Average minimum rank of symmetric matrices described by a graph

Leslie Hogben

Iowa State University and
American Institute of Mathematics

16^{th} Conference of the International Linear Algebra Society
Pisa, Italy, June 22,2010

Matrices and Graphs

$S_n(\mathbb{R})$ is the set of $n \times n$ real symmetric matrices.

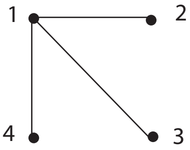
The *graph* $\mathcal{G}(A) = (V, E)$ of $A \in S_n(\mathbb{R})$ is

- $V = \{1, \ldots, n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of A is ignored.

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{G}(A)$



Minimum rank

The family of symmetric matrices described by a graph is

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}.$$

The minimum rank of graph G:

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank}\, A.$$

Problem Determine the minimum rank of a graph G.

Examples:

Path: $\text{mr}(P_n) = n - 1$. Complete graph: $\text{mr}(K_n) = 1$.

$$A = \begin{bmatrix} ? & * & 0 & \dots & 0 & 0 \\ * & ? & * & \dots & 0 & 0 \\ 0 & * & ? & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & ? & * \\ 0 & 0 & 0 & \dots & * & ? \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$*$ is nonzero, ? is indefinite

Basic properties of minimum rank

- It is easy to obtain a matrix $A \in \mathcal{S}(G)$ with $\operatorname{rank} A = |G| - 1$ (translate).
- If G is the disjoint union of graphs G_i then

$$\operatorname{mr}(G) = \sum \operatorname{mr}(G_i)$$

- Only connected graphs are studied.
- If G is connected,
  - $\operatorname{mr}(G) = 0$ iff G is single vertex.
  - $\operatorname{mr}(G) = 1$ iff $G = K_n,\ n \geq 2$.
- $\operatorname{mr}(G) = |G| - 1$ if and only if G is a path. [Fiedler 69]

Results & techniques

Through the work of many people:
Minimum rank is known for

- Trees
- Unicyclic graphs
- $|G| \leq 7$
- $\text{mr}(G) \leq 2$
- $\text{mr}(G) \geq |G| - 2$

Techniques for computing minimum rank

- cut-vertex reduction
- join reduction
- Colin de Verdière parameters
- zero forcing

Expected value and average minimum rank

Joint work with Tracy Hall, Ryan Martin, Bryan Shader.

- average minimum rank of graphs of order n: sum over all labeled graphs of order n of minimum ranks of graphs, divided by number of graphs of order n,

$$\operatorname{avemr}(n) = \frac{\sum_{|G|=n} \operatorname{mr}(G)}{2^{\binom{n}{2}}}.$$

- $G(n,p)$ is the Erdős-Rényi random graph on n vertices with edge probability p.
- $\mathbf{E}[\operatorname{mr}(G(n,p))]$ is the expected value of minimum rank
- $\operatorname{avemr}(n) = \mathbf{E}[\operatorname{mr}(G(n,1/2))]$

Main results for average minimum rank

Theorem

For n sufficiently large,

$$0.146907n < \operatorname{avemr}(n) < 0.5n + \sqrt{7n \ln n}.$$

Theorem

With probability approaching 1 as $n \to \infty$,

$$|\operatorname{mr}(G(n, 1/2)) - \operatorname{avemr}(n)| < \sqrt{n \ln \ln n}.$$

Expected value of minimum rank is tightly concentrated about the mean

Theorem (Alon, Spencer 2000)

Let $p \in (0,1)$.
Let f *be a graph invariant such that for any graphs* G *and* H, *if* $x \in V(G) = V(H)$ *and* $G - x = H - x$, *then* $|f(G) - f(H)| \leq 1$.
Let $\mu = \mathbf{E}[f(G(n,p))]$.
Then, for any $\beta > 0$,

$$\Pr[|f(G(n,p)) - \mu| > \beta\sqrt{n-1}] < 2e^{-\beta^2/2}.$$

Proof uses Azuma's inequality for martingales and the vertex exposure martingale.

Corollary

Let $p \in (0,1)$ *be fixed.*

$$\Pr[|\operatorname{mr}(G(n,p)) - \mathbf{E}[\operatorname{mr}(G(n,p))]| > 2\sqrt{n \ln \ln n}]$$

$$< 2\left(\frac{1}{\ln n}\right)^{1/8}.$$

The following two results are well-known, and can be derived from the Chernoff-Hoeffding bound.

$e(G)$ denotes the number of edges of G.

Theorem

Let p be fixed and let G be distributed according to $G(n, p)$. Then,

$$e(G) \le p\binom{n}{2} + n\sqrt{2\ln n},$$

with probability at least $1 - n^{-2}$.

$$e(G) \ge p\binom{n}{2} - n\sqrt{2\ln n}$$

with probability at least $1 - n^{-2}$.

$\delta(G)$ denotes the minimum degree and $\Delta(G)$ denotes the maximum degree of G.

Theorem

Let p be fixed and let G be distributed according to $G(n, p)$. Then,

$$pn - \sqrt{6n \ln n} \leq \delta(G) \leq \Delta(G) \leq pn + \sqrt{6n \ln n}$$

with probability at least $1 - 2n^{-2}$.

Lower bound for expected minimum rank

- zero-pattern $\zeta(\mathbf{x})$ of the real vector $\mathbf{x} = (x_1, \ldots, x_\ell)$: $(0, *)$-vector obtained from $\mathbf{x}$ by replacing its nonzero entries by $*$.
- support of $\mathbf{z} = (z_1, \ldots, z_\ell)$: $S(\mathbf{z}) = \{i : z_i \neq 0\}$.

The following result is a variant of a theorem of [Rónyai, Babai, Ganapathy 01]

Theorem

If $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is an m-tuple of polynomials in k variables over a field F with $m \geq k$, each f_i of degree at most d, then the number of zero-patterns $\mathbf{z} = \zeta(\mathbf{f}(\mathbf{x}))$ with $|S(\mathbf{z})| \leq s$ is at most

$$\binom{k+sd}{k}.$$

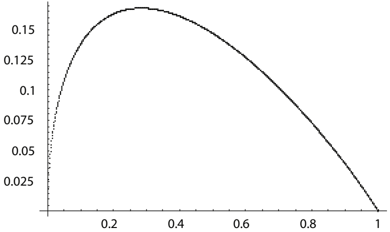
- Every real symmetric $n \times n$ matrix of rank at most r can be expressed in the form

$$X^T D_i X$$

  for some i such that $0 \leq i \leq r$, where
  - $D_i = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ is an $r \times r$ diagonal matrix with i diagonal entries equal to 1 and $r - i$ equal to -1 and
  - X is an $r \times n$ real matrix.
- Let each entry of X be a variable.
- The total number of variables is rn.
- Each entry of the matrix $X^T D_i X$ is a polynomial of degree at most 2.

For a fixed value of $p (0 < p < 1)$, let $c(p)$ be the solution to

$$\frac{(c+p)^{2c+2p}}{(c)^{2c}(p)^{2p}} p^p (1-p)^{(1-p)} = 1.$$



The graph of $c(p)$

Theorem

Let G be distributed according to $G(n,p)$ for a fixed p, $0 < p < 1$. For any $c < c(p)$,

$$\mathbf{E}[\text{mr}(G)] > cn$$

for n sufficiently large.

Corollary

For n sufficiently large, the average minimum rank over all labeled graphs of order n satisfies

$$\text{avemr}(n) > 0.146907n.$$

An upper bound for expected minimum rank

- Recall delta conjecture: $\mathrm{mr}(G) \leq |G| - \delta(G)$.
- Could use this and known bound on $\delta(G)$ to bound $\mathrm{mr}(G)$ above.
- Replace delta conjecture by analogous result for vertex connectivity:
  $\kappa(G)$ = the smallest number k such that there is a set of vertices S, with $|S| = k$, for which $G - S$ is disconnected ($G \leq K_n$).
  By convention, $\kappa(K_n) = n - 1$.

Terminology taken from [Lovász, Saks, Schrijver 89] – nonstandard for minimum rank problems.

- orthogonal representation of G of dimension d: $\varphi : V(G) \to \mathbb{R}^d$ such that if $v \not\sim w$, then $\varphi(v)$ and $\varphi(w)$ are orthogonal.
- Zero representation is orthogonal.
- faithful orthogonal representation of G of dimension d: $\varphi : V(G) \to \mathbb{R}^d$ such that $v \not\sim w$ if and only if $\varphi(v)$ and $\varphi(w)$ are orthogonal.
- Every faithful orthogonal representation of dimension d gives rise to a positive semidefinite matrix of rank d and vice versa.

Theorem (Lovász, Saks, Schrijver 89)

A graph G has a faithful orthogonal representation of dimension $|G| - \kappa(G)$.

Corollary

$\mathrm{mr}(G) \leq |G| - \kappa(G)$.

Theorem

Let G be distributed according to $G(n, p)$.

- *For n sufficiently large, the expected value of minimum rank satisfies* $\mathbf{E}[\mathrm{mr}(G)] \leq (1-p)n + \sqrt{7n \ln n}$.
- *For n sufficiently large, the average minimum rank over all labeled graphs of order n satisfies*

$$\mathrm{avemr}(n) \leq 0.5n + \sqrt{7n \ln n}.$$

The proof uses

- $\mathrm{mr}(G) \leq |G| - \kappa(G)$,
- with probability at least $1 - n^{-2}$

$$e(G(n,p)) \leq p\binom{n}{2} + n\sqrt{2\ln n},$$

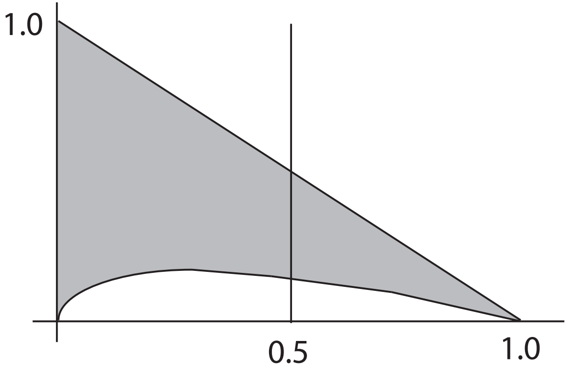
- $\delta \leq \frac{2e(G)}{n}$,
- and the relationship (on average) between the connectivity $\kappa(G)$ and the minimum degree $\delta(G)$.

Bollobás, and Bollobás and Thomason proved that for $G \sim G(n,p)$, regardless of p, then $\Pr[\kappa(G) < \delta(G)] \to 0$ as $n \to \infty$.
The following simpler result is obtained by similar means.

Lemma

Let $p \in (0,1)$ be fixed and G be distributed according to $G(n,p)$. If n is sufficiently large, then

$$\Pr[\kappa(G) < \delta(G)] \leq 3n^{-2}.$$



The bounds for $\frac{\mathbf{E}[\mathrm{mr}(G(n,p))]}{n}$ as a function of p

Thank you!