete the columns whose indices are in Z to obtain Y'.
- Compute $\mathrm{tri}(Y')$ by elimination:
  - Vertex v forcing vertex w means the v, w entry is the only nonzero entry of row v.
  - Delete row v and column w and add 1 to $\mathrm{tri}(Y')$.
- Thus $\mathrm{tri}(\mathbb{D}) \geq \mathrm{tri}(Y') = |\mathbb{D}| - \mathrm{Z}_o(\mathbb{D})$.

Example

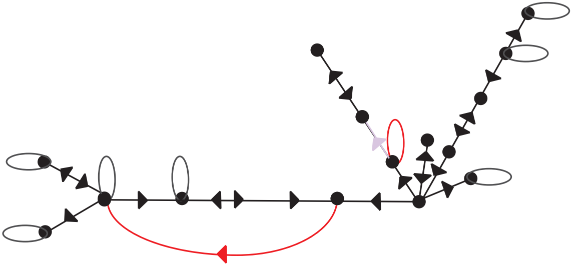
$$\begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ * & * & * \end{bmatrix} \to \begin{bmatrix} 0 & * \\ 0 & * \\ * & * \end{bmatrix} \to \begin{bmatrix} 0 \\ * \end{bmatrix} \quad \textit{so} \ \mathrm{tri}(\mathcal{Y}(\mathbb{D})) \geq 2.$$

Edit Distance to nonsingularity

Let Y be a square pattern and let $\mathbb{D}$ be a digraph.

- (row) edit distance to nonsingularity, $\mathrm{ED}(Y)$: the minimum number of rows that must be changed to obtain a pattern that requires nonsingularity.
- $\mathrm{ED}(\mathbb{D}) = \mathrm{ED}(\mathcal{Y}(\mathbb{D}))$.

Example



Theorem (AIM Square)

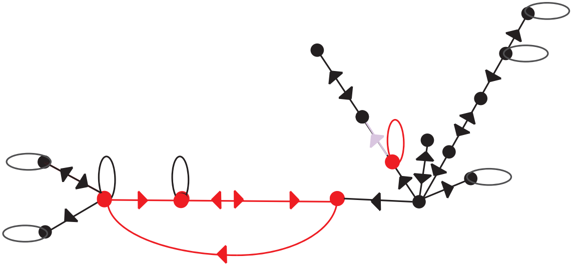
For any ditree $\mathbb{T}$, $\mathrm{ED}(\mathbb{T}) \leq \mathcal{P}(\mathbb{T})$.

Proof:

- Let $P = \{P_1, \ldots, P_k\}$ be a set of vertex-disjoint paths such that $\mathbb{T} - V_P$ requires nonsingularity
- Let v_i be the first vertex and w_i the last vertex of P_i.
- Edit row w_i (i.e., edit the out-neighborhood of w_i) so that the only out-neighbor of w_i is v_i.
- This involves k edits and produces a digraph $\mathbb{D}$.
- $\mathbb{D}$ requires nonsingularity, which implies $\mathrm{ED}(\mathbb{T}) \leq k = \mathcal{P}(\mathbb{T})$.

Example

$$\mathrm{ED}(\mathbb{T}) \le \mathcal{P}(\mathbb{T}) \le 2$$



Theorem (AIM Square)

$\text{tri}(\mathbb{D}) + \text{ED}(\mathbb{D}) = |\mathbb{D}|$.

Proof is similar to $\text{tri}(\mathbb{D}) + \text{Z}_o(\mathbb{D}) = |\mathbb{D}|$.

Corollary

For a ditree, $\text{ED}(\mathbb{T}) = \text{Z}_o(\mathbb{T}) = \mathcal{P}(\mathbb{T})$.

Proof: $\mathcal{P}(\mathbb{T}) \le \text{Z}_o(\mathbb{T}) = \text{ED}(\mathbb{T}) \le \mathcal{P}(\mathbb{T})$.

It remained to prove $M(\mathbb{T}) = \mathcal{P}(\mathbb{T})$.

The AIM Square gave a complicated argument that used the loop tree result and used the triangle number to reduce the problem for ditrees to the strong (symmetric) components.

Question

Is $\mathcal{P}(\mathbb{D}) \leq M(\mathbb{D})$ for every digraph?

If so (and if proved) this would provide an alternate proof of $M(\mathbb{T}) = \mathcal{P}(\mathbb{T})$:
$\mathcal{P}(\mathbb{T}) \leq M(\mathbb{T}) \leq \mathrm{Z}_o(\mathbb{T}) = \mathrm{ED}(\mathbb{T}) \leq \mathcal{P}(\mathbb{T})$.

Conversion of an asymmetric minimum rank problem to a symmetric minimum rank problem

Here Y need not be square.

Theorem (AIM Square)

Let Y be an $m \times n$ pattern Y such that every row and column of Y has a nonzero entry.

$\mathbb{D}_Y$ is the symmetric digraph having pattern $\begin{bmatrix} * & Y \\ Y^T & * \end{bmatrix}$, and G_Y is the underlying simple graph of $\mathbb{D}_Y$. Then

$$\mathrm{mr}(Y) = \mathrm{mr}(\mathbb{D}_Y) = \mathrm{mr}(G_Y).$$

Symmetric minimum rank problem

Minimum rank is characterized for:

- trees [Nylen 96], [Johnson, Leal-Duarte 99]
- unicyclic graphs [Barioli, Fallat, Hogben 05]
- all small graphs ($|G| \leq 7$) [ISU group]
- extreme minimum rank:
  $\text{mr}(G) = 0, 1, 2$: [Barrett, van der Holst, Loewy 04]
  $\text{mr}(G) = |G| - 1, |G| - 2$: [Fiedler 69],
  [Hogben, van der Holst 07], [Johnson, Loewy, Smith]
- many families of graphs [AIM 08]

Reduction techniques for

- cut-set of order 1 [Barioli, Fallat, Hogben 04] and order 2 [van der Holst 08]
- joins [Barioli, Fallat 06]

Bounds for minimum rank/maximum nullity:

- $M(G) \leq Z(G)$ [AIM 08]
- $\mu(G) \leq M(G)$ [Colin de Verdière 93]
  $\xi(G) \leq M(G)$ [Barioli, Fallat, Hogben 05]

Minimum rank graph catalogs

- Minimum rank of many families of graphs determined at the 06 AIM Workshop.
- On-line catalogs of minimum rank for small graphs and families developed.
- The ISU group determined the order of all graphs of order 7.

Minimum rank of families of graphs
http://aimath.org/pastworkshops/catalog2.html

Minimum rank of small of graphs
http://aimath.org/pastworkshops/catalog1.html

Related problems

- Could consider asymmetric/diagonal free matrices described by simple digraphs
- Could consider rectangular patterns (describing asymmetric matrices with diagonal constrained) - limited amount done
- Could consider other families of matrices such as positive semidefinite - some of the symmetric/diagonal free results have been extended to positive definite.
- Could consider matrices over other fields - many of the symmetric/diagonal free results been extended to other fields.

Thank You!