Average minimum rank of symmetric matrices described by a graph

Leslie Hogben

Iowa State University and American Institute of Mathematics

16th Conference of the International Linear Algebra Society
Pisa, Italy, June 22, 2010
Introduction

Matrices and Graphs
Minimum rank

Average minimum rank

Main results for average minimum rank
Tight concentration about the mean
Parameters of random graphs
Lower bound for expected minimum rank
An upper bound for expected minimum rank
All bounds
$S_n(\mathbb{R})$ is the set of $n \times n$ real symmetric matrices.

The graph $G(A) = (V, E)$ of $A \in S_n(\mathbb{R})$ is

- $V = \{1, \ldots, n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of $A$ is ignored.

**Example:**

$$A = \begin{bmatrix}
2 & -1 & 3 & 5 \\
-1 & 0 & 0 & 0 \\
3 & 0 & -3 & 0 \\
5 & 0 & 0 & 0
\end{bmatrix}$$
Minimum rank

The family of symmetric matrices described by a graph is

\[ S(G) = \{ A \in S_n(\mathbb{R}) : G(A) = G \} \].

The minimum rank of graph \( G \):

\[ \text{mr}(G) = \min_{A \in S(G)} \text{rank} A. \]

**Problem** Determine the minimum rank of a graph \( G \).
Examples:
Path: $\text{mr}(P_n) = n - 1$. Complete graph: $\text{mr}(K_n) = 1.$

$$A = \begin{bmatrix}
? & * & 0 & \ldots & 0 & 0 \\
* & ? & * & \ldots & 0 & 0 \\
0 & * & ? & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & ? & * \\
0 & 0 & 0 & \ldots & * & ?
\end{bmatrix}$$

$$B = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix}$$

* is nonzero, ? is indefinite
Basic properties of minimum rank

- It is easy to obtain a matrix $A \in \mathcal{S}(G)$ with $\text{rank } A = |G| - 1$ (translate).
- If $G$ is the disjoint union of graphs $G_i$ then
  
  $$\text{mr}(G) = \sum \text{mr}(G_i)$$

- Only connected graphs are studied.
- If $G$ is connected,
  
  - $\text{mr}(G) = 0$ iff $G$ is a single vertex.
  - $\text{mr}(G) = 1$ iff $G = K_n, \ n \geq 2$.

- $\text{mr}(G) = |G| - 1$ if and only if $G$ is a path. [Fiedler 69]
Results & techniques

Through the work of many people:
Minimum rank is known for

- Trees
- Unicyclic graphs
- $|G| \leq 7$
- $\text{mr}(G) \leq 2$
- $\text{mr}(G) \geq |G| - 2$

Techniques for computing minimum rank

- cut-vertex reduction
- join reduction
- Colin de Verdière parameters
- zero forcing
Joint work with Tracy Hall, Ryan Martin, Bryan Shader.

- **average minimum rank** of graphs of order $n$: sum over all labeled graphs of order $n$ of minimum ranks of graphs, divided by number of graphs of order $n$,

$$\text{avemr}(n) = \frac{\sum_{|G|=n} \text{mr}(G)}{2^{\binom{n}{2}}}.$$

- $G(n, p)$ is the Erdős-Rényi random graph on $n$ vertices with edge probability $p$.

- $\mathbb{E}[\text{mr}(G(n, p))]$ is the expected value of minimum rank

- $\text{avemr}(n) = \mathbb{E}[\text{mr}(G(n, 1/2))]$
Main results for average minimum rank

Theorem
For $n$ sufficiently large,

$$0.146907n < \text{avemr}(n) < 0.5n + \sqrt{7n \ln n}.$$  

Theorem
With probability approaching 1 as $n \to \infty$,

$$|\text{mr}(G(n, 1/2)) - \text{avemr}(n)| < \sqrt{n \ln \ln n}.$$
Expected value of minimum rank is tightly concentrated about the mean

**Theorem (Alon, Spencer 2000)**

Let $p \in (0, 1)$.

Let $f$ be a graph invariant such that for any graphs $G$ and $H$, if $x \in V(G) = V(H)$ and $G - x = H - x$, then $|f(G) - f(H)| \leq 1$.

Let $\mu = \mathbb{E}[f(G(n, p))]$.

Then, for any $\beta > 0$,

$$\Pr[|f(G(n, p)) - \mu| > \beta \sqrt{n - 1}] < 2e^{-\beta^2/2}.$$ 

Proof uses Azuma’s inequality for martingales and the vertex exposure martingale.
Corollary

Let \( p \in (0, 1) \) be fixed.

\[
\Pr[|\mr(G(n, p)) - \mathbf{E}[\mr(G(n, p))]| > 2\sqrt{n \ln \ln n}] < 2 \left( \frac{1}{\ln n} \right)^{1/8}.
\]
The following two results are well-known, and can be derived from the Chernoff-Hoeffding bound.

\( e(G) \) denotes the number of edges of \( G \).

**Theorem**

*Let \( p \) be fixed and let \( G \) be distributed according to \( G(n, p) \). Then,

\[
e(G) \leq p \binom{n}{2} + n\sqrt{2 \ln n},
\]

with probability at least \( 1 - n^{-2} \).

\[
e(G) \geq p \binom{n}{2} - n\sqrt{2 \ln n}
\]

with probability at least \( 1 - n^{-2} \).*
\( \delta(G) \) denotes the minimum degree and \( \Delta(G) \) denotes the maximum degree of \( G \).

**Theorem**

Let \( p \) be fixed and let \( G \) be distributed according to \( G(n, p) \). Then,

\[
\begin{align*}
 pn - \sqrt{6n \ln n} & \leq \delta(G) \leq \Delta(G) \leq pn + \sqrt{6n \ln n} \\
\end{align*}
\]

with probability at least \( 1 - 2n^{-2} \).
Lower bound for expected minimum rank

- **zero-pattern** $\zeta(x)$ of the real vector $x = (x_1, \ldots, x_\ell)$: $(0, \ast)$-vector obtained from $x$ by replacing its nonzero entries by $\ast$.

- **support** of $z = (z_1, \ldots, z_\ell)$:
  
  $S(z) = \{i : z_i \neq 0\}$.

The following result is a variant of a theorem of [Rónyai, Babai, Ganapathy 01]

**Theorem**

If $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is an $m$-tuple of polynomials in $k$ variables over a field $F$ with $m \geq k$, each $f_i$ of degree at most $d$, then the number of zero-patterns $z = \zeta(f(x))$ with $|S(z)| \leq s$ is at most

$$\binom{k + sd}{k}.$$
Every real symmetric $n \times n$ matrix of rank at most $r$ can be expressed in the form

$$X^T D_i X$$

for some $i$ such that $0 \leq i \leq r$, where

- $D_i = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ is an $r \times r$ diagonal matrix with $i$ diagonal entries equal to 1 and $r - i$ equal to $-1$ and
- $X$ is an $r \times n$ real matrix.

Let each entry of $X$ be a variable.

The total number of variables is $rn$.

Each entry of the matrix $X^T D_i X$ is a polynomial of degree at most 2.
For a fixed value of $p (0 < p < 1)$, let $c(p)$ be the solution to

$$
\frac{(c + p)^{2c+2p}}{(c)^{2c} (p)^{2p}} p^p (1 - p)^{(1-p)} = 1.
$$

The graph of $c(p)$
Theorem
Let $G$ be distributed according to $G(n, p)$ for a fixed $p$, $0 < p < 1$. For any $c < c(p)$,

$$E[\text{mr}(G)] > cn$$

for $n$ sufficiently large.

Corollary
For $n$ sufficiently large, the average minimum rank over all labeled graphs of order $n$ satisfies

$$\text{avemr}(n) > 0.146907n.$$
An upper bound for expected minimum rank

- Recall delta conjecture: \( \text{mr}(G) \leq |G| - \delta(G) \).
- Could use this and known bound on \( \delta(G) \) to bound \( \text{mr}(G) \) above.
- Replace delta conjecture by analogous result for vertex connectivity:
  \( \kappa(G) = \) the smallest number \( k \) such that there is a set of vertices \( S \), with \( |S| = k \), for which \( G - S \) is disconnected (\( G \leq K_n \)).
  By convention, \( \kappa(K_n) = n - 1 \).

- **orthogonal representation of** $G$ **of dimension** $d$:
  $\varphi : V(G) \rightarrow \mathbb{R}^d$ such that if $v \not\sim w$, then $\varphi(v)$ and $\varphi(w)$ are orthogonal.

- Zero representation is orthogonal.

- **faithful orthogonal representation of** $G$ **of dimension** $d$:
  $\varphi : V(G) \rightarrow \mathbb{R}^d$ such that $v \not\sim w$ if and only if $\varphi(v)$ and $\varphi(w)$ are orthogonal.

- Every faithful orthogonal representation of dimension $d$ gives rise to a positive semidefinite matrix of rank $d$ and vice versa.
Theorem (Lovász, Saks, Schrijver 89)

A graph $G$ has a faithful orthogonal representation of dimension $|G| - \kappa(G)$.

Corollary

$mr(G) \leq |G| - \kappa(G)$. 
Theorem

Let $G$ be distributed according to $G(n, p)$.

- For $n$ sufficiently large, the expected value of minimum rank satisfies $\mathbb{E}[\text{mr}(G)] \leq (1 - p)n + \sqrt{7n \ln n}$.

- For $n$ sufficiently large, the average minimum rank over all labeled graphs of order $n$ satisfies

$$\text{avemr}(n) \leq 0.5n + \sqrt{7n \ln n}.$$
The proof uses

- $\text{mr}(G) \leq |G| - \kappa(G)$,
- with probability at least $1 - n^{-2}$

$$e(G(n, p)) \leq p\left(\binom{n}{2}\right) + n\sqrt{2 \ln n},$$

- $\delta \leq \frac{2e(G)}{n}$,
- and the relationship (on average) between the connectivity $\kappa(G)$ and the minimum degree $\delta(G)$. 
Bollobás, and Bollobás and Thomason proved that for $G \sim G(n, p)$, regardless of $p$, then $\Pr[\kappa(G) < \delta(G)] \to 0$ as $n \to \infty$.

The following simpler result is obtained by similar means.

**Lemma**

Let $p \in (0, 1)$ be fixed and $G$ be distributed according to $G(n, p)$. If $n$ is sufficiently large, then

$$\Pr[\kappa(G) < \delta(G)] \leq 3n^{-2}.$$
The bounds for $\frac{E[\text{mr}(G(n,p))]}{n}$ as a function of $p$. 
Thank you!