Generic maximum nullity of a graph

Leslie Hogben* Bryan Shader†

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Abstract

For a graph G of order n, the maximum nullity of G is defined to be the largest possible nullity over all real symmetric $n \times n$ matrices A whose (i,j)th entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. Maximum nullity and the related parameter minimum rank of the same set of matrices have been studied extensively. A new parameter, generic maximum nullity, is introduced. Generic maximum nullity provides insight into the structure of the null-space in a matrix realizing maximum nullity of a graph. It is shown that generic maximum nullity is bounded above by edge connectivity.

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1 Introduction

The (real symmetric) minimum rank problem for a simple graph asks us to determine the minimum rank among real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G, or equivalently to determine of the maximum nullity (or maximum multiplicity of an eigenvalue) among the same family of matrices.

All graphs discussed in this paper are simple, meaning no loops or multiple edges, undirected, finite, and have nonempty vertex sets. The *order* of a graph G, denoted $|G|$, is the number of vertices of G. The set of $n \times n$ real symmetric matrices will be denoted by S_n. For $A \in S_n$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i,j\} : \ a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The set of real symmetric matrices of a graph G is $\mathcal{S}(G) = \{A \in S_n : \ \mathcal{G}(A) = G\}$. The *minimum rank* of a graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(G)\}.$$

The *maximum nullity* of G is

$$M(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(G)\}$$

where $\mathrm{null}(A)$ is the dimension of the null space $\ker(A)$ of A. Clearly $M(G) + \mathrm{mr}(G) = |G|$. If $A \in \mathcal{S}(G)$ and $\alpha \in \mathbb{R}$, then $A + \alpha I \in \mathcal{S}(G)$, so the maximum multiplicity of any eigenvalue is the same as maximum multiplicity of eigenvalue 0, i.e., the maximum nullity. See [3] for a survey

*Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu) and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).
†Department of Mathematics, University of Wyoming, University of Wyoming, Laramie, WY 82071, USA (bshader@uwyo.edu).

of known results and discussion of the motivation for the minimum rank problem; an extensive bibliography is also provided there.

If $W, U \subseteq \{1, 2, \ldots, n\}$ and $B \in S_n$, then $B[W, U]$ denotes the *submatrix* of B having rows indexed by W and columns indexed by W. In case $W = U$, this is a *principal submatrix* and is denoted by $B[W]$; the complementary principal submatrix obtained from B by deleting the rows and columns indexed by W is denoted $B(W)$. In the special case when $W = \{k\}$, we use $B(k)$ to denote $B(W)$.

A graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V, E' \subseteq E$. The subgraph $G[W]$ of $G = (V, E)$ *induced* by $W \subseteq V$ is the subgraph with vertex set W and edge set $\{\{i, j\} \in E \mid i, j \in W\}$; $G(W)$ is used to denote $G[V \setminus W]$, obtained from G by deleting all the vertices in W and all edges incident with these vertices. If $S \subseteq E$, the subgraph $G - S$ is the subgraph obtained by deleting the edges in S, i.e., the graph with vertex set V and edge set $E \setminus S$. A path on n vertices, a cycle on n vertices, and a complete graph on n vertices will be denoted by P_n, C_n, and K_n respectively.

A graph is *connected* if there is a path from any vertex to any other vertex. A *component* of a graph is a maximal connected subgraph. A set W of vertices of G is a *separating set* or *vertex cut* if $G(W)$ has more than one component. The *vertex connectivity* of G, denoted $\kappa_v(G)$ is the minimum size of a separating set of G. A set S of edges of a graph G (with $|G| > 1$) is a *disconnecting set* if $G - S$ has more than one component. The *edge connectivity* of G, denoted $\kappa_e(G)$ is the minimum size of a disconnecting set of G. Given $W, U \subset V(G)$, the set of edges of G having one endpoint in W and the other in U is denoted $[W, U]$. An *edge cut* is a set of edges of the form $[W, V(G) \setminus W]$ for some $W \subset V(G)$. Every edge cut is a disconnecting set but not every disconnecting set is an edge cut. However, a minimal disconnecting set is an edge cut (cf. [5, p. 152]).

The *degree* of a vertex is the number of edges incident with the vertex, and the *minimum degree* over all vertices of a graph G will be denoted by $\delta(G)$. Since the edges incident with a vertex of minimal degree constitute a disconnecting set, and since the endpoints in W of each edge in an edge cut $[W, V(G) \setminus W]$ constitutes a separating set,

$$\kappa_v(G) \leq \kappa_e(G) \leq \delta(G).$$

These inequalities can be strict (e.g., see [5, p. 153])

A $n \times k$ real matrix X is *generic* if every square submatrix of X is nonsingular. A generic matrix could be called a *totally nonsingular* in analogy with the definition of a *totally positive* matrix as a matrix all of whose minors are positive. Clearly a totally positive matrix is generic. Notice that any submatrix of a generic matrix is generic. The *generic nullity* of a nonzero matrix $A \in \mathbb{R}^{n \times n}$ is

$$GN(A) = \max\{k : X \in \mathbb{R}^{n \times k}, AX = 0, \text{ and } X \text{ is generic}\}$$

(the generic nullity of an $n \times n$ zero matrix is n). The *maximum generic nullity* of a graph G is

$$GM(G) = \max\{GN(A) : A \in \mathcal{S}(G)\}.$$

The maximum generic nullity of a graph can be strictly less than the maximum nullity. In this case, the null space of an optimal matrix is often highly structured.

Example 1.1. Let $G =$ G130 be the graph shown in Figure 1 (the numbering of graphs is taken from [4]). Since G can be covered by two copies of K_3 and one K_2, $\operatorname{mr}(G) \leq 3$ and since G has an induced P_4, $\operatorname{mr}(G) \geq 3$. Thus $M(G) = 6 - 3 = 3$.

We claim that $GM(G) \leq 1$. Suppose to the contrary that there is a generic 6×2 matrix $X = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ whose columns x_i are in the nullspace of $A \in \mathcal{S}(G)$. The nonzero pattern of

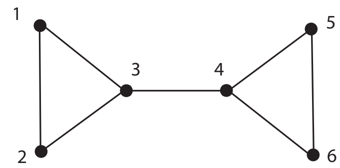

Figure 1: The graph $G =$ G130 in Example 1.1

$A \in \mathcal{S}(G)$ is

$$\begin{bmatrix} ? & * & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 & 0 \\ * & * & ? & * & 0 & 0 \\ 0 & 0 & * & ? & * & * \\ 0 & 0 & 0 & * & ? & * \\ 0 & 0 & 0 & * & * & ? \end{bmatrix},$$

where $*$ denotes a nonzero entry and ? denotes an entry about which nothing is known. Columns 2, 3 and 4 (and columns 3, 4, and 5) are clearly independent. If $\text{rank}(A) = 3$ then the first two columns have rank 1 and the last two columns have rank 1. So there are vectors in the null space of A of the forms $y = [*, *, 0, 0, 0, 0]^T$ and $z = [0, 0, 0, 0, *, *]^T$. Any linear combination of y and z has the 3rd and 4th coordinates equal to 0. Since there cannot be a relationship between the rows of X, x_1, x_2, y, z are linearly independent, which implies $\text{rank}(A) \leq 2$, a contradiction. If $\text{rank}(A) = 4$ then either the first two rows have rank 1 and the last two rows have rank 1; assume the first case. Then there is a vector in the null space having all the last four entries 0, hence independent of x_1 and x_2, and again a contradiction is obtained.

The matrix in $\mathcal{S}(G)$ with nonzero off-diagonals equal to 1 and row sums 0 has nullspace spanned by the vector of all 1s, and hence $GM(G) = 1$.

Our main result about maximum generic nullity is that for any connected graph G, $GM(G) \leq \kappa_e(G)$. This will be established in Section 2 using methods based on the ideas in Example 1.1. Using the methods of [1], it is easy to show that $GM(G) \leq \delta(G)$, but we do not include that proof since $\kappa_e(G) \leq \delta(G)$.

2 Maximum generic nullity and edge connectivity

A *nonzero pattern* $C = [c_{ij}]$ is a $m \times n$ matrix whose entries c_{ij} are elements of $\{*, 0\}$. The number of $*$ (nonzero entries) in C is denoted by $\text{nz}(C)$. Given a pattern $C = [c_{ij}]$, we let $\mathcal{Q}(C)$ denote the set of all matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ such that $a_{ij} \neq 0$ if and only if $c_{ij} = *$. Note that (unlike the set of symmetric matrices described by a graph), here the diagonal is constrained by the nonzero pattern. The *minimum rank* of a nonzero pattern C is

$$\text{mr}(C) = \min\{\text{rank}(A) : A \in \mathcal{Q}(C)\}.$$

Theorem 2.1. *If C is an $m \times n$ nonzero pattern that does not have any zero row or zero column, $\text{mr}(C) \geq m + n - \text{nz}(C)$.*

Proof. Note that arbitrary permutation of rows or columns of C does not affect $\text{mr}(C)$. For fixed m and n, the proof is by induction on $\text{nz}(C)$. The base case is any C (without zero row or column) such that for every nonzero entry, it is the only nonzero in its row or the only nonzero in its column.

That is, no row and column permutation of C contains a 2×2 submatrix $\begin{bmatrix} * & ? \\ * & * \end{bmatrix}$. By row and column permutations, any such a C can be put into the following form:

$$\begin{array}{r|cccccccccccccc}
 & * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
a_1 & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & * & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\hline
 & 0 & * & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
a_2 & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & 0 & * & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\hline
 & & & & & & & \vdots & & & & & & & \\
\hline
 & 0 & 0 & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
a_s & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & 0 & 0 & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
\hline
 & 0 & 0 & \dots & 0 & * & \dots & * & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * & \dots & 0 & \dots & 0 \\
 & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & * & \dots & * \\
 & & & & & \multicolumn{3}{c}{\underbrace{\hphantom{0 \quad \dots \quad 0}}_{b_1}} & \multicolumn{3}{c}{\underbrace{\hphantom{0 \quad \dots \quad 0}}_{b_2}} & & \multicolumn{3}{c}{\underbrace{\hphantom{0 \quad \dots \quad 0}}_{b_t}}
\end{array}$$

$$\begin{aligned}
\operatorname{nz}(C) &= a_1 + \cdots + a_s + b_1 + \cdots + b_t \\
m &= a_1 + \cdots + a_s + t \\
n &= s + b_1 + \cdots + b_t \\
m + n - \operatorname{nz}(C) &= t + s \\
&= \operatorname{mr}(C)
\end{aligned}$$

Now assume C contains a 2×2 submatrix $\begin{bmatrix} * & ? \\ * & * \end{bmatrix}$. Consider the nonzero pattern C' obtained from C by replacing one $*$ by 0 so the 2×2 submatrix is now $\begin{bmatrix} * & ? \\ 0 & * \end{bmatrix}$. Then by the induction hypothesis applied to C',

$$\operatorname{mr}(C) \geq \operatorname{mr}(C') - 1 \geq m + n - \operatorname{nz}(C') - 1 = m + n - (\operatorname{nz}(C) - 1) - 1 = m + n - \operatorname{nz}(C). \quad \square$$

Theorem 2.2. *If G is connected, then $GM(G) \leq \kappa_e(G)$.*

Proof. Let $S \geq 1$ be a minimal disconnecting set for G (so $\kappa_e(G) = |S|$). Since S is an edge cut, $S = [W, \overline{W}]$ for some $W \subset V$. Let $W_1 = W$ and $W_2 = \overline{W}$. Number the vertices of G so that the vertices of W_1 are $1, \ldots, |W|_1$, all vertices of W_1 incident with an edge of S are last among the vertices of W_1, and all vertices of W_2 incident with an edge of S are first among the vertices of W_2.

Let $A \in \mathcal{S}(G)$ be such that $GN(A) = GM(G)$. Let $A_i = A[W_i]$ and then A can be partitioned as

$$A = \left[\begin{array}{c|c|c} \widehat{A}_1 & 0 & 0 \\ \hline D & C & 0 \\ \hline F & E & \widehat{A}_2 \end{array}\right]$$

where $A_1 = \begin{bmatrix} \widehat{A}_1 \\ D \end{bmatrix}$, $A_2 = \begin{bmatrix} E & \widehat{A}_2 \end{bmatrix}$, C is $d \times e$, $\widehat{A}_1$ is $(n_1 - d) \times n_1$ and $\widehat{A}_2$ is $n_2 \times (n_2 - e)$. Let $\operatorname{rank}(\widehat{A}_i) = r_i$. Then

$$\begin{aligned} \operatorname{rank}(A) &\geq r_1 + \operatorname{mr}(C) + r_2 \\ &\geq r_1 + r_2 + d + e - \operatorname{nz}(C) \\ &= r_1 + r_2 + d + e - \kappa_e(G). \end{aligned}$$

$$\begin{aligned} n_1 + n_2 - \operatorname{null}(A) &\geq r_1 + \operatorname{mr}(C) + r_2 \\ n_1 - r_1 + n_2 - r_2 - \operatorname{mr}(C) &\geq \operatorname{null}(A) \end{aligned} \tag{1}$$

Now consider the vectors that must be in $\ker(A)$. Since $\operatorname{rank}(\widehat{A}_2) = r_2$, there exist $k_2 = n_2 - e - r_2$ independent vectors $\hat{y}_i \in \mathbb{R}^{n_2 - e}$ such that $\widehat{A}_2 \hat{y}_i = 0$. If we let $y_i = \begin{bmatrix} 0 \\ 0 \\ \hat{y}_i \end{bmatrix}$ (where the first zero vector is of length $n_1 - d$ and the second is of length $d+e$), then $y_i \in \ker(A)$, $i = 1, \ldots, n_2 - e - r_2$. Since $\operatorname{rank}(\widehat{A}_1) = r_1$, there exist $k_2 = n_1 - d - r_1$ independent vectors $\hat{x}_i \in \mathbb{R}^{n_1 - d}$ such that $\hat{x}_i^T \widehat{A}_1 = 0$. Since A_1 is symmetric, if we let $x_i = \begin{bmatrix} \hat{x}_i \\ 0 \\ 0 \end{bmatrix}$ (where the first zero vector is of length $d+e$ and the second is of length $n_2 - e$), then $x_i \in \ker(A)$, $i = 1, \ldots, n_1 - d - r_1$.

Any $z \in \operatorname{Span}(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2})$ has $d + e$ zeros. No set of columns in a generic basis can have a dependence relation, so either

(a) there are $d + e$ additional vectors z_i in the null space of A (and the generic vectors may be a subset of $\operatorname{Span}(x_1, \ldots, x_{k_1}, y_1, \ldots, y_{k_2}, z_1, \ldots, z_{d+e})$), or

(b) there are $GM(G)$ additional vectors in the null space of A.

But (a) is impossible, because if so we would have

$$n_1 - r_1 + n_2 - r_2 - 1 \geq n_1 - r_1 + n_2 - r_2 - \operatorname{mr}(C) \geq \operatorname{null}(A) \geq n_1 - d - r_1 + n_2 - e - r_2 + d + e = n_1 - r_1 + n_2 - r_2.$$

So there are $GM(G)$ additional independent vectors in the null space of A. By considering these vectors together with the x_i and y_i, and equation (1), we have

$$\begin{aligned} \operatorname{null}(A) &\geq n_1 - d - r_1 + n_2 - e - r_2 + GM(G) \\ n_1 - r_1 + n_2 - r_2 - \operatorname{mr}(C) &\geq n_1 - d - r_1 + n_2 - e - r_2 + GM(G) \\ -\operatorname{mr}(C) &\geq -d - e + GM(G) \end{aligned}$$

By Theorem 2.1, $\operatorname{mr}(C) \geq d + e - \operatorname{nz}(C) = d + e - \kappa_e(G)$,

$$\begin{aligned} \kappa_e(G) - d - e &\geq -d - e + GM(G) \\ \kappa_e(G) &\geq GM(G) \quad \square \end{aligned}$$

It is possible to have $GM(G) < \kappa_e(G)$, as the next example shows.

Example 2.3. The graph H shown in Figure 2, has $GM(G) = 2 < 3 = \kappa_e(G)$. We assume there

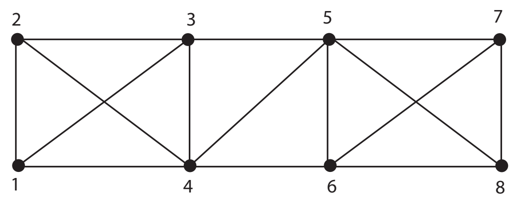

Figure 2: The graph H in Example 2.3

is a generic 8×3 matrix $X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$ whose columns x_i are in the nullspace of $A \in \mathcal{S}(G)$ and derive a contradiction. The nonzero pattern of $A \in \mathcal{S}(G)$ is

$$A = \left[\begin{array}{cccc|cc|cc} ? & * & * & * & 0 & 0 & 0 & 0 \\ * & ? & * & * & 0 & 0 & 0 & 0 \\ \hline * & * & ? & * & * & 0 & 0 & 0 \\ * & * & * & ? & * & * & 0 & 0 \\ \hline 0 & 0 & * & * & ? & * & * & * \\ 0 & 0 & 0 & * & * & ? & * & * \\ 0 & 0 & 0 & 0 & * & * & ? & * \\ 0 & 0 & 0 & 0 & * & * & * & ? \end{array}\right].$$

Columns 2, 3, 4 and 5, and columns 4, 5, 6, and 7 are clearly independent. If $\operatorname{rank}(A) = 4$ then the first two columns have rank 1 and the last two columns have rank 1. As in Example 1.1 a contradiction is obtained. So assume $\operatorname{rank}(A) = 5$.

$$5 = \operatorname{rank}(A) \geq \operatorname{rank}(A[\{1,2\},\{1,2,3,4\}) + \operatorname{rank}(A[\{3,4\},\{5,6\}) + \operatorname{rank}(A[\{5,6,7,8\},\{7,8\}).$$

Since $\operatorname{rank}(A[\{3,4\},\{5,6\}) = 2$, either the first two rows have rank 1 or the last two columns have rank 1. In the former case, since A is symmetric, the first two columns have rank 1, and thus there is a vector in the null space of A of the form $y = [*,*,0,0,0,0,0,0]^T$. But since X is generic, there is no relation among the rows of X, so y is independent of x_1, x_2, x_3, and a contradiction is obtained.

3 Maximum generic nullity and Vandermonde matrices

In the section we develop techniques for computation of maximum generic nullity and show that $GM(G) = \kappa_e(G) = \delta(G)$ for all graphs of order at most five.

When constructing a $n \times k$ matrix to show that the generic nullity of A is at least k, the next proposition shows that it is enough to construct Y such that $AY = 0$ and every $k \times k$ submatrix of Y is nonsingular.

Proposition 3.1. *For a real $n \times k$ matrix Y, if all $k \times k$ submatrices are nonsingular then there exists a real nonsingular $k \times k$ matrix B such that $X = YB$ is generic.*

Proof. Given $Y = [y_{ij}]$, let F be the field extension of the rational numbers generated by all the y_{ij}. Choose k^2 real numbers β_{ij} that are algebraically independent over F and let $B = [\beta_{ij}]$. Then for $1 \leq r \leq k$, the determinant of an $r \times r$ submatrix is a nonzero polynomial over F in β_{ij} and so is nonzero. $\square$

In the study of maximum nullity, it is customary to consider only connected graphs, since if the connected components of G are $G_i, i = 1, \ldots, h$, then $M(G) = \sum_{i=1}^{h} M(G_i)$. We can also reduce the study of maximum generic nullity to the study of the connected components, but with a different relationship.

Proposition 3.2. *If $G_i, i = 1, \ldots, h$ are connected disjoint graphs and $|G_i| \geq 2$ for $i = 1, \ldots, h$, then*

$$GM(\bigcup_{i=1}^{h} G_i \cup mK_1) \leq \min_{i-1}^{h} GM(G_i).$$

Proof. Number the vertices of G_1 first, then G_2, etc. Let $n_i = |G_i|$. If $A \in \mathcal{S}(G)$, then $A = A_1 \oplus \cdots \oplus A_h \oplus D$, where $A_i \in \mathcal{S}(G_i)$ and D is diagonal. In fact, order for A to have a generic null vector, $D = 0$. Let X be a generic $n \times k$ matrix such that $AX = 0$ and partition X as $X = \begin{bmatrix} X_1 \\ \vdots \\ X_h \\ X_{h+1} \end{bmatrix}$ where there are n_i rows in X_i and m rows of X_{h+1}. Then $A_i X_i = 0$ for $i = 1, \ldots, h$. Since any nonempty submatrix of a generic matrix is generic and $A_i \neq 0$,

$$k \leq \min_{i-1}^{h} GN(A_i) \leq \min_{i-1}^{h} GM(G_i). \quad \square$$

One might expect that the inequality in Proposition 3.2 should be an equality (and we do not know of any cases of strict inequality). One way to establish equality for many graphs is through the use of Vandermonde matrices. Given k real numbers $\alpha_1, \ldots, \alpha_k$ we define the $n \times k$ *Vandermonde matrix* $V_n(\alpha_1, \ldots, \alpha_k) = [\alpha_j^{i-1}]$. If $0 < \alpha_1 < \cdots < \alpha_k$, then $V_n(\alpha_1, \ldots, \alpha_k)$ is totally positive [2, p. 21-3]. Given k real numbers $\alpha_1, \ldots, \alpha_k$ and n nonnegative integers $m_1, \ldots, m_n$, we define the $n \times k$ *generalized Vandermonde matrix* $V(\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_n) = [\alpha_j^{m_i}]$. A matrix is a generalized Vandermonde matrix if and only if it is submatrix of a (larger) Vandermonde matrix. Thus, if $0 < \alpha_1 < \cdots < \alpha_k$ and $0 \leq m_1 < \cdots < m_n$, then $V(\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_n) = [\alpha_j^{m_i}]$ is totally positive and hence generic.

When trying to exhibit a generic matrix of maximum nullity it is often convenient to search for a Vandermonde matrix, and we will see that for every graph G of order $n \leq 5$ it is always possible to use the Vandermonde matrix $V_n(1, 2, \ldots, GM(G))$ as the generic matrix.

Proposition 3.3. *Let $G = \cup_{i=1}^{h} G_i$ where $n_i = |G_i| \geq 2$ but the G_i are not assumed disjoint. If there exist positive real numbers $\alpha_1 < \cdots < \alpha_k$ such that for every generalized Vandermonde matrix $V_i = V(\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_{n_i})$ there exists $A_i \in \mathcal{S}(G_i)$ such that $A_i V_i = 0$, then*

$$GM(G) \geq \min_{i=1}^{h} GM(G_i).$$

Proof. If the vertices of G_i are $v_1, \ldots, v_{n_i} \in \{1, \ldots, n\}$, choose $A_i \in \mathcal{S}(G_i)$ such that $A_i V_i = 0$ for $V_i = V(\alpha_1, \ldots, \alpha_k; v_1 - 1, \ldots, v_{n_i} - 1)$. Embed A_i in the appropriate place an $n \times n$ matrix $\widehat{A}_i$. Then $\widehat{A}_i V = 0$ for $V = V_n(\alpha_1, \ldots, \alpha_k)$. Choose real numbers $\beta_1, \ldots, \beta_h$ so that no subtractive cancellation occurs in $A = \sum_{i-1}^{h} \beta_i \widehat{A}_i$. Then $A \in \mathcal{S}(G)$ and $AV = 0$. $\square$

In fact, by choosing different exponents in the proof, the property that for every generalized Vandermonde matrix $V_i = V(\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_n)$ there exists $A \in \mathcal{S}(G)$ such that $AV = 0$ is obtained.

Corollary 3.4. *Let $G_i, i = 1, \ldots, h$ are connected disjoint graphs and $|G_i| \geq 2$ for $i = 1, \ldots, h$, If there exist positive real numbers $\alpha_1 < \cdots < \alpha_k$ such that for every generalized Vandermonde matrix $V_i = V(\alpha_1, \ldots, \alpha_k; m_1, \ldots, m_{n_i})$ there exists $A_i \in \mathcal{S}(G_i)$ such that $A_i V_i = 0$, then*

$$GM(G) = \min_{i=1}^{h} GM(G_I).$$

We now establish the hypotheses of Proposition 3.3 for some families of graphs.

Proposition 3.5. *For any generic X $n \times (n-1)$ matrix, there exist a matrix $A \in \mathcal{S}(K_n)$ such that $AX = 0$. In particular, for any nonnegative integers $m_1 \leq \cdots \leq m_n$, there exists $A \in \mathcal{S}(K_n)$ such that $A V(1, 2, \ldots, n-1; m_1, \ldots, m_n) = 0$*

Proof. Since X is $n \times (n-1)$, there exists a vector $a \in \mathbb{R}^n$ such that $a^T X = 0$. Since X is generic, all entries of a are nonzero. Let $A = aa^T$. □

Corollary 3.6. *If G is K_n with an edge deleted, then $GM(G) = M(G) = n - 1$.*

Proof. G is the union of two copies of K_{n-1}. □

Proposition 3.7. *$GM(C_n) = M(C_n) = 2$. Furthermore, for $\alpha > 1$ and any nonnegative integers $m_1 \leq \cdots \leq m_n$, there exists $A \in \mathcal{S}(C_n)$ such that $A V(1, \alpha; m_1, \ldots, m_n) = 0$.*

Proof. Let

$$a_{i,i+1} = \frac{\alpha^{m_1} - \alpha^{m_2}}{\alpha^{m_i} - \alpha^{m_{i+1}}} \text{ and } a_{ii} = \frac{(\alpha^{m_2} - \alpha^{m_1})(\alpha^{m_{i-1}} - \alpha^{m_{i+1}})}{(\alpha^{m_{i+1}} - \alpha^{m_i})(\alpha^{m_i} - \alpha^{m_{i-1}})}.$$

where the index $n + 1$ is interpreted as 1 and 0 is interpreted as n. □

Corollary 3.8. *If G is a union of cycles then $GM(G) \geq 2$ and 2 can be realized by a Vandermonde matrix. If G is a union of copies of K_r then $GM(G) \geq r - 1$ and $r - 1$ can be realized by a Vandermonde matrix.*

Corollary 3.9. *If G is connected and $|G| \leq 5$, then $GM(G) = \kappa_e(G) = \delta(G)$ and this can be realized by a Vandermonde matrix.*

Proof. Any graph having $\delta(G)$=1 satisfies $1 = GM(G) = \kappa_e(G) = \delta(G)$. Every connected graph of order at most 5 that has $\delta(G) = 2$ is a union of cycles and thus has $2 = GM(G) = \kappa_e(G) = \delta(G)$. A connected graph having order 5 or less and $\delta(G) = 3$ is K_4 or is one of those shown in Figure 3.

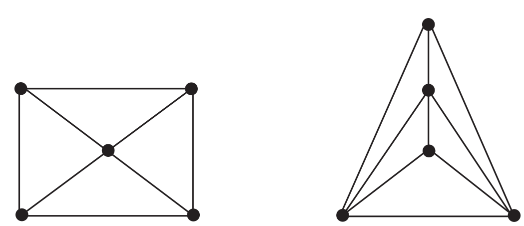

Figure 3: $W_5 = G50$ $G51$

$G51$ is K_5 with an edge deleted and is thus a union of two copies of K_4. Let

$$A = \begin{bmatrix} -\frac{20736}{23375} & \frac{36}{25} & -\frac{6}{11} & -\frac{36}{935} & \frac{66}{2125} \\ \frac{36}{25} & -\frac{12}{5} & 1 & 0 & -\frac{1}{25} \\ -\frac{6}{11} & 1 & -\frac{6}{11} & \frac{1}{11} & 0 \\ -\frac{36}{935} & 0 & \frac{1}{11} & -\frac{12}{187} & \frac{1}{85} \\ \frac{66}{2125} & -\frac{1}{25} & 0 & \frac{1}{85} & -\frac{6}{2125} \end{bmatrix}.$$

Then $A \in \mathcal{S}(W_5)$ and $A\,V_5(1,2,3) = 0$. Order 5 and $\delta(G) = 4$ implies G is K_5. □

The next example shows that it is possible to have $\kappa_v(G) < GM(G)$.

Example 3.10. The bowtie G42, shown in Figure 4, has $GM(\text{G42}) = \kappa_e(\text{G42}) = 2 > 1 = \kappa_v(\text{G42})$.

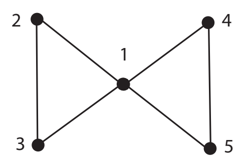

Figure 4: The graph $G =$ G42 in Example 3.10

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