

# Maximum generic nullity of a graph\*

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## Abstract

For a graph  $G$  of order  $n$ , the maximum nullity of  $G$  is defined to be the largest possible nullity over all real symmetric  $n \times n$  matrices  $A$  whose  $(i, j)$ th entry (for  $i \neq j$ ) is nonzero whenever  $\{i, j\}$  is an edge in  $G$  and is zero otherwise. Maximum nullity and the related parameter minimum rank of the same set of matrices have been studied extensively. A new parameter, maximum generic nullity, is introduced. Maximum generic nullity provides insight into the structure of the null-space of a matrix realizing maximum nullity of a graph. It is shown that maximum generic nullity is bounded above by edge connectivity and below by vertex connectivity. Results on random graphs are used to show that as  $n$  goes to infinity almost all graphs have equal maximum generic nullity, vertex connectivity, edge connectivity, and minimum degree.

**Keywords:** minimum rank, maximum nullity, maximum corank, maximum generic nullity, graph, rank, nullity, corank, symmetric matrix, orthogonal representation.

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## 1 Introduction

The (real symmetric) minimum rank problem for a simple graph asks us to determine the minimum rank among real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph  $G$ , or equivalently to determine the maximum nullity (or maximum multiplicity of an eigenvalue) among the same family of matrices.

All graphs discussed in this paper are simple, meaning no loops or multiple edges, undirected, finite, and have nonempty vertex sets. The *order* of a graph  $G$ , denoted  $|G|$ , is the number of vertices of  $G$ . The set of  $n \times n$  real symmetric matrices will be denoted by  $S_n$ . For  $A \in S_n$ , the *graph* of  $A$ , denoted  $\mathcal{G}(A)$ , is the graph with vertices  $\{1, \dots, n\}$  and edges  $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$ . Note that the diagonal of  $A$  is ignored in determining  $\mathcal{G}(A)$ . The set of real symmetric matrices of a graph  $G$  is  $\mathcal{S}(G) = \{A \in S_n : \mathcal{G}(A) = G\}$ . The *minimum rank* of a graph  $G$  is

$$\text{mr}(G) = \min\{\text{rank}(A) : A \in \mathcal{S}(G)\}.$$

The *maximum nullity* of  $G$  is

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}$$

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where  $\text{null}(A)$  is the dimension of the null space,  $\ker(A)$ , of  $A$ . Clearly  $M(G) + \text{mr}(G) = |G|$ . If  $A \in \mathcal{S}(G)$  and  $\alpha \in \mathbb{R}$ , then  $A + \alpha I \in \mathcal{S}(G)$ , so the maximum multiplicity of any eigenvalue is the same as maximum multiplicity of eigenvalue 0, i.e., the maximum nullity. See [4] for a survey of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there.

If  $W, U \subseteq \{1, 2, \dots, n\}$  and  $B \in S_n$ , then  $B[W, U]$  denotes the *submatrix* of  $B$  having rows indexed by  $W$  and columns indexed by  $U$ . In case  $W = U$ , this is a *principal submatrix* and is denoted by  $B[W]$ ; the complementary principal submatrix obtained from  $B$  by deleting the rows and columns indexed by  $W$  is denoted  $B(W)$ . In the special case when  $W = \{k\}$ , we use  $B(k)$  to denote  $B(W)$ .

A graph  $G' = (V', E')$  is a *subgraph* of graph  $G = (V, E)$  if  $V' \subseteq V, E' \subseteq E$ . The subgraph  $G[W]$  of  $G = (V, E)$  *induced* by  $W \subseteq V$  is the subgraph with vertex set  $W$  and edge set  $\{\{i, j\} \in E \mid i, j \in W\}$ ;  $G(W)$  is used to denote  $G[V \setminus W]$ , obtained from  $G$  by deleting all the vertices in  $W$  and all edges incident with these vertices. This follows the notational convention in the minimum rank literature. In graph theory  $G(W)$  is usually denoted by  $G - W$ . If  $S \subseteq E$ , the subgraph  $G - S$  is the subgraph obtained by deleting the edges in  $S$ , i.e., the graph with vertex set  $V$  and edge set  $E \setminus S$ . A path on  $n$  vertices, a cycle on  $n$  vertices, and a complete graph on  $n$  vertices will be denoted by  $P_n, C_n$ , and  $K_n$ , respectively.

A graph is *connected* if there is a path from any vertex to any other vertex. A *component* of a graph is a maximal connected subgraph. A set  $W$  of vertices of  $G$  is a *separating set* or *vertex cut* if  $G(W)$  has more than one component. The *vertex connectivity* of  $G$ , denoted  $\kappa_v(G)$  is the minimum size of a separating set of  $G$ . A set  $S$  of edges of a graph  $G$  (with  $|G| > 1$ ) is a *disconnecting set* if  $G - S$  has more than one component. The *edge connectivity* of  $G$ , denoted  $\kappa_e(G)$  is the minimum size of a disconnecting set of  $G$ . Given  $W, U \subset V(G)$ , the set of edges of  $G$  having one endpoint in  $W$  and the other in  $U$  is denoted  $[W, U]$ . An *edge cut* is a set of edges of the form  $[W, V(G) \setminus W]$  for some  $W \subset V(G)$ . Every edge cut is a disconnecting set but not every disconnecting set is an edge cut. However, a minimum disconnecting set (i.e., a subset  $S$  of edges such that  $G - S$  is disconnected and  $|S| = \kappa_e(G)$ ) is an edge cut (cf. [11, p. 152]).

The *degree* of a vertex is the number of edges incident with the vertex, and the *minimum degree* over all vertices of a graph  $G$  will be denoted by  $\delta(G)$ . It is known that

$$\kappa_v(G) \leq \kappa_e(G) \leq \delta(G).$$

and these inequalities can be strict (e.g., see [11, pp. 152–153]).

A  $n \times k$  real matrix  $X$  is *generic* if every square submatrix of  $X$  is nonsingular. A generic matrix could be called a *totally nonsingular* in analogy with the definition of a *totally positive* matrix as a matrix all of whose minors are positive. Clearly a totally positive matrix is generic. Notice that any submatrix of a generic matrix is generic. The *generic nullity* of a nonzero matrix  $A \in \mathbb{R}^{n \times n}$  is

$$GN(A) = \max\{k : X \in \mathbb{R}^{n \times k}, AX = 0, \text{ and } X \text{ is generic}\}$$

(the generic nullity of an  $n \times n$  zero matrix is  $n$ ). The *maximum generic nullity* of a graph  $G$  is

$$GM(G) = \max\{GN(A) : A \in \mathcal{S}(G)\}.$$

The maximum generic nullity of a graph can be strictly less than the maximum nullity. In this case, the null space of a matrix of maximum nullity is often highly structured, as in Example 1.1 below.

**Example 1.1.** Let  $G = G130$  be the graph shown in Figure 1 (the numbering of graphs is taken from [10]). Since  $G$  can be covered by two copies of  $K_3$  and one  $K_2$ ,  $\text{mr}(G) \leq 3$  and since  $G$  has

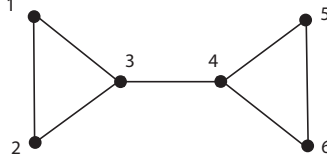


Figure 1: The graph  $G = G130$  in Example 1.1

an induced  $P_4$ ,  $\text{mr}(G) \geq 3$ . Thus  $M(G) = 6 - 3 = 3$ .

We assume there is a generic  $6 \times 2$  matrix  $X = [x_1 \ x_2]$  whose columns  $x_i$  are in the nullspace of  $A \in \mathcal{S}(G)$  and derive a contradiction, thus showing that  $GM(G) = 1$ . The nonzero pattern of  $A \in \mathcal{S}(G)$  is

$$\begin{bmatrix} ? & * & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 & 0 \\ * & * & ? & * & 0 & 0 \\ 0 & 0 & * & ? & * & * \\ 0 & 0 & 0 & * & ? & * \\ 0 & 0 & 0 & * & * & ? \end{bmatrix},$$

where  $*$  denotes a nonzero entry and  $?$  denotes an entry about which nothing is known. Columns 2, 3 and 4 (and columns 3, 4, and 5) are clearly independent.

If  $\text{rank}(A) = 3$  then the first two columns are linearly dependent and the last two columns are linearly dependent. So there are nonzero vectors in the null space of  $A$  of the forms  $y = [*, *, 0, 0, 0, 0]^T$  and  $z = [0, 0, 0, 0, *, *]^T$ . Suppose that  $ax_1 + bx_2 + cy + dz = 0$ . Then

$$X[\{3, 4\}, \{1, 2\}] \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since  $X$  is generic,  $a = b = 0$ , and it follows that  $c = d = 0$ .

If  $\text{rank}(A) = 4$  then we claim that either the first two columns are linearly dependent or the last two columns are linearly dependent; assume the first case. Then there is a nontrivial linear combination of the first two columns which is equal to zero, and hence a nonzero vector in the null space having all the last four entries 0. This vector is independent of  $x_1$  and  $x_2$ , and again a contradiction is obtained. To establish the claim, note first that if the first four columns of  $A$  are dependent, then the first two columns are necessarily dependent. If the first four columns are independent, then the first four rows are also independent (since  $A$  is symmetric). In this case, in order to have  $\text{rank}(A) = 4$ , the last two rows must be in the span of the first four, forcing the last two columns to be dependent. This establishes the claim and completes the argument.

Our main result about maximum generic nullity is that for every connected graph  $G$ ,

$$GM(G) \leq \kappa_e(G).$$

This will be established in Section 2 using methods based on the ideas in Example 1.1. Using the methods of [1], it is easy to show that  $GM(G) \leq \delta(G)$ , but we do not include that proof since  $\kappa_e(G) \leq \delta(G)$ . In Section 4 it is shown that for every graph  $G$ ,

$$\kappa_v(G) \leq GM(G),$$

and graph theoretic results are used to show that as  $n$  goes to infinity almost all graphs have equal maximum generic nullity, vertex connectivity, edge connectivity, and minimum degree.

## 2 Maximum generic nullity and edge connectivity

A *nonzero pattern*  $C = [c_{ij}]$  is a  $m \times n$  matrix whose entries  $c_{ij}$  are elements of  $\{*, 0\}$ . The number of  $*$  (nonzero entries) in  $C$  is denoted by  $\text{nz}(C)$ . Given a pattern  $C = [c_{ij}]$ , we let  $\mathcal{Q}(C)$  denote the set of all matrices  $A = [a_{ij}] \in \mathbb{R}^{m \times n}$  such that  $a_{ij} \neq 0$  if and only if  $c_{ij} = *$ . Note that (unlike the set of symmetric matrices described by a graph), here the diagonal is constrained by the nonzero pattern. The *minimum rank* of a nonzero pattern  $C$  is

$$\text{mr}(C) = \min\{\text{rank}(A) : A \in \mathcal{Q}(C)\}.$$

**Theorem 2.1.** *If  $C$  is an  $m \times n$  nonzero pattern that does not have any zero row or zero column,  $\text{mr}(C) \geq m + n - \text{nz}(C)$ .*

*Proof.* Note that arbitrary permutation of rows or columns of  $C$  does not affect  $\text{mr}(C)$ . For fixed  $m$  and  $n$ , the proof is by induction on  $\text{nz}(C)$ . The base case is any  $C$  (without zero row or column) such that for every nonzero entry, it is the only nonzero in its row or the only nonzero in its column.

That is, no row and column permutation of  $C$  contains a  $2 \times 2$  submatrix  $\begin{bmatrix} * & ? \\ * & * \end{bmatrix}$ . By row and column permutations, any such a  $C$  can be put into the following form:

$$\begin{array}{c} a_1 \left\{ \begin{array}{cccccccccccc} * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right. \\ a_2 \left\{ \begin{array}{cccccccccccc} * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & * & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{array} \right. \\ \vdots \\ a_s \left\{ \begin{array}{cccccccccccc} 0 & 0 & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & * & \dots & * & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & * & \dots & * \end{array} \right. \\ \underbrace{\hspace{10em}}_{b_1} \quad \underbrace{\hspace{10em}}_{b_2} \quad \underbrace{\hspace{10em}}_{b_t} \end{array}$$

$$\begin{aligned} \text{nz}(C) &= a_1 + \dots + a_s + b_1 + \dots + b_t \\ m &= a_1 + \dots + a_s + t \\ n &= s + b_1 + \dots + b_t \\ m + n - \text{nz}(C) &= t + s \\ &= \text{mr}(C) \end{aligned}$$

Now assume  $C$  contains a  $2 \times 2$  submatrix  $\begin{bmatrix} * & ? \\ * & * \end{bmatrix}$ . Consider the nonzero pattern  $C'$  obtained from  $C$  by replacing one  $*$  by 0 so the  $2 \times 2$  submatrix is now  $\begin{bmatrix} * & ? \\ 0 & * \end{bmatrix}$ . Then by the induction

hypothesis applied to  $C'$ ,

$$\text{mr}(C) \geq \text{mr}(C') - 1 \geq m + n - \text{nz}(C') - 1 = m + n - (\text{nz}(C) - 1) - 1 = m + n - \text{nz}(C). \quad \square$$

**Theorem 2.2.** *If  $G$  is connected, then  $\text{GM}(G) \leq \kappa_e(G)$ .*

*Proof.* Let  $S$  be a minimum disconnecting set for  $G$  with  $|S| \geq 1$  (so  $\kappa_e(G) = |S|$ ). Since  $S$  is an edge cut,  $S = [W, \overline{W}]$  for some  $W \subset V$ . Let  $W_1 = W$  and  $W_2 = \overline{W}$ . Number the vertices of  $G$  so that the vertices of  $W_1$  are  $1, \dots, |W|_1$ , all vertices of  $W_1$  incident with an edge of  $S$  are last among the vertices of  $W_1$ , and all vertices of  $W_2$  incident with an edge of  $S$  are first among the vertices of  $W_2$ .

Let  $A \in \mathcal{S}(G)$  be such that  $\text{GN}(A) = \text{GM}(G)$ . Let  $A_i = A[W_i]$ . Then  $A$  can be partitioned as

$$A = \left[ \begin{array}{c|c|c} \widehat{A}_1 & 0 & 0 \\ \hline D & C & 0 \\ \hline F & E & \widehat{A}_2 \end{array} \right]$$

where  $A_1 = \begin{bmatrix} \widehat{A}_1 \\ D \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} E & \widehat{A}_2 \end{bmatrix}$ ,  $C$  is  $d \times e$ ,  $\widehat{A}_1$  is  $(n_1 - d) \times n_1$  and  $\widehat{A}_2$  is  $n_2 \times (n_2 - e)$ . Note that  $\widehat{A}_1$  or  $\widehat{A}_2$  may be empty. Let  $r_i = \text{rank}(\widehat{A}_i)$ . Then

$$\begin{aligned} \text{rank}(A) &\geq r_1 + \text{mr}(C) + r_2 \\ &\geq r_1 + r_2 + d + e - \text{nz}(C) \\ &= r_1 + r_2 + d + e - \kappa_e(G). \end{aligned}$$

Now consider the vectors that must be in  $\ker(A)$ . Since  $\text{rank}(\widehat{A}_2) = r_2$ , there exist  $k_2 = n_2 - e - r_2$  independent vectors  $\hat{y}_i \in \mathbb{R}^{n_2 - e}$  such that  $\widehat{A}_2 \hat{y}_i = 0$ . If we let  $y_i = \begin{bmatrix} 0 \\ \hat{y}_i \end{bmatrix}$  (where the first zero vector is of length  $n_1 - d$  and the second is of length  $d + e$ ), then  $y_i \in \ker(A)$ ,  $i = 1, \dots, k_2$ . Since  $\text{rank}(\widehat{A}_1) = r_1$ , there exist  $k_1 = n_1 - d - r_1$  independent vectors  $\hat{x}_i \in \mathbb{R}^{n_1 - d}$  such that  $\hat{x}_i^T \widehat{A}_1 = 0$ . Since  $A$  is symmetric, if we let  $x_i = \begin{bmatrix} \hat{x}_i \\ 0 \end{bmatrix}$  (where the first zero vector is of length  $d + e$  and the second is of length  $n_2 - e$ ), then  $x_i \in \ker(A)$ ,  $i = 1, \dots, k_1$ .

Let  $n = n_1 + n_2$  be the number of vertices of  $G$ . Extend  $\{x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}\}$  to a basis  $\{x_1, \dots, x_{k_1}, y_1, \dots, y_{k_2}, z_1, \dots, z_k\}$  for  $\ker(A)$ . Then

$$\text{null}(A) = k_1 + k_2 + k = n - d - e - r_1 - r_2 + k.$$

Adding this to the inequality  $\text{rank}(A) \geq r_1 + r_2 + \text{mr}(C)$  gives

$$n \geq n - d - e + k + \text{mr}(C).$$

Since  $\text{mr}(C) \geq 1$ , we conclude that  $k \leq d + e - 1$ .

Let  $g = \text{GM}(G)$  and let  $X$  be a generic  $n \times g$  matrix in  $\ker(A)$ . Then

$$X = \begin{bmatrix} x_1 & \cdots & x_{k_1} & y_1 & \cdots & y_{k_2} & z_1 & \cdots & z_k \end{bmatrix} R$$

for some  $n \times g$  matrix  $R$ . Let  $\widehat{X}$  be the matrix obtained by deleting the first  $n_1 - d$  rows and the last  $n_2 - e$  rows of  $X$  and define  $\hat{z}_i$  ( $i = 1, \dots, k$ ), to be the vectors obtained by deleting the first  $n_1 - d$  and the last  $n_2 - e$  entries of  $z_i$ . Then  $\widehat{X} = [0 \cdots 0 \hat{z}_1 \cdots \hat{z}_k] R$ . Since  $\widehat{X}$  is a generic  $(d + e) \times g$  matrix,  $\min\{d + e, g\} = \text{rank } \widehat{X} \leq \text{rank}[\hat{z}_1 \cdots \hat{z}_k] = k$ . Since  $d + e > k$ ,  $g \leq k$ . Then  $\text{null}(A) = n - d - e - r_1 - r_2 + k \geq n - d - e - r_1 - r_2 + g$ . Adding this to  $\text{rank } A \geq r_1 + r_2 + d + e - \kappa_e(G)$ , we have  $n \geq n + g - \kappa_e(G)$ , and  $\text{GM}(G) = g \leq \kappa_e(G)$ .  $\square$

It is possible to have  $GM(G) < \kappa_e(G)$ , as the next example shows.

**Example 2.3.** The graph  $H$  shown in Figure 2, has  $GM(G) = 2 < 3 = \kappa_e(G)$ . We assume there

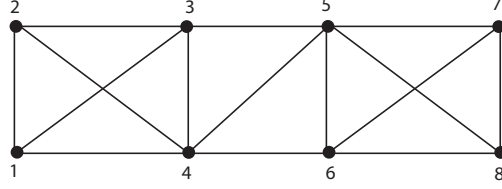


Figure 2: The graph  $H$  in Example 2.3

is a generic  $8 \times 3$  matrix  $X = [x_1 \ x_2 \ x_3]$  whose columns  $x_i$  are in the nullspace of  $A \in \mathcal{S}(G)$  and derive a contradiction. The nonzero pattern of  $A \in \mathcal{S}(G)$  is

$$A = \begin{bmatrix} ? & * & * & * & 0 & 0 & 0 & 0 \\ * & ? & * & * & 0 & 0 & 0 & 0 \\ \hline * & * & ? & * & * & 0 & 0 & 0 \\ * & * & * & ? & * & * & 0 & 0 \\ \hline 0 & 0 & * & * & ? & * & * & * \\ 0 & 0 & 0 & * & * & ? & * & * \\ 0 & 0 & 0 & 0 & * & * & ? & * \\ 0 & 0 & 0 & 0 & * & * & * & ? \end{bmatrix}.$$

Columns 4, 5, 6, and 7, and columns 2, 3, 4 and 5 are clearly independent. If  $\text{rank}(A) = 4$  then the first two columns are linearly dependent and the last two columns are linearly dependent. As in Example 1.1 a contradiction is obtained. So assume  $\text{rank}(A) = 5$ .

$$5 = \text{rank}(A) \geq \text{rank}(A[\{1, 2\}, \{1, 2, 3, 4\}]) + \text{rank}(A[\{3, 4\}, \{5, 6\}]) + \text{rank}(A[\{5, 6, 7, 8\}, \{7, 8\}]).$$

Since  $\text{rank}(A[\{3, 4\}, \{5, 6\}]) = 2$ , either the first two rows are linearly dependent or the last two columns are linearly dependent. In the former case, since  $A$  is symmetric, the first two columns are linearly dependent, and thus there is a vector in the null space of  $A$  of the form  $y = [*, *, 0, 0, 0, 0, 0, 0]^T$ . But since  $X$  is generic, there is no relation among the columns of  $X$ , so  $y$  is independent of  $x_1, x_2, x_3$ , and a contradiction is obtained.

### 3 Maximum generic nullity and Vandermonde matrices

In this section we develop techniques for computation of maximum generic nullity and show that  $GM(G) = \kappa_e(G) = \delta(G)$  for all connected graphs of order at least two and at most five.

When constructing a  $n \times k$  matrix to show that the generic nullity of  $A$  is at least  $k$ , the next proposition shows that it is enough to construct  $Y$  such that  $AY = 0$  and every  $k \times k$  submatrix of  $Y$  is nonsingular.

**Proposition 3.1.** *For a real  $n \times k$  matrix  $Y$  with  $n \geq k$ , if all  $k \times k$  submatrices are nonsingular then there exists a real nonsingular  $k \times k$  matrix  $B$  such that  $X = YB$  is generic.*

*Proof.* Given  $Y = [y_{ij}]$ , let  $F$  be the field extension of the rational numbers generated by all the  $y_{ij}$ . Choose  $k^2$  real numbers  $\beta_{ij}$  that are algebraically independent over  $F$  and let  $B = [\beta_{ij}]$ . Now

consider an  $r \times r$  submatrix  $X[\alpha, \beta]$  where  $1 \leq r \leq k$ . By the Cauchy-Binet formula,

$$\det X[\alpha, \beta] = \sum_{\gamma} \det Y[\alpha, \gamma] \det B[\gamma, \beta]$$

where the sum is over all subsets of  $\{1, 2, \dots, k\}$  of cardinality  $r$ . Since each  $k \times k$  submatrix of  $Y$  is invertible, some  $Y[\alpha, \gamma]$  is nonsingular. Thus  $\det X[\alpha, \beta]$  is a nonzero polynomial over  $F$  in the  $\beta_{ij}$ 's. Since the  $\beta_{ij}$ 's are algebraically independent,  $\det X[\alpha, \beta]$  is nonzero.  $\square$

In the study of maximum nullity, it is customary to consider only connected graphs, since if the connected components of  $G$  are  $G_i, i = 1, \dots, h$ , then  $M(G) = \sum_{i=1}^h M(G_i)$ . We can also reduce the study of maximum generic nullity to the study of the connected components, but with a different relationship.

**Proposition 3.2.** *If  $G_i, i = 1, \dots, h$  are connected disjoint graphs and  $|G_i| \geq 2$  for  $i = 1, \dots, h$ , then*

$$GM\left(\bigcup_{i=1}^h G_i \cup mK_1\right) \leq \min\{GM(G_i) : i = 1, \dots, h\}.$$

*Proof.* Number the vertices of  $G_1$  first, then  $G_2$ , etc. Let  $n_i = |G_i|$ . If  $A \in \mathcal{S}(G)$ , then  $A = A_1 \oplus \dots \oplus A_h \oplus D$ , where  $A_i \in \mathcal{S}(G_i)$  and  $D$  is diagonal. In fact, order for  $A$  to have a generic null vector,  $D = 0$ . Let  $X$  be a generic  $n \times k$  matrix such that  $AX = 0$  and partition  $X$  as

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_h \\ X_{h+1} \end{bmatrix} \text{ where there are } n_i \text{ rows in } X_i \text{ and } m \text{ rows of } X_{h+1}. \text{ Then } A_i X_i = 0 \text{ for } i = 1, \dots, h.$$

Since any nonempty submatrix of a generic matrix is generic and  $A_i \neq 0$ ,

$$k \leq \min\{GN(A_i) : i = 1, \dots, h\} \leq \min\{GM(G_i) : i = 1, \dots, h\}. \quad \square$$

One might expect that the inequality in Proposition 3.2 should be an equality (and we do not know of any cases of strict inequality). One way to establish equality for many graphs is through the use of Vandermonde matrices. Given  $k$  real numbers  $\alpha_1, \dots, \alpha_k$  we define the  $n \times k$  Vandermonde matrix  $V_n(\alpha_1, \dots, \alpha_k) = [\alpha_j^{i-1}]$ . If  $0 < \alpha_1 < \dots < \alpha_k$ , then  $V_n(\alpha_1, \dots, \alpha_k)$  is totally positive [3, p. 21-3]. Given  $k$  real numbers  $\alpha_1, \dots, \alpha_k$  and  $n$  nonnegative integers  $m_1, \dots, m_n$ , we define the  $n \times k$  generalized Vandermonde matrix  $V(\alpha_1, \dots, \alpha_k; m_1, \dots, m_n) = [\alpha_j^{m_i}]$ . A matrix is a generalized Vandermonde matrix if and only if it is submatrix of a (larger) Vandermonde matrix. Thus, if  $0 < \alpha_1 < \dots < \alpha_k$  and  $0 \leq m_1 < \dots < m_n$ , then  $V(\alpha_1, \dots, \alpha_k; m_1, \dots, m_n) = [\alpha_j^{m_i}]$  is totally positive and hence generic.

When trying to exhibit a generic matrix of maximum nullity it is often convenient to search for a Vandermonde matrix, and we will see that for every graph  $G$  of order  $n \leq 5$  it is always possible to use the Vandermonde matrix  $V_n(1, 2, \dots, GM(G))$  as the generic matrix.

**Proposition 3.3.** *Let  $G = \cup_{i=1}^h G_i$  where  $n_i = |G_i| \geq 2$  but the  $G_i$  are not assumed disjoint. If there exist positive real numbers  $\alpha_1 < \dots < \alpha_k$  such that for every generalized Vandermonde matrix  $V_i = V(\alpha_1, \dots, \alpha_k; m_1, \dots, m_{n_i})$  there exists  $A_i \in \mathcal{S}(G_i)$  such that  $A_i V_i = 0$ , then*

$$GM(G) \geq \min\{GM(G_i) : i = 1, \dots, h\}.$$

*Proof.* If the vertices of  $G_i$  are  $v_1, \dots, v_{n_i} \in \{1, \dots, n\}$ , choose  $A_i \in \mathcal{S}(G_i)$  such that  $A_i V_i = 0$  for  $V_i = V(\alpha_1, \dots, \alpha_k; v_1 - 1, \dots, v_{n_i} - 1)$ . Let  $\widehat{A}_i$  be the  $n \times n$  matrix obtained by embedding  $A_i$  in the appropriate place in an  $n \times n$  matrix. Then  $\widehat{A}_i V = 0$  for  $V = V_n(\alpha_1, \dots, \alpha_k)$ . It is then possible to choose real numbers  $\beta_1, \dots, \beta_h$  so that for all  $r$  and  $s$  the  $(r, s)$ -entry of  $A = \sum_{i=1}^h \beta_i \widehat{A}_i$  is 0 if and only if the  $(r, s)$ -entry of each  $\widehat{A}_i$  is 0. Thus,  $A \in \mathcal{S}(G)$  and  $AV = 0$ .  $\square$

**Corollary 3.4.** *Let  $G_i, i = 1, \dots, h$  be connected disjoint graphs and  $|G_i| \geq 2$  for  $i = 1, \dots, h$ . If there exist positive real numbers  $\alpha_1 < \dots < \alpha_k$  such that for every generalized Vandermonde matrix  $V_i = V(\alpha_1, \dots, \alpha_k; m_1, \dots, m_{n_i})$  there exists  $A_i \in \mathcal{S}(G_i)$  such that  $A_i V_i = 0$ , then*

$$GM(G) = \min\{GM(G_i) : i = 1, \dots, h\}.$$

We now establish the hypotheses of Proposition 3.3 for some families of graphs.

**Proposition 3.5.** *For any generic  $X$   $n \times (n - 1)$  matrix, there exist a matrix  $A \in \mathcal{S}(K_n)$  such that  $AX = 0$ . In particular, for any nonnegative integers  $m_1 \leq \dots \leq m_n$ , there exists  $A \in \mathcal{S}(K_n)$  such that  $AV(1, 2, \dots, n - 1; m_1, \dots, m_n) = 0$ . Moreover,  $GM(K_n) = n - 1$  for  $n \geq 2$ .*

*Proof.* Since  $X$  is  $n \times (n - 1)$ , there exists a nonzero vector  $a \in \mathbb{R}^n$  such that  $a^T X = 0$ . Since  $X$  is generic, all entries of  $a$  are nonzero. Let  $A = aa^T$ .  $\square$

**Corollary 3.6.** *If  $G$  is  $K_n$  with an edge deleted, then  $GM(G) = M(G) = n - 2$ .*

*Proof.*  $G$  is the union of two copies of  $K_{n-1}$ .  $\square$

**Proposition 3.7.**  *$GM(C_n) = M(C_n) = 2$ . Furthermore, for  $\alpha > 1$  and any nonnegative integers  $m_1 \leq \dots \leq m_n$ , there exists  $A \in \mathcal{S}(C_n)$  such that  $AV(1, \alpha; m_1, \dots, m_n) = 0$ .*

*Proof.* Let

$$a_{i,i+1} = \frac{1}{\alpha^{m_i} - \alpha^{m_{i+1}}} \text{ and } a_{ii} = \frac{\alpha^{m_{i+1}} - \alpha^{m_{i-1}}}{(\alpha^{m_{i+1}} - \alpha^{m_i})(\alpha^{m_i} - \alpha^{m_{i-1}})}.$$

where the index  $n + 1$  is interpreted as 1 and 0 is interpreted as  $n$ .  $\square$

**Corollary 3.8.** *If  $G$  is a union of cycles then  $GM(G) \geq 2$ , with equality if the union is disjoint. If  $G$  is a union of copies of  $K_r$  then  $GM(G) \geq r - 1$  with equality if the union is disjoint. In all these cases, maximum generic nullity can be realized by a matrix having a Vandermonde matrix as a generic null space matrix.*

**Corollary 3.9.** *If  $G$  is connected and  $2 \leq |G| \leq 5$ , then  $GM(G) = \kappa_e(G) = \delta(G)$  and maximum generic nullity can be realized by a matrix having a Vandermonde matrix as a generic null space matrix.*

*Proof.* Any graph having  $\delta(G)=1$  satisfies  $1 = GM(G) = \kappa_e(G) = \delta(G)$ . Every connected graph of order at most 5 that has  $\delta(G) = 2$  is a union of cycles and thus has  $2 = GM(G) = \kappa_e(G) = \delta(G)$ . A connected graph having order 5 or less and  $\delta(G) = 3$  is  $K_4$  or is one of those shown in Figure 3.

$G_{51}$  is  $K_5$  with an edge deleted and is thus a union of two copies of  $K_4$ . Let

$$A = \begin{bmatrix} -\frac{20736}{23375} & \frac{36}{25} & -\frac{6}{11} & -\frac{36}{935} & \frac{66}{2125} \\ \frac{36}{25} & -\frac{12}{5} & 1 & 0 & -\frac{1}{25} \\ -\frac{6}{11} & 1 & -\frac{6}{11} & \frac{1}{11} & 0 \\ -\frac{36}{935} & 0 & \frac{1}{11} & -\frac{12}{187} & \frac{1}{85} \\ \frac{66}{2125} & -\frac{1}{25} & 0 & \frac{1}{85} & -\frac{6}{2125} \end{bmatrix}.$$

Then  $A \in \mathcal{S}(W_5)$  and  $AV_5(1, 2, 3) = 0$ . Order 5 and  $\delta(G) = 4$  implies  $G$  is  $K_5$ .  $\square$



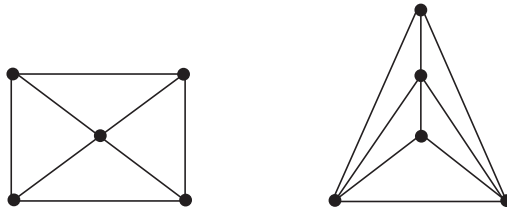


Figure 3:  $W_5 = G50$

$G51$

## 4 Maximum generic nullity and vertex connectivity

In the section we show that vertex connectivity bounds maximum generic nullity from below, and give an example where these two parameters differ.

For a graph  $G$ , an *orthogonal representation of  $G$  of dimension  $d$  (or in  $\mathbb{R}^d$ )* is a set of vectors in  $\mathbb{R}^d$ , one corresponding to each vertex, with the property that if two vertices are nonadjacent, then their corresponding vectors are orthogonal. Trivially, every graph has an orthogonal representation in any dimension (by associating the zero vector with every vertex). A *faithful orthogonal representation of  $G$  of dimension  $d$*  is an orthogonal representation such that if two vertices are adjacent, then their corresponding vectors are not orthogonal. In the minimum rank literature, the term “orthogonal representation” is often used for what is here called a faithful orthogonal representation, following the notation of [7]. An orthogonal representation of  $G$  in  $\mathbb{R}^d$  is in *general-position* if every subset of  $d$  vectors is linearly independent. Let  $\text{mr}_+(G)$  denote the minimum rank among all symmetric positive semidefinite matrices  $A$  such that  $\mathcal{G}(A) = G$ , and let  $M_+(G)$  denote the maximum nullity among all such matrices. Clearly  $\text{mr}_+(G) \leq \text{mr}(G)$  and  $M_+(G) \geq M(G)$ . It is well known (and easy to see) that every faithful orthogonal representation of dimension  $d$  gives rise to a positive semidefinite matrix of rank  $d$  and vice versa.

The following result of Lovász, Saks and Schrijver [7], [8] relates vertex connectivity to maximum nullity. We use the version stated by van der Holst in [6].

**Theorem 4.1.** [6, Theorem 3], [7, Corollary 1.4] *For a graph  $G$  with  $n$  vertices,  $G$  is  $(n - d)$ -connected if and only if  $G$  has a general-position faithful orthogonal representation in  $\mathbb{R}^d$ .*

**Corollary 4.2.** *For any graph  $G$ ,*

$$\kappa_v(G) \leq GM(G).$$

*Proof.* Let  $G$  be a graph of order  $n$  and let  $k = \kappa_v(G)$ . By Theorem 4.1, there exists a general-position faithful orthogonal representation in  $\mathbb{R}^{n-k}$ . Let the vector representing vertex  $v_i$  be denoted by  $b_i$ , and define  $B = [b_1, \dots, b_n]$  and  $A = B^T B$ . Then  $A \in \mathcal{S}(G)$  and  $\text{rank}(A) = n - k$ . Let  $x_1, \dots, x_k$  be a basis for  $\ker(A) = \ker(B)$  and define  $X = [x_1, \dots, x_k]$ . We claim that every  $k \times k$  submatrix of  $X$  is nonsingular, which implies  $k \leq GM(G)$  by Proposition 3.1.

Suppose not. Then there exists a set of indices  $i_1, \dots, i_k$  such that  $X[\{i_1, \dots, i_k\}, \{1, \dots, k\}]$  is singular. So there exists a nonzero vector  $w = [0, \dots, 0, w_{i_1}, 0, \dots, 0, w_{i_k}, 0, \dots, 0]^T \in \mathbb{R}^n$  such that  $w^T X = 0$ . Since  $X \in \mathbb{R}^{n \times k}$ ,  $BX = 0$ ,  $\text{rank}(B) = n - k$ , and  $\text{rank}(X) = k$ , the rows of  $B$  are a basis for the right null space of  $X$ . Thus there exists a vector  $u \in \mathbb{R}^{n-k}$  such that  $u^T B = w^T$ . Let  $R = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$ , so  $w_j = 0$  for  $j \in R$  and  $|R| = n - k$ . Thus  $u^T B[\{1, \dots, n-k\}, R] = 0^T$ . Since  $B[\{1, \dots, n-k\}, R]$  is nonsingular,  $u^T = 0^T$ , contradicting  $u^T B = w^T \neq 0^T$ .  $\square$

The next example shows that it is possible to have  $\kappa_v(G) < GM(G)$ .

**Example 4.3.** The bowtie G42, shown in Figure 4, has  $GM(G42) = \kappa_e(G42) = 2 > 1 = \kappa_v(G42)$ .

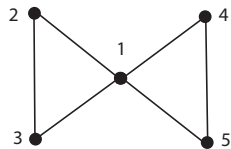


Figure 4: The graph  $G = G42$  in Example 4.3

Bollobás and Thomason [2] proved that for a random graph  $G$  on  $n$  vertices having edge probability  $p$ , the probability that  $\kappa_v(G) < \delta(G)$  goes to 0 as  $n$  goes to infinity. This result is very general and does not require  $p$  to be fixed. A simplification of the Bollobás and Thomason proof for fixed  $p$  is given in [5]. Choosing a graph at random from all graphs of order  $n$  is the same as choosing a random graph of order  $n$  with edge probability  $p = 1/2$ . Thus Theorem 2.2 and Corollary 4.2, together with the Bollobás and Thomason result, show that as  $n$  goes to infinity, almost all graphs have

$$\kappa_v(G) = GM(G) = \kappa_e(G) = \delta(G).$$

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