

The Copositive Completion Problem

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Abstract

An $n \times n$ real symmetric matrix A is called (strictly) copositive if $x^T Ax \geq 0$ (> 0) whenever $x \in \mathbf{R}^n$ satisfies $x \geq 0$ ($x \geq 0$ and $x \neq 0$). The (strictly) copositive matrix completion problem asks which partial (strictly) copositive matrices have a completion to a (strictly) copositive matrix. We prove that every partial (strictly) copositive matrix has a (strictly) copositive matrix completion and give a lower bound on the values used in the completion.

Key words: copositive, strictly copositive, matrix completion, partial matrix

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An $n \times n$ real symmetric matrix A is called copositive if $x^T Ax \geq 0$ whenever $x \in \mathbf{R}^n$ satisfies $x \geq 0$ (entry-wise), and is called strictly copositive if $x^T Ax > 0$ whenever $x \geq 0$ and $x \neq 0$. The copositive matrices arise in a number of ways (e.g. they constitute the cone theoretic dual of the completely positive matrices [2]), and have received notable study [1], [3], [4], [5], [7]. Checking a given matrix may be carried out definitively [8], [9], [10], [11] but is generally computationally time-consuming. Since the vector argument for the quadratic form of a principal submatrix may be embedded into an argument for the quadratic form for the full matrix by insertion of 0's, copositivity and strict copositivity, are inherited by principal submatrices. Thus, the diagonal entries of a (strictly) copositive matrix are nonnegative (positive).

A partial matrix is one in which some entries are specified, while the remaining entries are unspecified and free to be chosen. A completion of a partial matrix

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is a choice of values for the unspecified entries, resulting in a conventional matrix, and a matrix completion problem asks which matrices have completions with a desired property. The (strictly) copositive matrix completion problem asks which partial symmetric matrices have a (strictly) copositive completion. Our purpose here is to answer these two questions. We assume, without loss of generality, that the diagonal entries are specified. An obvious necessary condition that a symmetric partial matrix B have a (strictly) copositive completion is that every fully specified principal submatrix of B be (strictly) copositive. Such a partial matrix is called partial (strictly) copositive. We show that in each case, the necessary condition is sufficient. Thus, the copositive problems are rather like the combinatorially symmetric P -matrix completion problem [6] and quite different from the positive (semi-)definite completion problem, for which complicated additional conditions are needed when the graph of the specified entries is not chordal.

We first analyse the copositive completion problems in the case of one symmetrically placed pair of unspecified entries. Since the property of (strict) copositivity is permutation similarity invariant, we may assume that the unspecified entry is in the upper right and lower left corners, without loss of generality.

Theorem 1 Let $A = \begin{pmatrix} a & b^T & ? \\ b & A' & c \\ ? & c^T & d \end{pmatrix}$ be a partial copositive matrix. Then $A = \begin{pmatrix} a & b^T & s \\ b & A' & c \\ s & c^T & d \end{pmatrix}$ is a copositive matrix for $s \geq \sqrt{ad}$. If A is partial strictly copositive then A with $s \geq \sqrt{ad}$ is strictly copositive. Furthermore, \sqrt{ad} is best possible in general.

PROOF. Let $x = (x_1, x', x_n)^T \geq 0$, where $x' \in \mathbf{R}^{n-2}$ and $x_1, x_n \in \mathbf{R}$. Then $x^T Ax = ax_1^2 + x'^T A' x' + dx_n^2 + 2sx_1 x_n + 2x_1 x'^T b + 2x_n x'^T c$. A is partial copositive, so if $x_n = 0$, $x^T Ax = ax_1^2 + x'^T A' x' + 2x_1 x'^T b \geq 0$, for any $x' \geq 0$ and $x_1 \geq 0$. Let $f(x_1) = ax_1^2 + 2x_1 x'^T b + x'^T A' x'$, then $f(x_1) \geq 0$, for any $x_1 \geq 0$.

If $x'^T b < 0$ then by choosing x_1 as large as desired, we must have $a > 0$. Then f has a minimum at $x_1 = -\frac{x'^T b}{a}$, and we have $f(-\frac{x'^T b}{a}) = x'^T A' x' - \frac{(x'^T b)^2}{a} \geq 0$.

Similarly, if $x'^T c < 0$ we have $x'^T A' x' - \frac{(x'^T c)^2}{d} \geq 0$.

If $x'^T b \geq 0$, then for $s \geq 0$ we have

$$x^T Ax = ax_1^2 + x'^T A' x' + dx_n^2 + 2sx_1 x_n + 2x_1 x'^T b + 2x_n x'^T c,$$

$$\begin{aligned}
&\geq x'^T A' x' + dx_n^2 + 2x_n x'^T c, \\
&= (0, x'^T, x_n) A (0, x', x_n)^T \geq 0.
\end{aligned}$$

Similarly, if $x'^T c \geq 0$ and $s \geq 0$ then $x^T A x \geq 0$.

Assume $x'^T b < 0$ and $x'^T c < 0$, and without loss of generality $\frac{x'^T b}{\sqrt{a}} \geq \frac{x'^T c}{\sqrt{d}}$.

If $s \geq \sqrt{ad}$ then

$$\begin{aligned}
x^T A x &\geq ax_1^2 + x'^T A' x' + dx_n^2 + 2\sqrt{ad}x_1x_n + 2x_1x'^T b + 2x_nx'^T c, \\
&= (\sqrt{a}x_1 + \sqrt{d}x_n)^2 + x'^T A' x' + 2\sqrt{a}x_1 \frac{x'^T b}{\sqrt{a}} + 2\sqrt{d}x_n \frac{x'^T c}{\sqrt{d}}, \\
&\geq (\sqrt{a}x_1 + \sqrt{d}x_n)^2 + 2(\sqrt{a}x_1 + \sqrt{d}x_n) \frac{x'^T c}{\sqrt{d}} + x'^T A' x', \\
&= [\sqrt{a}x_1 + \sqrt{d}x_n + \frac{x'^T c}{\sqrt{d}}]^2 - (\frac{x'^T c}{\sqrt{d}})^2 + x'^T A' x' \geq 0.
\end{aligned}$$

If A is partial strictly copositive then for $s \geq 0$, $x'^T b \geq 0$ implies $x^T A x \geq (0, x'^T, x_n) A (0, x', x_n)^T = (x'^T, x_n) \begin{pmatrix} A' & c \\ c^T & d \end{pmatrix} (x', x_n)^T > 0$. Similarly, $x'^T c \geq 0$ implies $x^T A x \geq (x_1, x'^T, 0) A (x_1, x', 0)^T = (x_1, x'^T)^T \begin{pmatrix} a & b^T \\ b & A' \end{pmatrix} (x_1, x')^T > 0$. If $x'^T b < 0$, then $(x_1, x') \neq 0$, so $f(x_1) > 0$, for any $x_1 \geq 0$, and so $x'^T A' x' - \frac{(x'^T b)^2}{a} > 0$, and if $x'^T c < 0$ then $x'^T A' x' - \frac{(x'^T c)^2}{d} > 0$. Then, as in the copositive case, for $s \geq \sqrt{ad}$ if $x'^T b < 0$ and $x'^T c < 0$, we have $x^T A x > 0$.

To show that no general improvement in the bound is possible take $A = \begin{pmatrix} 1 & -1 & s \\ -1 & 1 & -1 \\ s & -1 & 1 \end{pmatrix}$, and consider $x^T A x$, where $x = (1, 1, x_3)$ with x_3 small. In the strictly copositive case, take $A + \epsilon I$, $\epsilon > 0$, and argue by continuity. \square

Remark: Note that if $b, c \geq 0$, then $s \geq -\sqrt{ad}$ suffices, as $\begin{pmatrix} 0 & b^T \\ b & A' \end{pmatrix}$ and $\begin{pmatrix} A' & c \\ c^T & 0 \end{pmatrix}$ are copositive and $\begin{pmatrix} a & -\sqrt{ad} \\ -\sqrt{ad} & d \end{pmatrix}$ is copositive. We have $s \geq -\sqrt{ad}$, necessarily. It is an interesting question to determine the minimum s that yields copositivity, in terms of the specific data.

Now, consider a general partial (strictly) copositive matrix $B = (b_{ij})$, with fully specified diagonal and focus upon a particular symmetrically placed pair of unspecified entries, say b_{pq} , b_{qp} . This pair completes at least one, and perhaps several, maximal partial (strictly) copositive principal submatrices. For each of these, there is a common numerical value, namely $b_{ij} = b_{ji} = \sqrt{b_{ii}b_{jj}}$, that completes each to a (strictly) copositive matrix and, thus, produces a new partial (strictly) copositive matrix \hat{B} with one fewer pairs of unspecified entries. Thus, sequentially, B has a completion A that is (strictly) copositive. We note that no unspecified entry need be chosen any greater than the geometric mean of the two corresponding (specified) diagonal entries. We conclude

Theorem 2 *Let B be a partial (strictly) copositive $n \times n$ matrix. Then, there is a completion A of B that is (strictly) copositive.*

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