

Graph Theoretic Methods for Matrix Completion Problems

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Abstract A pattern is a list of positions in an $n \times n$ real matrix. A matrix completion problem for the class of \mathcal{P} -matrices asks whether every partial \mathcal{P} -matrix whose specified entries are exactly the positions of the pattern can be completed to a \mathcal{P} -matrix. We survey the current state of research on \mathcal{P} -matrix completion problems for many subclasses of P_0 -matrices, including positive definite matrices, M -matrices, inverse M -matrices, P -matrices, and matrices defined by various sign symmetry and positivity conditions on P_0 - and P -matrices. Graph-theoretic techniques used to study completion problems are discussed. Several new results are also presented, including the solution to the M_0 -matrix completion problem and the sign symmetric P_0 -matrix completion problem.

1. Classes of Matrices

A partial matrix is a matrix in which some entries are specified and others are not. A completion of a partial matrix is a specific choice of values for the unspecified entries. A pattern for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1, \dots, n\} \times \{1, \dots, n\}$. Note that a pattern may be empty. A partial matrix specifies the pattern if its specified entries are exactly those listed in the pattern. For a particular type of matrices, the matrix completion problem for patterns asks which patterns of positions have the property that any partial matrix of this type that specifies the pattern can be completed to a matrix of desired type. For example, for positive definite matrices, we ask which patterns have the property that any partial positive definite matrix specifying the pattern can be completed to a positive definite matrix.

This question was answered in [GJSW] through the use of graph theoretic techniques. In 1990 [J2] provided a survey of matrix completion problems, focusing on positive definite completions, rank completions and contraction completions. These papers led to a large number of results on matrix completion problems for patterns in recent years, including inverse M -matrices [JS1], [JS2], [H1], [H3], M -matrices

[H2], P-matrices [JK], [DH], and classes defined by sign symmetry and positivity conditions on P_0 -matrices [FJTU]. Although these papers use many of the same techniques, the terminology varies, with the same property or construction having multiple names. In this paper we attempt to clarify the terminology (Sections 1 and 2), describe some useful techniques (Sections 3, 4, and 5), illustrate these techniques by proving several new results (Sections 6, 7, 8, and 9), and survey the current state of knowledge about matrix completion problems for a variety of classes of matrices (Section 10). For a few classes, work has been done on completing individual partial matrices, but the focus of this paper is on the study of the completion problem for patterns of matrices.

A principal minor is the determinant of a principal submatrix. For a subset of $\{1, \dots, n\}$, the principal submatrix $A[\]$ is obtained from A by deleting all rows and columns not in $\]$. Similarly, the principal subpattern $Q[\]$ is obtained from a pattern Q by deleting all positions whose first or second coordinate is not in $\]$. Subpatterns are described more fully in Section 3.

The real matrix A is called positive stable (respectively, positive semistable) if all the eigenvalues of A have positive (non-negative) real part. A real matrix with the property that any principal minor is positive (respectively, non-negative) is called a P-matrix (P_0 -matrix). The class of P-matrices generalizes many important classes of matrices that arise in the study of positive stability, such as positive definite matrices, M-matrices, inverse M-matrices, Fischer and Koteljanskii matrices, all of which are discussed in [HJ2].

All the classes we discuss in this paper are listed in Table 1, along with their definitions. These 24 classes of matrices are all contained in the class of P_0 -matrices and are all called positivity classes. With the exception of classes of inverse M-matrices, each of the classes listed in Table 1 has a characterization as a class of P or P_0 -matrices with some additional conditions. These equivalent characterizations are listed with the standard definitions in Table 1, and we will use these characterizations as definitions. The property of being an inverse M-matrix is inherited by principal submatrices [HJ2], so an inverse M-matrix is a P-matrix.

Additional classes of matrices are obtained by imposing various symmetry conditions, including restrictions on the signs of entries. The symmetry conditions we will discuss are:

- weakly sign symmetric, abbreviated wss, which requires $a_{ij} a_{ji} \geq 0$ for each pair i, j
- sign symmetric, abbreviated ss, which requires $a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each pair i, j
- non-negative, which requires $a_{ij} \geq 0$ for all i, j
- positive, which requires $a_{ij} > 0$ for all i, j
- symmetric, which requires $a_{ij} = a_{ji}$ for each pair i, j

Table 1: Definitions of Classes of matrices

Class	Definition of a -matrix A	Definition of a partial -matrix A Every fully specified principal submatrix of A is a -matrix and whenever the listed entries are specified,
P_0 -matrices	determinant of every principal submatrix ≥ 0	
$P_{0,1}$ -matrices	P_0 -matrix and all diagonal entries > 0	
P-matrices	determinant of every principal submatrix > 0	
wss P_0 -matrices	P_0 -matrix and $a_{ij} a_{ji} \geq 0$ for each i, j	$a_{ij} a_{ji} \geq 0$
wss $P_{0,1}$ -matrices	$P_{0,1}$ -matrix and $a_{ij} a_{ji} \geq 0$ for each i, j	$a_{ij} a_{ji} \geq 0$
M_0 -matrices	positive semistable and $a_{ij} \leq 0$ for all $i \neq j$ P_0 -matrix and $a_{ij} \leq 0$ for all $i \neq j$	$a_{ij} \leq 0$ if $i \neq j$
wss P-matrices	P-matrix and $a_{ij} a_{ji} \geq 0$ for each i, j	$a_{ij} a_{ji} \geq 0$
M-matrices	positive stable and $a_{ij} \leq 0$ for all $i \neq j$ P-matrix and $a_{ij} \leq 0$ for all $i \neq j$	$a_{ij} \leq 0$ if $i \neq j$
Fischer matrices	P-matrix and for $\alpha = 1, 2, \dots, n$ $\det(A[\alpha]) \leq \det(A[\alpha-1]) \det(A[\alpha+1])$	$a_{ij} a_{ji} \geq 0$
Koteljanskii matrices	P-matrix & (with $\det(A[\alpha])$ defined to be 1) $\det(A[\alpha]) \det(A[\alpha+1]) \leq \det(A[\alpha-1]) \det(A[\alpha+2])$	$a_{ij} a_{ji} \geq 0$
ss P_0 -matrices	P_0 -matrix and $a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each i, j	$a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$
ss $P_{0,1}$ -matrices	$P_{0,1}$ -matrix and $a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each i, j	$a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$
ss P-matrices	P-matrix and $a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$ for each i, j	$a_{ij} a_{ji} > 0$ or $a_{ij} = 0 = a_{ji}$
non-negative P_0 -matrices	P_0 -matrix and $a_{ij} \geq 0$ for all i, j	$a_{ij} \geq 0$
non-negative $P_{0,1}$ -matrices	$P_{0,1}$ -matrix and $a_{ij} \geq 0$ for all i, j	$a_{ij} \geq 0$
non-negative P-matrices	P-matrix and $a_{ij} \geq 0$ for all i, j	$a_{ij} \geq 0$
inverse M-matrices	inverse of an M-matrix	$a_{ij} \geq 0$
positive $P_{0,1}$ -matrices	$P_{0,1}$ -matrix and $a_{ij} > 0$ for all i, j	$a_{ij} > 0$
positive P-matrices	P-matrix and $a_{ij} > 0$ for all i, j	$a_{ij} > 0$
positive semidefinite	symmetric and $x^t A x \geq 0$ symmetric P_0 -matrix	$a_{ji} = a_{ij}$
positive definite	symmetric and $x^t A x > 0$ for all $x \neq 0$ symmetric P-matrix	$a_{ji} = a_{ij}$
symmetric M_0 -matrices	symmetric positive semistable and $a_{ij} \leq 0$ for all $i \neq j$ symmetric P_0 -matrix and $a_{ij} \leq 0$ for all $i \neq j$	$a_{ji} = a_{ij}$ and $a_{ij} \leq 0$ if $i \neq j$
symmetric M-matrices	symmetric positive stable and $a_{ij} \leq 0$ for all $i \neq j$ symmetric P-matrix and $a_{ij} \leq 0$ for all $i \neq j$	$a_{ji} = a_{ij}$ and $a_{ij} \leq 0$ if $i \neq j$
symmetric inverse M-matrices	symmetric and inverse of an M-matrix	$a_{ji} = a_{ij}$ and $a_{ij} \geq 0$ if $i \neq j$

When we want to refer to any one of the classes of matrices listed in Table 1, we will say “for a class of matrices” or “for all of the classes of matrices” (without qualification). On other occasions we refer to only some of these classes, and then we will say “for one of the classes...”

For all of the classes of matrices, membership in the class is inherited by principal submatrices. Thus in order for a partial n -matrix to have a completion to a n -matrix, it is certainly necessary that every fully specified principal submatrix be a n -matrix. For one of the classes P_0 -, $P_{0,1}$ -, and P -matrices, this is sufficient to define a partial n -matrix. Some of the classes, such as M -matrices and ss P -matrices, have sign patterns, so any specified entries must satisfy the sign pattern. For each of the classes, the definition of a partial matrix n -matrix is listed in the third column of Table 1, in the form: every fully specified principal submatrix is a n -matrix, and _____, where _____ is whatever additional condition is listed in the third column of Table 1.

For a class of matrices, we say a pattern has n -completion if every partial n -matrix specifying the pattern can be completed to a n -matrix (note that an empty pattern has n -completion for every n). The matrix completion problem for the class of n -matrices is to determine which patterns have n -completion.

2. Graphs and Digraphs of Patterns

In recent years graphs and digraphs have been used very effectively to study matrix completion problems. One usually begins by assuming that the pattern includes all diagonal positions. The case in which some diagonal positions are omitted is discussed later.

For a class of symmetric matrices, such as positive definite matrices, since the j,i -entry is equal to the i,j -entry, it makes sense to require the pattern to be positionally symmetric, i.e., if position (i,j) is in the pattern, then so is (j,i) . Note that positionally symmetric patterns are also called combinatorially symmetric patterns. Both terms are in use ("positionally symmetric" in [JS1], [JS2], [H1], [H2], [H3], [DH], "combinatorially symmetric" in [JK], [J2], [FJTU]); here we will use the term "positionally symmetric". Patterns that are positionally symmetric can be studied by means of their graphs, and one usually begins with this case ([JS1], [JS2], [JK], [FJTU]). The definitions of graph-theoretic terms will follow [CO] unless otherwise noted.

For our purposes, a graph $G=(V(G),E(G))$ is a finite nonempty set of positive integers $V(G)$, whose members are called vertices, and a set $E(G)$ of (unordered) pairs $\{v,u\}$ of vertices, called edges. This differs from standard use in that we require vertices to be positive integers (since we will be using them to represent matrices). When it is clear what graph G is under discussion, we will use V for $V(G)$ and E for $E(G)$. If $\{v,u\}$ is an

edge of G , then we say that v and u are adjacent in G and $\{v,u\}$ is incident with both v and u . The order of G is the number of vertices of G .

A subgraph of the graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$, where $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$ (note that $\{v,u\} \in E(H)$ requires $v,u \in V(H)$ since H is a graph). If W is a subset of V , the subgraph induced by W , $\langle W \rangle$, is the graph $(W, E(W))$ with $E(W)$ the set of all the edges of G between the vertices in W . A subgraph induced by a subset of vertices is also called an induced subgraph. A warning about terminology: The terms “subgraph” and “subdigraph” (see below) have been used in some papers ([JS1], [H2]) to mean induced subgraph (or induced subdigraph). This use is undesirable and contrary to usage in graph theory.

Let A be a (fully specified) symmetric $n \times n$ matrix. The nonzero-graph of A is the graph having as vertex set $\{1, \dots, n\}$, and, as its set of edges, the set of (unordered) pairs $\{i, j\}$ such that both i and j are vertices with $i \neq j$ and $a_{ij} \neq 0$. If G is the nonzero-graph of A , then the nonzero-graph of the principal submatrix $A[i]$ is $\langle i \rangle$.

In many situations, positions omitted from a pattern are viewed in the same way as 0 entries in a matrix. To facilitate this analogy, we make the following definition: The characteristic matrix of a pattern for $n \times n$ matrices is the $n \times n$ matrix C such that $c_{ij} = 1$ if the position (i, j) is in the pattern and $c_{ij} = 0$ if (i, j) is not in the pattern.

For a positionally symmetric pattern Q that includes all diagonal positions, the graph of Q (pattern-graph) is the nonzero-graph of its characteristic matrix C . Equivalently, it is the graph having $\{1, \dots, n\}$ as its vertex set and, as its set of edges, the set of (unordered) pairs $\{i, j\}$ such that position (i, j) (and therefore also (j, i)) is in Q . If G is the pattern-graph of Q , then the pattern-graph of a principal subpattern $Q[i]$ is $\langle i \rangle$, which is the nonzero-graph of $C[i]$.

If a class of matrices does not allow the j, i -entry to be determined from the i, j -entry, the general \square -completion problem entails the study of patterns that are not positionally symmetric (even if the class has some mild symmetry properties such as sign symmetry). For a pattern without positional symmetry, a directed graph (digraph) must be used.

A digraph $D = (V(D), E(D))$ is a finite set of positive integers $V(D)$, whose members are called vertices, and a set $E(D)$ of ordered pairs (v, u) of vertices, called arcs (also called directed edges). The terms adjacent, incident, order, subdigraph and induced subdigraph are defined analogously to the corresponding terms for graphs. Note that [CO] uses “to” and “from” rather than “with” in conjunction with “adjacent” and “incident” to distinguish the ends of an arc, but we do not need this distinction.

Let A be a (fully specified) $n \times n$ matrix. The nonzero-digraph of A is the digraph having vertex set $\{1, \dots, n\}$ and, as arcs, the ordered pairs (i, j) where $i \neq j$ and $a_{ij} \neq 0$. The nonzero-digraph of $A[i]$ is $\langle i \rangle$. For a

pattern Q that contains all diagonal positions, the digraph of Q (pattern-digraph) is the nonzero-digraph of its characteristic matrix C . The pattern-digraph $Q[\]$ is $\langle \rangle$. A partial matrix that specifies a pattern is also referred to as specifying the digraph or graph of the pattern.

The positionally symmetric case remains important even when digraphs are used. Reduction to this case is a very effective strategy for showing that a particular digraph has $\bar{0}$ -completion by showing that for any partial $\bar{0}$ -matrix specifying the digraph it is possible to specify additional entries to obtain a partial $\bar{0}$ -matrix specifying a positionally symmetric pattern that has $\bar{0}$ -completion. This technique is used in [H1] and [H3] for inverse M-matrix completions, in [H2] for M-matrices, and [DH] for P-matrices.

For some classes $\bar{0}$, the case of a pattern that includes all diagonal positions is the only part of the problem that needs to be studied, because a pattern has $\bar{0}$ -completion if and only if the principal subpattern defined by the diagonal positions included in the pattern has $\bar{0}$ -completion ([GJSW], [JK], [H2]). However, one must be careful when determinants can be 0 or other conditions apply, as we shall see in Section 4.

For a pattern that does not include the diagonal, we need to revise our digraph notation. A marked directed graph (mardigraph) $D = (V(D), E(D), M(D))$ is a digraph $(V(D), E(D))$ together with a function $M(D)$ from $V(D)$ into the set of marks $\{S, U\}$. A vertex is referred to as specified or unspecified according as it is assigned by $M(D)$ to S or U . The nonzero-mardigraph of the $n \times n$ matrix A is the nonzero-digraph of A in which the vertex i is marked specified if $a_{ii} \neq 0$ and unspecified if $a_{ii} = 0$. The mardigraph of a pattern (pattern-mardigraph) for $n \times n$ matrices is the nonzero-mardigraph of its characteristic matrix. A submardigraph $H = (V(H), E(H), M(H))$ of D is a subdigraph whose $M(H)$ is the restriction of $M(D)$ to $V(H)$. Other digraph terms (order, induced subdigraph) are applied to mardigraphs in the obvious way. The nonzero-mardigraph of $A[\]$ is $\langle \rangle$. The pattern-mardigraph $Q[\]$ is $\langle \rangle$. A partial matrix that specifies a pattern is also referred to as specifying the mardigraph of the pattern.

In many cases, we need to say something that applies to graphs, digraphs and/or mardigraphs; we will use the term "mar/di/graph" to mean "mardigraph or digraph or graph," "mar/digraph" to mean "mardigraph or digraph," and "di/graph" to mean "digraph or graph."

Diagrams help to visualize a mar/di/graph. For di/graphs, the vertices are diagrammed as black dots. The edges are drawn as line segments, oriented by arrows for a digraph. Figure 1(a) represents the graph of the positionally symmetric pattern $\{(1,1), (1,2), (1,4), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,1), (4,3), (4,4)\}$. When both arcs (i,j) and (j,i) are present in a digraph, the arrows can be omitted, and these are represented by a double line ([CO] continues to show arrows, and

curves the lines). Figure 1(b) represents the digraph of $\{(1,1), (1,2), (1,3), (2,1), (2,2), (3,2), (3,3)\}$. Note that the diagram of a digraph of a positionally symmetric pattern would have a double line everywhere the diagram of the graph has a single line. When diagramming a mardigraph, a specified vertex will be indicated by a solid black dot (\bullet) and an unspecified vertex will be indicated by a hollow circle (\circ). Figure 1(c) represents the mardigraph of the pattern $\{(1,1), (1,2), (1,3), (2,1), (2,3), (3,3)\}$. Note that this is consistent with the usage for di/graphs, because when we used di/graph for patterns, the pattern included all diagonal positions. When describing a set of mardigraphs, a vertex that may be either specified or unspecified is diagrammed as $\bullet\circ$ (cf. Figure 8).

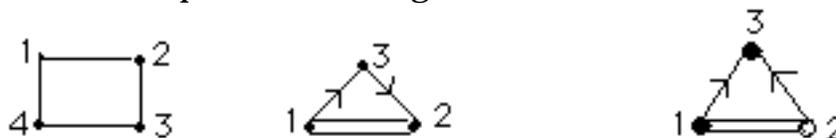


Figure 1 (a) A graph (b) A digraph (c) A mardigraph

Relabeling the vertices of a mar/di/graph corresponds to performing a permutation similarity on the pattern. Since all the classes we are considering are closed under permutation similarity, we are free to relabel mar/di/graphs as desired, or use unlabeled mar/di/graphs.

Up to this point we have looked at mar/di/graphs as derived from patterns, but we can also look at patterns derived from mardigraphs. Let $D = (V, E, M)$ be a mardigraph. The pattern associated with D is the pattern $Q_D = E \cup \{(v,v) : v \in V \text{ and } M(v) = S\}$. The mardigraph of Q_D is D . Analogously, there is a pattern that includes all diagonal positions associated with a digraph on $\{1, \dots, n\}$ and a positionally symmetric pattern (that includes all diagonal positions) associated with a graph on $\{1, \dots, n\}$. Thus a partial matrix specifies a mar/di/graph D if it specifies the pattern associated with D . For a class \mathcal{C} of matrices, a mar/di/graph D is said to have \mathcal{C} -completion if the pattern associated with D has \mathcal{C} -completion.

The underlying graph D' of a mar/digraph D is the graph obtained by replacing each arc (v,u) or pair of arcs (v,u) and (u,v) (if both are present) with the one edge $\{v,u\}$ (marks are ignored). Note that the underlying graph of the digraph of a positionally symmetric pattern that includes the diagonal is the graph of the pattern. Arc (v,u) (or arcs (v,u) and (u,v) if both are present) of D and edge $\{v,u\}$ of D' are said to correspond.

A walk in a mar/digraph D (respectively, graph G) is a sequence of arcs $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$ (edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}$). A walk is closed if $v_1 = v_k$. A path is a walk in which the vertices are distinct (except that possibly $v_1 = v_k$). The length of a path is the number of arcs (edges) in the path. A cycle is a closed path. The mar/digraph cycle $(v_1, v_2), (v_2, v_3), \dots, (v_k, v_1)$ or the graph cycle $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_k, v_1\}$ is sometimes denoted $v_1, v_2, \dots, v_k, v_1$. We will abuse notation to talk about the mar/di/graph induced by a path or cycle, i.e., if \mathcal{C} is the cycle $v_1,$

v_2, \dots, v_k, v_1 , then $\langle \rangle$ means $\langle \{v_1, v_2, \dots, v_k\} \rangle$. A cycle of length k is called a k -cycle. A Hamilton cycle of a mar/di/graph D is a cycle in D that includes every vertex of D .

A semiwalk (respectively, semipath, semicycle) of a mar/digraph D is a sequence of arcs whose corresponding edges form a walk (path, cycle) in the underlying graph D' . It is sometimes convenient to write the walk $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$ as $v_1 \quad v_2 \quad \dots \quad v_k$. In a semiwalk where the direction is known, it is denoted by \rightarrow or \leftarrow ; when it is unknown it is denoted \cdot .

A chord of the cycle $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$ is an edge $\{v_s, v_t\}$ not in the cycle (with $1 \leq s, t \leq k$). A chord of a cycle in a mar/digraph is an arc whose corresponding edge is a chord of the corresponding cycle in the underlying graph. A graph is chordal if every cycle of length ≥ 4 has a chord; a mar/digraph is chordal if its underlying graph is chordal.

In a mar/digraph D , (v, u) is called the reverse of (u, v) (whether it is present in D or not). The term "reverse" has been used in [DH]; the reverse has also been referred to as the opposite [H3]. An arc (v, u) in D is symmetric in D if its reverse is also in D ; otherwise (v, u) is asymmetric in D . A submar/digraph H of D is symmetric (respectively, asymmetric) if every arc of H is symmetric (asymmetric) in D . A mar/digraph D is homogeneous if it is either symmetric or asymmetric; otherwise it is nonhomogeneous.

A cycle \cdot in a mar/digraph D is called simple if the arc set of the induced submar/digraph of \cdot is \cdot , i.e., D does not contain the reverse of any arc in \cdot and D does not contain any chord of \cdot , or the length of \cdot is 2. Note that the term "simple" has also been used to mean a chordless cycle in a graph [FJTU].

A graph is connected if there is a path from any vertex to any other vertex (this includes a graph of order 1); otherwise it is disconnected. A component of a graph is a maximal connected subgraph. A mar/digraph is connected if its underlying graph is connected (equivalently, if there is a semipath from any vertex to any other vertex), and a component of the mar/digraph is a component of the underlying graph. Connectedness ignores the orientation of arcs in mar/digraphs, so when orientation needs to be considered, the concept of strong connectedness is used: A mar/digraph is strongly connected if there is a path from any vertex to any other vertex. Clearly, a strongly connected mar/digraph is connected, although the converse is false.

A di/graph is complete if it includes all possible arcs (edges) between its vertices. The vertices are also required to be specified in order for a mardigraph to be called complete. A complete subdi/graph has traditionally been referred to as a clique [JS1], [FJTU], [H1]. Thus, a submardigraph F of a mardigraph D is called a clique if all vertices of F are specified and for any $v, u \in V(F)$, $(v, u), (u, v) \in E(F)$. If Q is a pattern, D is

its mardigraph and F is the submardigraph induced by a subset T of $\{1, \dots, n\}$, then F is a clique if and only if the principal subpattern defined by T is $T \times T$.

3. Block Diagonal and Block Triangular Patterns

Graphs were used in [GJSW] to determine which positionally symmetric patterns have positive definite completion, and mar/di/graphs have subsequently been used for many other classes of matrices ([JS1], {FJTU}, [DH], [H1], [H2], [H3]). There are two fundamental ways to attack the \mathbb{R} -matrix completion problem: collecting mar/di/graphs that have \mathbb{R} -completion and collecting mar/di/graphs that don't. An effective strategy for mar/di/graphs that have \mathbb{R} -completion is to build new mar/di/graphs having \mathbb{R} -completion from old (i.e., ones known to have \mathbb{R} -completion). There are a number of different building techniques. We will discuss two, based on block diagonal and block triangular matrices, in this section, and another, vertex identification, in Section 5. To effectively make use of mar/di/graphs that don't have \mathbb{R} -completion, we need to have \mathbb{R} -completion be inherited by induced submar/di/graphs. This is equivalent to having \mathbb{R} -completion inherited by principal submatrices. We begin with this property.

Note that all of the classes under discussion that allow off-diagonal entries to be zero are closed under direct sums of matrices, i.e., if A_1, \dots, A_k are \mathbb{R} -matrices, then the block diagonal matrix with diagonal blocks A_1, \dots, A_k must also be in \mathbb{R} . This statement is immediate from properties of the determinant for all but Fischer and Koteljanskii matrices; for these classes see [FJTU].

Lemma 3.1 For any class \mathcal{C} of matrices except the class of positive $P_{0,1}$ -matrices, if the pattern Q has \mathbb{R} -completion, then any principal subpattern R has \mathbb{R} -completion. Equivalently, for these classes, if a mar/di/graph has \mathbb{R} -completion, then so does any induced submar/di/graph.

Proof: Note that for symmetric classes (positive semidefinite, positive definite, symmetric M -, symmetric M_0 -, symmetric inverse M -matrices) patterns must be positionally symmetric, i.e., mardigraphs are symmetric.

For classes \mathcal{C} that allow entries to be 0 (all of the above classes except positive P -matrices), if A is a partial \mathbb{R} -matrix specifying R , extend it to a partial matrix B specifying Q by assigning 1 to the diagonal and 0 to the off-diagonal entries that need to be specified. This creates a partial \mathbb{R} -matrix, because the class of \mathbb{R} -matrices is closed under direct sums. Since Q has \mathbb{R} -completion, we can complete B to a \mathbb{R} -matrix \hat{B} . The appropriate principal submatrix of \hat{B} then completes A . When entries are not allowed to be zero, but determinants are strictly positive, such as for positive P -

matrices, we can perturb the off-diagonal entries slightly and still have a P-matrix. ■

The one class exempted from Lemma 3.1 requires nonzero entries, and for this class there are problems with using the perturbation argument because determinants can be 0. As we shall see in the next section, strange things happen when the determinants can be 0, which leads to questions about whether principal submatrices inherit \mathcal{P} -completion for this class, but no examples where it is not inherited are known.

We now turn to building strategies. As we will need to discuss block structure for both matrices and patterns, we extend the terminology for partitioning matrices [HJ1] to patterns. Let Q be a pattern for $n \times n$ matrices. For subsets I and J of $\{1, \dots, n\}$ the block $Q[I, J]$ is the result of deleting all ordered pairs in Q whose first coordinate is not in I or whose second coordinate is not in J . When $I = J$, $Q[I, I]$ is denoted $Q[I]$ and is called a principal subpattern of Q . A partition of Q is a decomposition of Q into mutually exclusive blocks such that each position is in exactly one block. We concern ourselves only with the case in which the rows and columns are partitioned the same way, and the partition is of the form $\{I_1, \dots, I_K\}$ where $I_1 = \{1, \dots, s_1\}$, $I_2 = \{s_1+1, \dots, s_2\}, \dots$, $I_K = \{s_{K-1}+1, \dots, n\}$. In

this case we abuse notation to write $Q =$

$$\begin{array}{cccc} Q_{11} & Q_{12} & \dots & Q_{1K} \\ Q_{21} & Q_{22} & \dots & Q_{2K} \\ \vdots & \vdots & & \vdots \\ Q_{K1} & Q_{K2} & \dots & Q_{KK} \end{array}$$

where $Q_{IJ} = Q[I, J]$. We say $\{I_1, \dots, I_K\}$ provides a block structure for patterns of $n \times n$ matrices.

A pattern Q is block diagonal (for a given block structure) if and only if its characteristic matrix is block diagonal for the block structure.

In this case we abuse the notation to write $Q =$

$$\begin{array}{cccc} Q_{11} & ? & \dots & ? \\ ? & Q_{22} & \dots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \dots & Q_{KK} \end{array}$$

where $?$ indicates a rectangular set of positions not included in the pattern. Note that this is equivalent to saying that Q is the union of principal subpatterns Q_{11}, \dots, Q_{KK} . Also, it is possible that Q_{11}, \dots, Q_{KK} are missing some positions.

When studying the \mathcal{P} -completion problem by means of mar/di/graphs, one routinely assumes that the mar/di/graph is connected, and then deals with the more general case by concluding that if each component has \mathcal{P} -completion, then so does the whole mar/di/graph. This reduction is possible if and only if a pattern must have \mathcal{P} -completion if it is block diagonal and each diagonal block has \mathcal{P} -completion.

Lemma 3.2 For any class of matrices except the class of positive $P_{0,1}$ -matrices, if the pattern Q is permutation similar to a block diagonal pattern in which each diagonal block has \mathcal{C} -completion, then Q has \mathcal{C} -completion. Equivalently, if each component of a mar/di/graph has \mathcal{C} -completion, then so does the mar/di/graph.

Proof: Note that for symmetric classes patterns must be positionally symmetric, i.e., mardigraphs are symmetric.

Let Q be a block diagonal pattern such that the principal subpattern of each diagonal block has \mathcal{C} -completion. If A is a partial \mathcal{C} -matrix specifying Q , complete each of the principal submatrices specifying the diagonal blocks to a \mathcal{C} -matrix, and assign 0 to all entries outside the diagonal blocks. When entries are not allowed to be zero, but determinants are strictly positive, such as for positive P -matrices, we can perturb the off-diagonal entries slightly and still have a P -matrix. This completes A to a \mathcal{C} -matrix, so Q has \mathcal{C} -completion. Since all the classes under discussion are closed under permutation similarity, any pattern that is permutation similar to Q also has \mathcal{C} -completion. ■

Again, there are problems with this argument when determinants can be 0, as with the class of positive $P_{0,1}$ -matrices. Unless the questions of whether the class of positive $P_{0,1}$ -matrices has the properties that completion is inherited by principal subpatterns and closure under unions can be answered affirmatively, the study of this class is problematic.

All of the classes under discussion that are not symmetric and allow off-diagonal entries to be zero (except inverse M -matrices) have the property that if A_1, \dots, A_K are \mathcal{C} -matrices, then a block triangular matrix with diagonal blocks A_1, \dots, A_K must also be a \mathcal{C} -matrix. This is clear for all these classes except Fischer and Koteljanskii matrices; for these classes see [FJTU].

A pattern is called block triangular (for a particular block structure) if its characteristic matrix is block triangular (for that block structure). A pattern Q is called reducible if its characteristic matrix C is reducible, i.e.,

if there is a permutation matrix P such that $PCP^T = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$ where $C_{11},$

C_{22} are square and 0 denotes a matrix consisting entirely of 0s. A pattern is irreducible if it is not reducible, or, equivalently, if its characteristic matrix is irreducible. Any matrix, and hence any pattern, is permutation similar to a block triangular matrix with irreducible diagonal blocks. The nonzero-mar/digraph of a matrix A is strongly connected if and only if A is irreducible [V], and hence a pattern-mardigraph is strongly connected if and only if the pattern is irreducible.

Lemma 3.3 If the pattern Q is permutation similar to a block triangular pattern and each diagonal block has \mathcal{C} -completion, then Q has \mathcal{C} -completion for any of the classes P_0 -, $P_{0,1}$ -, P -, wss P_0 -, wss $P_{0,1}$ -, M_0 -, wss P -, M -, Fischer, Koteljanskii, ss P -, non-negative P_0 -, non-negative $P_{0,1}$ -, non-negative P -, positive P -matrices. Equivalently, for these classes, if every strongly connected induced submar/di/graph of the mar/di/graph G has \mathcal{C} -completion, then so does G .

Proof: Let Q be a block triangular pattern such that all diagonal blocks have \mathcal{C} -completion. Let A be a partial \mathcal{C} -matrix specifying Q . We can complete A to a \mathcal{C} -matrix \hat{A} by completing each diagonal block to a \mathcal{C} -matrix and setting all other unspecified entries to 0 (and perturbing the zero entries slightly for ss or positive P -matrices). Since all the classes under discussion are closed under permutation similarity, any pattern permutation similar to Q also has \mathcal{C} -completion. ■

The class of inverse M -matrices does not have the property in Lemma 3.3 [H1] (see also Section 6), nor do the classes of ss P_0 - or $P_{0,1}$ -matrices, as we see in the next example (an even simpler example for ss P_0 -matrices appears in Section 9):

Example 3.4 The partial matrix $A = \begin{matrix} & 4 & 2 & x \\ & 2 & 1 & y \\ & 4 & -1 & 1 \end{matrix}$ is a partial ss $P_{0,1}$ -matrix

specifying the pattern $Q = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (3,3)\}$. But A does not have a ss $P_{0,1}$ -completion because $\det A = -6x + 12y$ and $x > 0$ and $y < 0$ by sign symmetry. However, Q is block triangular with diagonal blocks $\{(1,1), (1,2), (2,1), (2,2)\}$ and $\{(3,3)\}$, both of which are complete. (Since A is a partial ss $P_{0,1}$ -matrix, it is also a partial ss P_0 -matrix.)

4. Patterns Omitting some Diagonal Positions

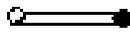
For some classes \mathcal{C} , a pattern has \mathcal{C} -completion if and only if the principal subpattern defined by the diagonal positions included in the pattern has \mathcal{C} -completion. This property is often described by saying that the problem of classifying patterns with \mathcal{C} -completion reduces to the classification of patterns that include the diagonal. This has been observed by numerous authors ([GJSW], [JK], [FJTU], [H2]), but care must be taken not to apply this where determinants can be 0 or other conditions apply, as the following examples illustrate.

Example 4.1 The partial matrix $A = \begin{matrix} & 0 & -1 \\ & -1 & z \end{matrix}$, which specifies the pattern

$Q = \{(1,1), (1,2), (2,1)\}$, is a partial P_0 -, wss P_0 -, M_0 -, ss P_0 -, positive semidefinite, and symmetric M_0 -matrix. The partial matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$,

which specifies Q , is a partial non-negative P_0 -matrix. But neither A nor B can be completed to a P_0 -matrix because $\det A = \det B = -1$. Therefore A cannot be completed to a P_0 -matrix for any of the classes P_0 -, wss P_0 -, M_0 -, ss P_0 -, positive semidefinite, symmetric M_0 -matrices. Similarly, B cannot be completed to a non-negative P_0 -matrix. However, the principal subpattern defined by the diagonal positions included in Q is $\{(1,1)\}$, which trivially has P_0 -completion.

This leads us to the following necessary condition for completion of patterns for these classes.

Lemma 4.2 Let \mathcal{C} be one of the classes P_0 -, wss P_0 -, M_0 -, ss P_0 -, non-negative P_0 -, positive semidefinite, and symmetric M_0 -matrices. If the pattern Q has \mathcal{C} -completion and Q omits position (j,j) and includes (i,i) , then Q omits at least one of the positions (i,j) and (j,i) . Equivalently, if a mardigraph D contains  as a submardigraph, then D does not have \mathcal{C} -completion.

For the classes listed in Lemma 4.2 and many others, the problem of classifying patterns with \mathcal{C} -completion does not reduce to the case of patterns that include the diagonal.

Example 4.3 The pattern $Q = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}$ has the mardigraph shown in Figure 2(a). The partial matrix $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & z \end{pmatrix}$ is a partial $P_{0,1}$ -matrix specifying Q for any of the classes $P_{0,1}$ -, wss $P_{0,1}$ -, ss $P_{0,1}$ -, non-negative $P_{0,1}$ -, positive $P_{0,1}$ - matrices.

But A cannot be completed to a $P_{0,1}$ -matrix because $\det A = -1$, and thus A cannot be completed to a $P_{0,1}$ -matrix for one of the listed classes. However, the principal subpattern defined by the diagonal positions included in the Q is $\{(1,1), (1,2), (2,1), (2,2)\}$, which trivially has $P_{0,1}$ -completion.

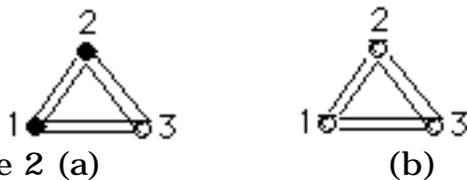


Figure 2 (a)

(b)

Example 4.4 The pattern $Q = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$ has the pattern-mardigraph shown in Figure 2(b). The partial matrix

$$A = \begin{bmatrix} z_1 & 0 & 1 \\ 1 & z_2 & 0 \\ 0 & 1 & z_3 \end{bmatrix}$$
 is a partial Koteljanskii matrix (and thus a partial Fischer matrix) specifying Q . But A cannot be completed to a Fischer matrix because with $I = \{1\}$ and $J = \{2,3\}$, $A[I, J] = [z_1]$, $A[J, I] = \begin{bmatrix} z_2 & 0 \\ 1 & z_3 \end{bmatrix}$, $A[I, I] = A$, and $\det A[I, J] = z_1 z_2 z_3 + 1 > z_1 z_2 z_3 = \det A[I, I] \det A[J, I]$. However, the principal subpattern defined by the diagonal positions included in Q is empty and so trivially has Fischer and Koteljanskii completion.

Example 4.5 The pattern $Q = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$ has the mardigraph shown in Figure 2(b). The partial matrix $A = \begin{bmatrix} ? & 1 & 0 \\ 1 & ? & 1 \\ 0 & 1 & ? \end{bmatrix}$ is a partial inverse M- and symmetric inverse M-matrix specifying Q . But A cannot be completed to an inverse M-matrix because it has nonzero entries $a_{12} = 1$, $a_{23} = 1$, but $a_{13} = 0$ [H3] (see also Section 6). However, the principal subpattern defined by the diagonal positions included in Q is empty, which trivially has (symmetric) inverse M-completion.

Theorem 4.6 For one of the classes P-, wss P-, M-, ss P-, non-negative P-, positive P-, positive definite, symmetric M-matrices, a pattern has \mathcal{C} -completion if and only if the principal subpattern defined by the diagonal positions included in the pattern has \mathcal{C} -completion. This statement is false for any other class of matrices in Table 1.

Proof: Note that for symmetric classes patterns must be positionally symmetric. The “only if” follows from Lemma 3.1.

Suppose Q is a pattern with the property that the principal submatrix defined by specified diagonal positions has \mathcal{C} -completion, for one of the listed classes. If Q contains at least one diagonal position, then by a permutation similarity we may assume that the diagonal positions included in Q are $1, \dots, k$; otherwise, let $k=0$.

Let A be a matrix that specifies Q . If $k>0$, then by hypothesis the principal submatrix $A[\{1, \dots, k\}]$ defined by $1, \dots, k$ can be completed to a \mathcal{C} -matrix. Complete it in A (call this matrix B); otherwise, let $B = A$.

Consider the principal submatrix $B[\{1, \dots, k, k+1\}]$. Since each class has a characterization as P-matrices, possibly with a symmetry condition, in order to complete $B[\{1, \dots, k, k+1\}]$ to a \mathcal{C} -matrix it is enough to complete it so that the determinant of every principal submatrix is positive and any symmetry conditions are satisfied.

All entries of $B[\{1, \dots, k, k+1\}]$ except some of those in the last row and column are specified. Let the unspecified $k+1, k+1$ -entry be denoted by z . Assign any other unspecified entries to satisfy the symmetry condition (if

any). If $k=0$, the determinant of $B[\{1, \dots, k, k+1\}]$ is z ; otherwise this determinant can be computed by expanding by minors on the last row, $\det B[\{1, \dots, k, k+1\}] = \pm b_{k+1, 1} M_{k+1, 1} \pm b_{k+1, 2} M_{k+1, 2} \dots + z M_{k+1, k+1}$ where $M_{k+1, i}$ is the $k+1, i$ -minor. $M_{k+1, k+1}$ is the determinant of $B[\{1, \dots, k\}]$ and thus is positive. By choosing z sufficiently large we can ensure $\det B[\{1, \dots, k, k+1\}]$ is positive.

This same technique works for the determinant of any principal submatrix containing the $(k+1, k+1)$ position. Since there are only finitely many such principal submatrices, z can be chosen sufficiently large to ensure all these determinants are positive. Complete the other unspecified diagonal entries in the same way.

The second statement follows from the examples preceding the theorem. ■

Although omitting some diagonal positions seemed to prevent γ -completion for some classes, the situation can be very different when all diagonal positions are omitted.

Theorem 4.7 For a class \mathcal{C} of matrices in Table 1, every mardigraph with all vertices unspecified has γ -completion if and only if \mathcal{C} is not one of Fischer, Koteljanskii, inverse M-, symmetric inverse M-matrices.

Proof: Note that for symmetric classes patterns must be positionally symmetric, i.e., mardigraphs are symmetric.

For any of the classes in Table 1 except Fischer, Koteljanskii, inverse M-, symmetric inverse M-matrices, the definition consists of (possibly) some symmetry conditions and the requirement that all principal minors be positive or non-negative. Set all the diagonal entries equal to z . Then any $k \times k$ principal minor will be $z^k + p(z)$ where $p(z)$ is a polynomial of degree $\leq k-1$, so choosing z large enough guarantees the principal minor is positive. Since there are only finitely many principal minors to consider, we can choose a sufficiently large z .

The converse follows from the examples preceding the theorem. ■

Notice that for classes where diagonal entries must be strictly positive, the examples that did not have completion were all 3×3 or larger matrices.

Lemma 4.8 Let \mathcal{C} be one of the classes $P_{0,1^-}$, P^- , wss $P_{0,1^-}$, wss P^- , M-, Fischer, Koteljanskii, ss $P_{0,1^-}$, ss P^- , non-negative $P_{0,1^-}$, non-negative P^- , inverse M-, positive $P_{0,1^-}$, positive P-matrices. Any pattern for 2×2 matrices has γ -completion, i.e., all mardigraphs of order 2 (shown in Figure 3) have γ -completion.

Proof: We first consider all the listed classes except Fischer, Koteljanskii, inverse M-, ss $P_{0,1^-}$, and positive $P_{0,1^-}$ -matrices. The mardigraphs in Figure 3(a), (b), (c), (d), (e), (f), (g), all represent block triangular patterns and

therefore have \mathcal{M} -completion by Lemma 3.3. Mardigraph 3(h) is complete and mardigraph 3(j) has \mathcal{M} -completion by Theorem 4.7. This leaves mardigraph 3(i). A partial matrix specifying mardigraph 3(i) has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & z \end{pmatrix}. \text{ Since diagonal elements must be greater than zero for all } \dots,$$

choosing z sufficiently large will complete A to a \mathcal{M} -matrix.

For the classes of Fischer and Koteljanskii matrices, note that a (partial) 2×2 matrix A is a (partial) Fischer matrix if and only if A is a (partial) Koteljanskii matrix if and only if A is a (partial) wss P-matrix.

For inverse M-, ss $P_{0,1}$ -, and positive $P_{0,1}$ -matrices, whenever a vertex is unspecified, choosing that entry sufficiently large will work because all diagonal entries are nonzero. This covers 3(b), 3(c), 3(e), 3(f), 3(g), 3(i) and 3(j). Mardigraph 3(h) is complete already. For mardigraphs 3(a) and 3(d), choosing the off-diagonal entry(ies) sufficiently small will work. ■

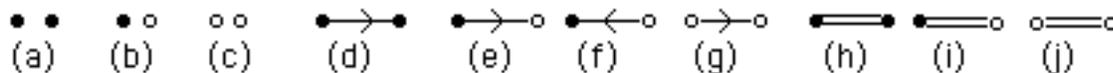


Figure 3 All possible mardigraphs of order 2

The property in Lemma 4.8 is also true for the classes of positive definite, symmetric M-, and symmetric inverse M-matrices when mardigraphs are restricted to being symmetric, and false for all other classes by Lemma 4.2.

5. Nonseparable Graphs and Vertex Identification

In Lemma 3.2 we saw that for all classes \mathcal{M} in Table 1 (except possibly positive P_0 -matrices) a mar/di/graph has \mathcal{M} -completion if and only if each component has \mathcal{M} -completion. Thus, in these cases, determining which mar/di/graphs have \mathcal{M} -completion amounts to determining which connected mar/di/graphs have \mathcal{M} -completion. In this section we examine the idea of building mardigraphs by identifying one vertex at a time. Results in [JS1], [JS2] and [FJTU] then allow us to reduce the problem considerably further (see Theorem 5.8 below).

The following graph-theoretic terminology and results are taken from [CO], and extended here to mardigraphs: If v is a vertex of mar/di/graph D , the deletion of v from D , denoted $D-v$, is the submar/di/graph induced by the set of all vertices of D except v . A cut-vertex of a connected mar/di/graph is a vertex whose deletion disconnects the mar/di/graph; more generally, a cut-vertex is a vertex whose deletion disconnects a component of a mar/di/graph. A connected mar/di/graph is nonseparable if it has no cut-vertices. A block B of a connected mar/di/graph D is a connected submar/di/graph that is

nonseparable and is maximal with respect to this property. The mardigraph shown in Figure 4 has ten blocks, labeled B_1 to B_{10} ; dotted lines (not part of the mardigraph) are used to indicate the blocks. A block is necessarily an induced submar/di/graph and the blocks of a mar/di/graph partition the arc (or edge) set of the mar/di/graph. Any two blocks have at most one vertex in common. Furthermore, if two blocks share a vertex, then this vertex is a cut-vertex. A nonseparable mar/di/graph has one block, itself. A block of D that contains exactly one cut-vertex of D is called an end-block of D . In Figure 4, blocks $B_1, B_2, B_4, B_6, B_8,$ and B_{10} are end blocks. A connected mar/di/graph with at least one cut-vertex has at least two end-blocks.

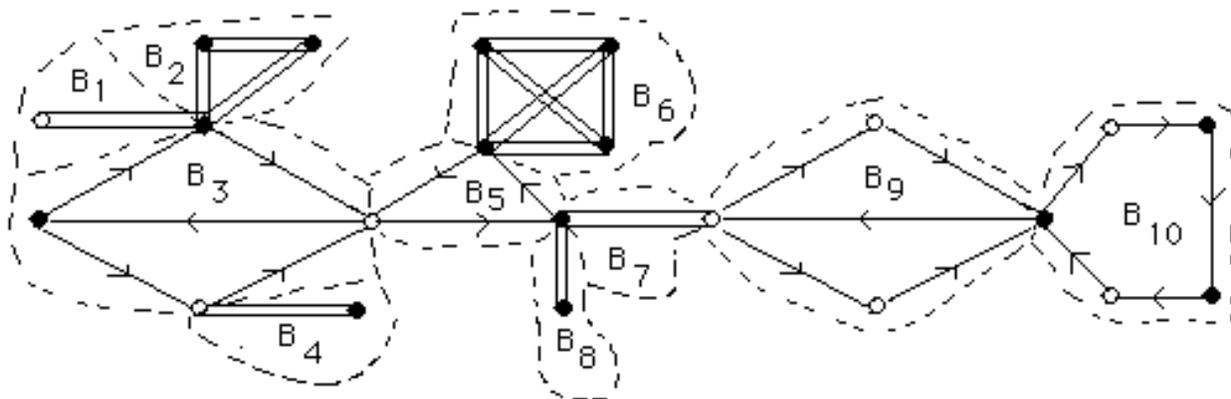


Figure 4 The blocks of a mardigraph

Note that blocks in mar/di/graphs give rise to “blocks” that overlap in one diagonal position in matrices or patterns. Thus the term “block” seems unfortunate, because the usage is different than that of block diagonal or block triangular patterns where blocks cannot overlap, but it is standard graph-theoretic usage, and has been used in this sense as “block-clique” in [JS1], [JS2].

Any mar/di/graph can be built from its blocks by adding on one block at a time. The process of “adding” is done by forming a disjoint union and then identifying one vertex from one component of the union with one vertex from the other component. If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are mar/di/graphs with disjoint vertex sets, then the union of G and H , written $G \cup H$, is the mar/di/graph that has vertex set $V(G) \cup V(H)$ and arc (edge) set $E(G) \cup E(H)$. Note that G and H are separate components of $G \cup H$, and $G \cup H$ is the mar/di/graph of the block diagonal pattern with diagonal blocks whose mar/di/graphs are G and H . If u and v are nonadjacent vertices of a graph G , the graph $G:u=v$ obtained by identifying u and v has vertex set $V(G:u=v) = V(G) - \{v\}$ and edge set $E(G:u=v) = \{e \mid e \in E(G) \text{ and } e \text{ is not incident with } v\} \cup \{(u,w) \mid (v,w) \in E(G)\}$. For digraphs, the arc (v,w) is replaced by (u,w) and (w,v) is replaced by (w,u) . In order to identify vertices in a mardigraph, both vertices must be marked the same way.

For a set of mar/di/graphs, an \ast -tree is constructed from disjoint connected mar/di/graphs B_1, \dots, B_s in \ast by identifying vertices:

Mardigraphs D_1, \dots, D_n are defined inductively by setting $D_1 = B_1$, and, once D_k is constructed, constructing D_{k+1} from D_k and B_{k+1} by identifying one vertex of D_k with one similarly marked vertex of B_{k+1} . Then D_s constructed this way is called block- or an -tree (or more specifically, an -tree constructed from B_1, \dots, B_s). We will tend to use the term -tree, except in the case of “block-clique,” which is widespread in the literature.

Theorem 5.1 Let be a set of mar/di/graphs. A mar/di/graph D is an -tree if and only if D is connected and every block of D is in .

Proof: Let $D = D_s$ be an -tree constructed from blocks B_1, \dots, B_s and let B be a block of D . Let k be the least integer such that $B \subseteq D_k$. Then B must equal B_k , because otherwise deleting the common vertex of B_k and D_{k-1} would disconnect B , contradicting the fact that B is nonseparable. So $B \subseteq B_k$ and by maximality, $B = B_k$.

The converse is proved by induction on the number of blocks of D . If D has only one block, then D is in and is thus an -tree. Assume the result for any mar/di/graph having fewer than k blocks. Let D be a mar/di/graph with $k > 1$ blocks such that each block of D is in . Since D has more than one block, D has a cut vertex and thus must have an end block. Let B be an end-block of D . Deletion of the one cut-vertex z of B disconnects D , and one of the components is a submar/di/graph of B . Let U be the set of vertices of D that are not in B , together with the cut-vertex z . Then $\langle U \rangle$ is connected and every block of $\langle U \rangle$ is a block of D , hence in . Since $\langle U \rangle$ has one fewer block than D , by the induction hypothesis $\langle U \rangle$ is an -tree, and D is the result of taking the disjoint union of $\langle U \rangle$ and B (changing the name of z in B to z') and identifying z and z' . ■

In order to use the ideas in Theorem 5.1 for -completion, we need to be able to deduce that if two mar/di/graphs D_1 and D_2 have -completion, then so does the result of taking their disjoint union and identifying one vertex of D_1 with one vertex of D_2 .

The simplest case is that in which both D_1 and D_2 are complete mar/di/graphs (i.e., cliques). We begin with this case, and assume that $V(D_1) = \{1, \dots, k\}$ and $V(D_2) = \{n+1, k+1, \dots, n\}$, and $D = (D_1 \cup D_2):k=n+1$. Such an example (for mardigraphs) is shown in Figure 5.

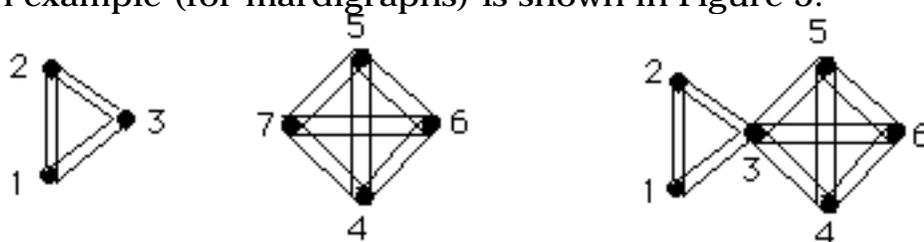


Figure 5 D_1 D_2 $D = (D_1 \cup D_2):k=n+1$

If we can complete any partial matrix specifying D to a -matrix, then we can complete any -tree where is the set of complete

mardigraphs (actually, graphs can be used here instead of mardigraphs). For the set of complete graphs, \mathcal{C} -trees are called clique-trees or block-clique graphs [JS1]. We will use the term block-clique to refer both to an \mathcal{C} -tree with the set of complete graphs, or to an \mathcal{C} -tree with the set of complete mardigraphs. (Clearly a pattern Q has a block-clique mardigraph if and only if Q is positionally symmetric, includes all diagonal positions, and has a block-clique graph.) The term 1-chordal is used [JS2], [FJTU] because the process of forming the disjoint union of cliques and identifying one vertex is presented as a special case of a method for constructing any chordal graph by forming a disjoint union and identifying a subclique of each graph (called a clique sum). This corresponds to a partial matrix with two fully specified blocks that are on the diagonal but overlap (possibly by more than one entry). Here we discuss only the case of one entry overlap. There has been substantial work done on block-clique graphs:

Theorem 5.2 [JS1], [JS2], [FJTU], [GJSW] For disjoint complete graphs G_1 and G_2 , let G be the graph constructed by taking the union of G_1 and G_2 and identifying a vertex of G_1 with a vertex of G_2 . Then G has \mathcal{C} -completion for any of the classes listed in Table 1 except positive $P_{0,1}$ -, M_0 -, M -, symmetric M_0 -, or symmetric M -matrices. (Note that patterns associated with graphs are positionally symmetric and include all diagonal positions.)

Proof: Without loss of generality we may assume that $V(G_1) = \{1, \dots, k\}$, $V(G_2) = \{n+1, k+1, \dots, n\}$, and $G = (G_1 \cup G_2):k=n+1$. Let A be a partial matrix specifying $G_1 \cup G_2$. A partial \mathcal{C} -matrix specifying such a G has the form

$$A = \begin{bmatrix} a_{11} & \dots & a_{1k-1} & a_{1k} & ? & \dots & ? \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k-11} & \dots & a_{k-1k-1} & a_{k-1k} & ? & \dots & ? \\ \hline a_{k1} & \dots & a_{kk-1} & a_{kk} & a_{kk+1} & \dots & a_{kn} \\ \hline ? & \dots & ? & a_{k+1k} & a_{k+1k+1} & \dots & a_{k+1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ ? & \dots & ? & a_{1k} & a_{1k+1} & \dots & a_{1n} \end{bmatrix}$$

which can be partitioned as $A = \begin{matrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{matrix}$. The two principal

submatrices specifying G_1 and G_2 are $M_1 = \begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix}$ and $M_2 = \begin{matrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{matrix}$.

Note that $A_{22} = [a_{kk}]$.

It is shown in [JS1] that if $A = \begin{matrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{matrix}$ is a partial inverse M-

matrix, then A can be completed to an inverse M-matrix by setting $X = A_{12} A_{22}^{-1} A_{23}$ and $Y = A_{32} A_{22}^{-1} A_{21}$. This gives a matrix whose inverse has 0 in the entries that were unspecified in A (this is called the zeros-in-the-inverse completion in [FJTU]). As the details are lengthy, the reader is referred to [JS1] for the proof that this completion is an inverse M-matrix. The same result is established in [JS2] for symmetric inverse M-matrices.

If $C = \begin{matrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{matrix}$ and $B = \begin{matrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{matrix}$ are partitioned matrices with

C_{22} and B_{22} both 1×1 , then $\begin{matrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22}+B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{matrix}$ is called the

1-subdirect sum of C and B, and is denoted $C \begin{matrix} 1 \\ 1 \end{matrix} B$ [FJ]. The 1-subdirect sum satisfies the determinant equality $\det(C \begin{matrix} 1 \\ 1 \end{matrix} B) = \det C_{11} \det B + \det C \det B_{33}$ [FJ]. The zero completion of A, obtained by setting $X = 0$ and $Y = 0$, is used when $a_{kk} = 0$ [FJTU]. If A is a partial \mathcal{P} -matrix with $a_{kk} = 0$, then the zero completion completes A to a \mathcal{P} -matrix for any of the classes \mathcal{P}_0^- , $\text{wss } \mathcal{P}_0^-$, $\text{ss } \mathcal{P}_0^-$, non-negative \mathcal{P}_0 -matrices. This is because any principal minor is non-negative (from the determinant equality when the kth row is present and because the principal submatrix is block diagonal when the kth row is absent).

The asymmetric completion of A, obtained by setting $X = A_{12} A_{22}^{-1} A_{23}$ and $Y = 0$ (provided $A_{22} = [a_{kk}]$ is nonzero), was introduced in [FJTU]. It is shown there that this completes a partial \mathcal{P} -matrix A to a \mathcal{P} -matrix for any of the classes \mathcal{P} in this theorem except inverse M-, symmetric inverse M-, positive (semi)definite and those \mathcal{P}_0 -matrices with $a_{kk} = 0$. There is a minor gap in the proof for the classes $\text{ss } \mathcal{P}_0^-$ and $\text{ss } \mathcal{P}_{0,1}$ -matrices.

Here we will examine why this completion works, and deal with these latter two classes. Let \hat{A} be the asymmetric completion of A. Then

$\det \hat{A} = \frac{\det M_1 \det M_2}{\det A_{22}}$ [FJTU]. That the principal minors of \hat{A} are

positive or non-negative follows from this equation when A_{22} is included, and from the fact that the principal submatrix is block triangular if A_{22} is excluded. Since A_{22} is 1×1 , it is included or excluded entirely. Thus the asymmetric completion will work for all the classes where entries can be 0 (more work needs to be done for Fischer and Koteljanskii matrices; see [FJTU]). For ss or positive \mathcal{P} -matrices it is clear we can perturb the zero elements slightly and still have a \mathcal{P} -matrix because determinants are positive. For $\text{ss } \mathcal{P}_{0,1}$ - or \mathcal{P}_0 -matrices (with $a_{kk} > 0$) we show here that perturbation still works. Whenever determinants are positive, again a slight perturbation will

work. When a determinant is 0, a perturbation can still be used because the determinant will remain zero. To see this, suppose $\det M_1 = 0$. Then there is a nonsingular $k \times k$ matrix B such that $B M_1$ has a row of zeros, and thus

$$B \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} \text{ has a row of zeros. Therefore } B \begin{pmatrix} A_{12}A_{22}^{-1}A_{23} \\ A_{23} \end{pmatrix} = B \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} A_{22}^{-1}A_{23}$$

has a row of zeros. Thus $(B \quad I_{n-k}) \begin{pmatrix} A_{11} & A_{12} & A_{12}A_{22}^{-1}A_{23} \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix}$ has a row of zeros

$$\text{and thus } \det \begin{pmatrix} A_{11} & A_{12} & A_{12}A_{22}^{-1}A_{23} \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix} = 0 \text{ regardless of } Y.$$

All listed classes have now been covered except the classes of positive (semi)definite matrices; for these classes see [GJSW], [JS3]. ■

Corollary 5.3 Every block-clique graph has \mathcal{H} -completion for any of the classes in Theorem 5.2.

Just as in Lemma 3.2, these completions work when M_1 and M_2 are partial matrices that have \mathcal{H} -completion, rather than fully specified matrices, by first completing M_1 and M_2 to \mathcal{H} -matrices. However, when the overlap diagonal entry is unspecified we need the class \mathcal{H} to have the property that if a positive real number is added to a diagonal entry of a \mathcal{H} -matrix, it remains a \mathcal{H} -matrix. This is known or easily established for all the classes in Theorem 5.2 except Koteljanskii (see the next lemma). For the remainder of this section, we will restrict our consideration to these classes of matrices (i.e., all classes listed in Table 1 except Koteljanskii and those listed as exceptions in Theorem 5.2).

Lemma 5.4 If a positive real number is added to a diagonal entry of a \mathcal{H} -matrix, it remains a \mathcal{H} -matrix.

Proof: For the classes of inverse M -matrices or symmetric inverse M -matrices see [J1], [JS2]. For the classes of positive (semi)definite matrices see [HJ1].

Let \mathcal{H} be any of the other listed classes. Let A be a \mathcal{H} -matrix (so A is necessarily a P_0 -matrix). Without loss of generality we may assume a positive number is added to the 1,1-entry, so let A' be obtained from A by adding a $\alpha > 0$ to a_{11} . Then $\det A' = \det A + \det A''$, where the first row of A'' is $(\alpha, 0, \dots, 0)$ and the remaining rows are the same as the corresponding rows of A , because the determinant is a multilinear function of the rows. $\det A'' = \alpha \det A[\{2, \dots, n\}] \geq 0$, so $\det A' \geq \det A$. The same argument can be applied to principal minors, so for any of the classes except Fischer the result is established.

Now consider the case in which A is a Fischer matrix. For any disjoint subsets I and J of $\{1, \dots, n\}$, $\det A[I, I] \leq \det A[I, J] \det A[J, I]$. If $1 \in I$, then clearly $\det A'[I, I] \leq \det A'[I, J] \det A'[J, I]$. If $1 \in J$, we may assume without loss of generality that $1 \in I$. Let $K = I - \{1\}$, so $I - \{1\} = K$. Then $\det A'[I, I] = \det A[K, K]$, $\det A'[I, J] = \det A[K, J] + a \det A[K, I]$, and $\det A'[J, I] = \det A[J, K] + a \det A[J, I]$. Since A is a Fischer matrix, $\det A[K, K] \leq \det A[K, J] \det A[J, K]$, so $\det A'[I, I] \leq \det A'[I, J] \det A'[J, I]$, and A' is a Fischer matrix. ■

Theorem 5.5 For disjoint mar/di/graphs D_1 and D_2 , let D be the mar/di/graph constructed by taking the union of D_1 and D_2 and identifying a vertex of D_1 with a (similarly marked) vertex of D_2 . If D_1 and D_2 both have \leq -completion, then D has \leq -completion.

Proof: Let A be a partial matrix specifying D . Let M_i be the principal submatrix specifying D_i (note that D_i is an induced submardigraph of D). Complete M_i to a \leq -matrix C_i (possible because D_i has \leq -completion). In the case that the common vertex of D_1 and D_2 was unspecified, this may result in two different values being assigned to this entry. Choose the larger, which results in both C_i being \leq -matrices, by Lemma 5.4. Then apply Theorem 5.2 to complete A to a \leq -matrix. ■

Corollary 5.6 A mardigraph D has \leq -completion if and only if each block of D has \leq -completion. If each mardigraph in \mathcal{M} has \leq -completion, then so does every \leq -tree.

The properties in Corollary 5.6 are not true for (symmetric) M - and (symmetric) M_0 -matrices, as we shall see in Section 7.

Our goal is to determine which mardigraphs have \leq -completion. We have now reduced the problem to the study of nonseparable mardigraphs. However, this is not terribly useful, because too many mardigraphs are nonseparable, such as the mardigraph in Figure 6.

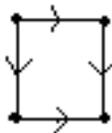


Figure 6 A nonseparable mardigraph

By Lemma 3.3, for many classes \mathcal{M} we can reduce the problem to determining which strongly connected mardigraphs have \leq -completion. Our goal is to find a collection \mathcal{S} of relatively simple strongly connected mardigraphs that have \leq -completion, and describe all strongly connected mardigraphs with \leq -completion as \leq -trees of these.

Lemma 5.7 A block B of a strongly connected mar/digraph D is strongly connected.

Proof: Let u and v be distinct vertices of B . Since D is strongly connected, there is a path from u to v in D . If involves a vertex not in B , then contains a vertex z of B that is a cut-vertex of D . Deletion of z disconnects D , and both u and v are in the same component of $D-z$ (since B is nonseparable). But a vertex can appear only once in , contradicting the fact that is a path from u to v . Thus must lie entirely in B and B is strongly connected. ■

Theorem 5.8 Let be one of the classes P_0 -, $P_{0,1}$ -, P -, wss P_0 -, wss $P_{0,1}$ -, wss P -, Fischer, ss P -, non-negative P_0 -, non-negative $P_{0,1}$ -, non-negative P -, positive P -matrices. A mardigraph has -completion if and only if every nonseparable strongly connected induced submardigraph has -completion.

Proof: If a mardigraph has -completion, then by Lemma 3.1 so does every induced submardigraph. For the converse, suppose D is a mar/di/graph with the property that every nonseparable strongly connected induced submardigraph has -completion. By Lemma 5.7, every block of D is strongly connected, and thus has -completion. So by Corollary 5.6, D has -completion. ■

Thus, if is any of the classes listed in Theorem 5.8, we have reduced the problem of determining which mardigraphs have -completion to that of determining which strongly connected nonseparable mardigraphs have -completion.

For one of the classes of P -, wss P -, ss P -, non-negative P -, positive P -matrices, by Theorem 4.6 we can further restrict our attention to strongly connected nonseparable mardigraphs with all vertices specified.

6. Inverse M-matrix Completions

We can apply the results of Section 5 to inverse M-matrices. In [JS1] it was shown that a graph (of a positionally symmetric pattern that includes the diagonal) has inverse M-completion if and only if it is block-clique. In [H1] it was shown that a pattern that includes all diagonal positions has inverse M-completion if and only if the induced subdigraph of any cycle is a clique and the induced subdigraph of any alternate path to a single arc is a clique. An alternate path to a single arc is a path $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$ where the digraph also contains (v_1, v_k) . A mar/digraph is path-clique if the induced submardigraph of any alternate path to a single arc is complete (note that for mardigraphs this requires all vertices specified). In [H3] it was shown that when a pattern need not include the diagonal, the mardigraph being path-clique is still necessary for inverse M-completion. From the path-clique requirement, it is

immediate that a block triangular pattern with complete diagonal blocks need not have inverse M-completion.

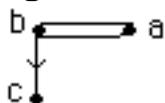
For a mardigraph, being path-clique together with the property that every strongly connected induced submardigraph has inverse M-completion is sufficient, reducing the problem of classifying mardigraphs having inverse M-completion to that of determining which path-clique strongly connected mardigraphs have inverse M-completion [H3].

Corollary 5.6 and Lemma 5.7 have reduced the problem of determining which path-clique strongly connected mardigraphs have inverse M-completion to that of determining which path-clique, strongly connected, nonseparable mardigraphs have inverse M-completion. The next (graph theoretic) result tells us a path-clique, strongly connected, nonseparable mardigraph must be homogeneous.

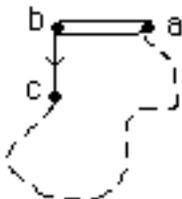
Theorem 6.1 If digraph D is strongly connected, nonseparable, and nonhomogeneous, then D is not path-clique.

Proof: The steps of the proof are illustrated in Figure 7.

Figure 7

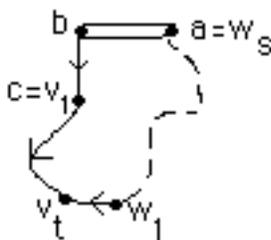


There are vertices a, b, c with $(a,b), (b,a), (b,c) \in E(D)$ and $(c,b) \notin E(D)$ (because D is nonhomogeneous and strongly connected). Call such a triple (a,b,c) a junction.



Since G is nonseparable, b is not a cut-vertex, so there is a semipath from c to a that does not include b . Such a semipath is called a bypass for junction (a,b,c) .

Label the vertices of a bypass $v_1=c, v_2, \dots, v_t$ as long as the orientation of the arc is (v_i, v_{i+1}) . Label the next vertex w_1 , i.e., arc (w_1, v_t) is in the semipath (unless $a = v_t$). Label the remaining vertices $w_2, \dots, w_s = a$.



Thus the bypass is labeled

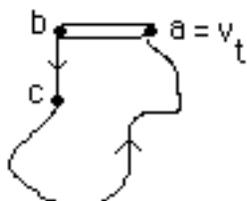
$$c = v_1 \ v_2 \ \dots \ v_t \ w_1 \ w_2 \ \dots \ w_s = a.$$

Let $V = \{v_1, \dots, v_t\}$ and $W = \{w_1, \dots, w_s\}$. Define the disorientation of this bypass to be s (if $a = v_t$, then the disorientation is 0).

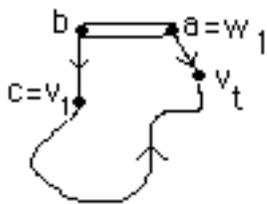
Among all junctions and bypasses, choose the junction (a,b,c) and bypass

$$c = v_1 \ v_2 \ \dots \ v_t \ w_1 \ w_2 \ \dots \ w_s = a$$

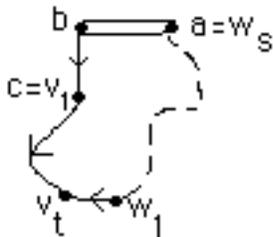
of minimal disorientation. There are 3 cases, disorientation = 0, disorientation = 1, and disorientation = $s > 1$.



Case 1: disorientation = 0: $c = v_1 \ v_2 \ \dots \ v_t = a$ is an alternate path to (b,a) and the induced subdigraph is not complete because $(c,b) \notin E(D)$, so D is not path-clique.

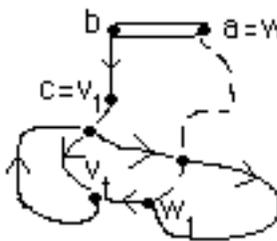


Case 2: disorientation = 1:
 $w_1 = a \quad b \quad c = v_1 \quad v_2 \quad \dots \quad v_t$ is an alternate path to (w_1, v_t) and the induced subdigraph is not complete, so D is not path-clique.



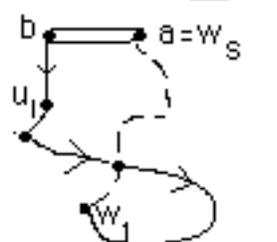
Case 3: disorientation = $s > 1$:

D is strongly connected so there is a path $v_t \quad z_1 \quad \dots \quad z_r \quad w_1$.

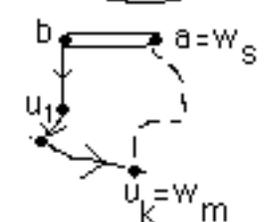


Suppose the cycle $= v_t \quad z_1 \quad \dots \quad z_r \quad w_1 \quad v_t$ is simple.

Because there is a walk $b \quad c = v_1 \quad v_2 \quad \dots \quad v_t \quad z_1 \quad \dots \quad z_r \quad w_1$ from b to w_1 , there is a path



$b \quad u_1 \quad u_2 \quad \dots \quad u_q = w_1$ from b to w_1 where each u_i is a v_j or z_j . Because \quad is simple, $(u_1, b) \in E(D)$, so (a, b, u_1) is a junction (regardless of whether $u_1 = c$ or not).



Let k be the least index such that $u_k \in W$. Define m by $u_k = w_m$. Then $b \quad u_1 \quad \dots \quad u_k = w_m \quad \dots \quad w_s = a$ is a bypass of junction (a, b, u_1) of disorientation $\leq s - m \leq s - 1 < s$, contradicting the original choice of bypass.

Therefore \quad is not simple. In a path-clique digraph any cycle is either simple or the subdigraph it induces is complete [H3]. The induced subdigraph of \quad is not simple because $(v_t, w_1) \in E(D)$, so D is not path-clique. ■

Corollary 6.2 A mardigraph has inverse M -completion if and only if it is path-clique and every strongly connected nonseparable induced submardigraph has inverse M -completion. A strongly connected, nonseparable mardigraph that has inverse M -completion must be homogeneous.

In [H3] it was also established that for \quad the set of all complete mardigraphs, all cycles with at least one unspecified vertex, and the cyclic double triangles in which every cycle contains at least one unspecified vertex (shown in Figure 8) every \quad -tree has inverse M -completion.

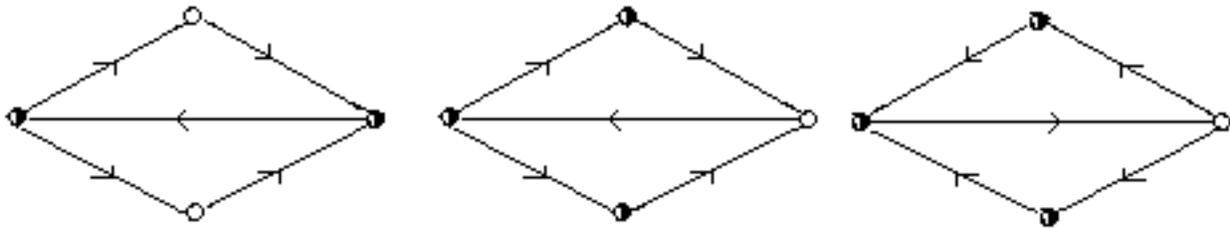


Figure 8 Cyclic double triangles having inverse M-completion

7. M_0 -matrices

It is well-known that a partial M-matrix (with all diagonal entries specified) can be completed to an M-matrix completion only if the zero completion of A is an M-matrix [JS1] (the zero completion A_0 of A is the result of setting all unspecified entries to 0).

If A is an M_0 -matrix, then for any $\epsilon > 0$, $A + \epsilon I$ is an M-matrix. If A is a partial M_0 -matrix (with all diagonal entries specified) that can be completed to an M_0 -matrix C , then $C + \epsilon I$ is an M-matrix, so the zero completion of $A + \epsilon I$ must be an M-matrix, and hence the zero completion of A is the limit of M-matrices and hence an M_0 -matrix.

Recall that in Section 4 we showed that if position (j,j) is omitted but (i,j) , (j,i) and (i,i) are included in a pattern, then the pattern does not have M_0 -completion. In particular, a positionally symmetric pattern has M_0 -completion only if the pattern includes all diagonal positions or omits all diagonal positions. Note that in contrast, a pattern has M-completion if and only if its principal subpattern defined by the diagonal positions has M-completion.

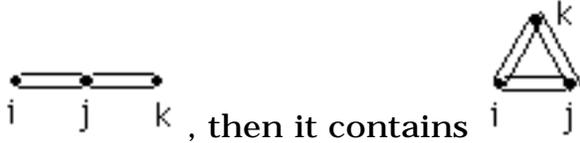
In Section 3 we showed a pattern has M_0 -completion if it is permutation similar to a block triangular pattern in which every irreducible diagonal block has M_0 -completion. Thus to complete the classification, we need only examine irreducible patterns, or equivalently, strongly connected mardigraphs.

Recall that a pattern that includes the diagonal has M-completion if and only if it is block triangular with each diagonal block complete, i.e., each strongly connected subdigraph is complete [H2]. For M_0 -patterns we will see that again we need block triangular, but each diagonal block can be complete or omit all diagonal positions, i.e., each strongly connected submardigraph is complete or all vertices are unspecified.

The result for patterns of M-matrices that include the diagonal is obtained from the following steps [H2]:

1) The induced subdigraph of any cycle of a digraph having M-completion is a clique.

2) If a digraph D having M -completion contains the subdigraph



3) If a strongly connected digraph has the properties (1) and (2), then it is a clique.

We begin by examining cycles in mardigraphs. Recall that a cycle in a digraph or mardigraph must respect the orientation of the arcs.

Theorem 7.1 Let D be a mardigraph that has contains a cycle with the properties a) has at least one specified vertex and b) the induced submardigraph of is not a clique. Then D does not have M_0 -completion.

Proof: If D omits the reverse of some arc in the cycle, then by choosing an appropriate relabeling of the vertices of D we may assume that is $1, 2, \dots, k, 1$, and arc $(1, k)$ is omitted. If D contains the reverse of every arc in the cycle, by choosing an appropriate relabeling of the vertices of D we may assume that is $1, 2, \dots, k, 1$, and one of the arcs $(1, s)$ with $2 < s < k$ is omitted. So in either case, is $1, 2, \dots, k, 1$, and one of the arcs $(1, s)$ with $2 < s \leq k$ is omitted, and at least one vertex t is specified.

Define a partial matrix A that specifies $\langle \{1, \dots, k\} \rangle$ by setting the $k, 1$ and $i, i+1$ -entries for $i=1, \dots, k-1$ equal to -1 and setting all other specified entries equal to 0 (including any specified diagonal entries). We claim A is a partial M_0 -matrix: Suppose $A[\]$ is a fully specified principal submatrix of A . Then 1 and s cannot both be in since the $1, s$ -entry is unspecified. If 1 is not in , then $A[\]$ is triangular with 0 s on the diagonal and thus $A[\]$ is an M_0 -matrix. If s is not in , then the $s-1$ row of $A[\]$ consists entirely of 0 s and thus $A[\]$ is an M_0 -matrix. So A is a partial M_0 -matrix.

But A cannot be completed to an M_0 -matrix, because the t, t -entry is 0 and for any choice of diagonal entries $d_1, \dots, d_{t-1}, d_{t+1}, \dots, d_k$ the zero

$$\begin{matrix}
 & d_1 & -1 & 0 & \cdot & \cdot & 0 \\
 & 0 & d_2 & -1 & \cdot & \cdot & 0 \\
 \text{completion is } A_0 = & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 0 & -1 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & -1 & 0 & 0 & \cdot & \cdot & d_n
 \end{matrix}$$

Expansion by cofactors on the first

column shows $\det A_0 = -1$. ■

Lemma 7.2 If a mardigraph D has M_0 -completion and contains the



submardigraph $\langle \{i, j, k\} \rangle$, then $\langle \{i, j, k\} \rangle$ is a clique or i, j , and k are all unspecified.

Proof: Suppose the mardigraph D contains the given submardigraph, $\langle \{i, j, k\} \rangle$ is not a clique and at least one of i, j, k is specified. Then if any of

the vertices is unspecified, D does not have M_0 -completion by Lemma 4.2. If all vertices are specified, then the partial matrix

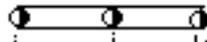
$$A = \begin{pmatrix} 1 & -0.9 & \# \\ -0.9 & 1 & -0.9 \\ \# & -0.9 & 1 \end{pmatrix} \quad \text{where } \# \text{ is either unspecified or } 0, \text{ is a partial } M\text{-matrix}$$

that specifies $\langle\{i,j,k\}\rangle$ (because at least one $\#$ is unspecified). Then the zero completion of A has determinant -0.62 , so $\langle\{i,j,k\}\rangle$ and hence D do not have M_0 -completion. ■

The third step is a graph theoretic result that does not refer to any class of matrices, although in [H2] the proof is contained in a theorem about M -matrices, and assumes all vertices are specified.

Theorem 7.3 Let D be a strongly connected mardigraph with the properties

1) The induced submardigraph of any cycle of D is a clique or all vertices are unspecified.



2) If D contains the submardigraph $\langle\{i,j,k\}\rangle$, then $\langle\{i,j,k\}\rangle$ is a clique or i , j , and k are all unspecified.

Then D is a clique or all vertices are unspecified.

Proof: Suppose D has properties (1) and (2) and contains a specified vertex v . We show by induction on the order of D that D must contain a Hamilton cycle. Then since D is the induced submardigraph of its Hamilton cycle, D is a clique by (1).

If the order of D is 1, then D is a clique (because the vertex is specified). So suppose order $D \geq 2$. Then D must contain a closed walk, because D includes two distinct vertices v and u , and there is a path from v to u and a path from u to v . Putting these together produces a closed walk including vertex v .

Since D contains a closed walk that includes the specified vertex v , D must contain a cycle that includes v : Begin the closed walk at v and continue until you return to vertex v or a vertex is repeated. If a vertex is repeated, edit out the vertices between the repetition. This process produces a cycle containing vertex v in a finite number of steps.

Since D is finite, D must contain a cycle of maximal length k among cycles containing vertex v . Note that since specified vertex v is in C , by property (1) C is a clique and thus all vertices of C are specified. We claim C contains all the vertices of D . If not, there is a vertex in D that is not in C . Since D is strongly connected, there is a path from v to this vertex and hence there is a vertex w not in C and a vertex z in C such that D includes arc (z,w) . Since D is strongly connected there is also a path in D from w to some vertex in C such that the path does not include any other vertices in C . Relabeling if necessary, z is 1, the original cycle is 1, 2, ..., k , 1, the vertex w not in C is $k+1$, which is adjacent to 1, and the

path from $k+1$ to s is $(k+1, k+2), \dots, (s-1, s)$ (s, m) (where m is in D and m may be equal to or different from 1). These two situations are illustrated in Figure 9.

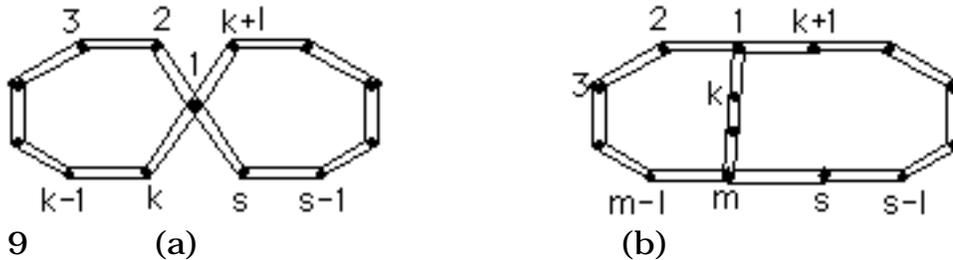


Figure 9 (a)

(b)

In these figures both arcs are present, because a cycle that contains a specified vertex is a clique, hence symmetric with all vertices specified, and vertex 1 is specified. Note that the subdigraph shown is not an induced subdigraph (the induced subdigraph of a cycle with specified vertex is a clique).

In either case, $1, 2, \dots, k, s, s-1, \dots, k+2, k+1, 1$ (which includes all vertices shown) will be a cycle if (k, s) and (s, k) are in G . In the case $m=1$ (Fig. 9(a)), D contains arcs $(k, 1)$, $(1, k)$, $(1, s)$ and $(s, 1)$. By property (2), the digraph must also contain (k, s) and (s, k) , since all vertices are specified. This is illustrated in Figure 10(a). In the case $m \neq 1$ (Fig. 9(b)), $1, k+1, \dots, s, m, m+1, \dots, k, 1$ is a cycle, and the subdigraph of any cycle with specified vertex is a clique, so G must contain the arcs (k, s) and (s, k) . This is illustrated in Figure 10(b).

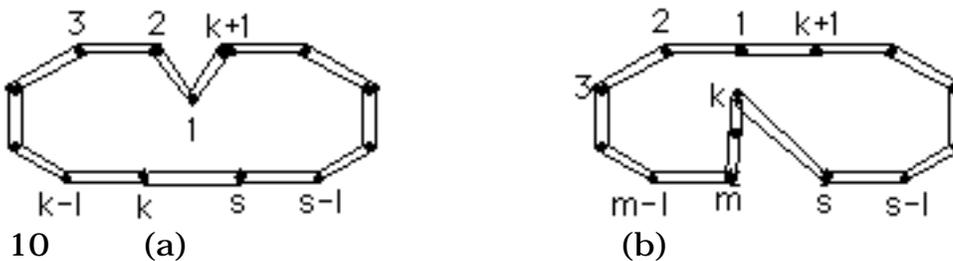


Figure 10 (a)

(b)

The length of the cycle $1, 2, \dots, k, s, s-1, \dots, k+2, k+1, 1$ is at least $k+1$ and hence greater than the length of D , contradicting the maximality of D . Thus a cycle of maximal length must include all vertices of D , so D is a clique. ■

Theorem 7.4 Let Q be a pattern and let D be its pattern-mardigraph. The following are equivalent:

- 1) The pattern Q has M_0 -completion.
- 2) The pattern Q is permutation similar to a block triangular pattern in which each diagonal block is a clique or omits all diagonal positions.
- 3) Any strongly connected induced submardigraph of D is a clique or all its vertices are unspecified.

Proof: Statement (3) is the mardigraph equivalent of statement (2), The implication (2) \Rightarrow (1) follows from Lemma 3.3 and Theorem 4.7. The implication (1) \Rightarrow (3) follows from Lemma 7.2 and Theorems 7.1 and 7.3. ■

8. $P_{0,1}$ -matrices, non-negative $P_{0,1}$ -matrices and non-negative P -matrices.

In [JK] it was shown that any positionally symmetric pattern has P -completion, and any pattern for 3×3 matrices has P -completion. In [FJTU] it was shown that any pattern for 3×3 matrices that omits an off-diagonal position has positive P -completion, and this was used to show that any symmetric n -cycle has positive P -completion. These arguments can be modified to establish the same results for the classes of non-negative P -matrices, non-negative $P_{0,1}$ -matrices and $P_{0,1}$ -matrices. In the discussion of positive P -matrices in [FJTU], nonzero elements on the diagonal and super-diagonal may be assumed to be 1, without loss of generality, because all elements are nonzero. Applying the same techniques to the classes, all diagonal entries are nonzero, but the case of zero entries on the super-diagonal must be dealt with separately.

Lemma 8.1 Let \mathcal{C} be one of the classes $P_{0,1}$ -, non-negative $P_{0,1}$ -, non-negative P -matrices. Then any pattern for 3×3 matrices that omits at least one off-diagonal position has \mathcal{C} -completion.

Proof: Setting any unspecified diagonal entries to 1 and following the

proof of Lemma 3.1 in [FJTU], we may assume $A = \begin{pmatrix} 1 & s & y \\ p & 1 & t \\ q & r & 1 \end{pmatrix}$ and

$\det A = 1 + qst + pry - ps - rt - qy$. Then if $s=0$ or $t=0$ or $qst \geq 1$, choose $y = 0$. If $s > 0$ and $t > 0$ and $qst < 1$, choose $y = st$. ■

Combining Theorem 4.6, [FJTU,3.1], and Lemma 8.1 yields

Corollary 8.2 Any pattern for 3×3 matrices has non-negative and positive P -completion.

Note that the statement of Corollary 8.2 for (non-negative) $P_{0,1}$ -matrices is false, by Example 4.3.

Lemma 8.3 Let \mathcal{C} be one of the classes $P_{0,1}$ -, nonnegative $P_{0,1}$ -, nonnegative P -matrices. Then a positionally symmetric pattern Q that includes the diagonal and whose graph is a 4-cycle has \mathcal{C} -completion. (The digraph of such a Q is a symmetric 4-cycle.)

Proof: Following the proof of Lemma 3.2 in [FJTU], we may assume that the graph of Q is the 4-cycle 1,2,3,4,1, and either $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = a_{14} = a_{41} = 0$ or $a_{12} > 0$ (by use of a permutation similarity). In the former case, setting all unspecified entries to 0 produces a positive diagonal matrix, which is certainly a \mathcal{C} -matrix. In the latter case, we may

As might be expected from the case of M-matrices, (weakly) sign symmetric classes behave very differently from non-negative classes. They do not have completion for all patterns of 3×3 matrices or (symmetric) n -cycles with specified vertices.

Example 9.1 The pattern $Q = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,3)\}$, whose mardigraph is a 3-cycle with all vertices specified, does not have \mathcal{P} -completion for any of the classes $wss P_0$ -, $wss P_{0,1}$ -, $wss P$ -, $ss P_0$ -, $ss P_{0,1}$ -,

$ss P$ -matrices. It is not possible to choose x, y, z such that $A = \begin{matrix} & & 1 & 1 & z \\ x & & 1 & 1 & \\ -2 & y & & 1 & \end{matrix}$

is a \mathcal{P} -matrix because $\det A = -1 + xyz + 2z - x - y$, and by sign symmetry, $x, y \geq 0$ and $z \leq 0$.

This example also shows that there are patterns for 3×3 matrices omitting one or two off-diagonal positions that fail to have \mathcal{P} -completion, because if $x = 0.5$ (and $y=0.5$) the partial \mathcal{P} -matrix still will not have \mathcal{P} -completion.

Example 3.4 in [FJTU] exhibits a partial \mathcal{P} -matrix A specifying a graph n -cycle ($n \geq 4$) that cannot be completed to a \mathcal{P} -matrix for any of $wss P_0$ -, $wss P_{0,1}$ -, $ss P_0$, $ss P_{0,1}$ -matrices. (Remember a graph can be used for a symmetric mardigraph with all vertices specified.) [FJTU, 3.3] gives an example showing that a graph 4-cycle does not have \mathcal{P} -completion for any of the classes $ss P$ -, $wss P$ -, Fischer and Koteljanskii matrices.

Furthermore, [FJTU] provides examples showing that the double triangle graph (see Figure 11(a)) does not have \mathcal{P} -completion for these and many other classes, and [JS1] notes that any graph that does not contain a double triangle or n -cycle ($n \geq 4$) as an induced subdigraph is block-clique. Hence the graphs that have \mathcal{P} -completion for any of these four classes are exactly the block-clique graphs [FJTU, Theorem 4.1].

In [FJTU] examples are given showing that the double triangle does not have \mathcal{P} -completion for any of the classes of $wss P_0$ -, $wss P_{0,1}$ -, $wss P$ -, Fischer, Koteljanskii, $ss P_0$ -, $ss P_{0,1}$ -, $ss P$ -, non-negative P_0 -, non-negative $P_{0,1}$ -, non-negative P -, positive $P_{0,1}$ -, or positive P -matrices. In [JS1] (respectively, [JS2]) examples are given showing that the double triangle does not have inverse M-completion (symmetric inverse M-completion).

From the result in [GJSW] that any chordal graph has positive definite completion (and the observation in [JS3] that these results extend to positive semidefinite completion for patterns that include the diagonal), we can conclude the double triangle has positive (semi)definite completion. From the result in [JK] that any graph has P -completion we can conclude the double triangle has P -completion, and it is also shown there that the double triangle does not have P_0 -completion.



Figure 11(a) Double triangle graph (b) Symmetric double triangle mardigraph

The double triangle graph does not have M- or M_0 -completion ([H2], §7). In fact, the following example shows that the double triangle does not have symmetric M- or M_0 -completion.

Example 9.2 The partial symmetric M-matrix $A = \begin{matrix} & & 1 & -0.9 & x & 0 \\ & & -0.9 & 1 & -0.9 & 0 \\ & & y & -0.9 & 1 & 0 \\ & & 0 & 0 & 0 & 1 \end{matrix}$

specifies the double triangle but cannot be completed to an M_0 -matrix because the zero completion of A has determinant -0.62 .

Graphs are used in [FJTU] because all patterns are assumed to contain the diagonal and be positionally symmetric. To avoid confusion, in Table 3 of the next section, graphs are described in mardigraph terminology (otherwise “n-cycle” would mean two different things). The mardigraph version of the double triangle graph is a symmetric double triangle with all vertices specified (shown in Figure 11(b)).

For the class of ss P_0 -matrices, a trivial example completes the classification of mardigraphs:

Example 9.3 The pattern $Q = \{(1,1), (2,1)\}$, which is triangular, does not have ss P_0 -completion, because $A = \begin{matrix} 0 & x \\ 1 & z \end{matrix}$ is a partial ss P_0 -matrix specifying Q that cannot be completed to a ss P_0 -matrix. Note that including the $(2,2)$ position in the pattern does not change the situation.

This example leads to the following necessary condition for ss P_0 -completion.

Lemma 9.4 If the pattern Q has ss P_0 -completion and Q includes position (j,i) and omits position (i,j) , then Q omits both of the positions (i,i) and (j,j) . Equivalently, if a mardigraph D contains any of $\bullet \rightarrow \bullet$ $\bullet \rightarrow \square$ $\bullet \leftarrow \square$ as an induced submardigraph, then D does not have ss P_0 -completion.

Theorem 9.5 A connected mardigraph has ss P_0 -completion if and only if it is block-clique or all vertices are unspecified.

Proof: Let D be a connected mardigraph with at least one specified vertex that has ss P_0 -completion. Working outward from the specified vertex, by Lemmas 9.4 and 4.2, D is symmetric with all vertices specified. Thus the associated pattern Q_D is positionally symmetric and Q_D includes all

diagonal positions. Then by the result of [FJTU, Theorem 4.1] cited previously, the graph of Q_D is block-clique, i.e., every block is complete. Thus D is a block-clique mardigraph (every block is complete).

Conversely, if D is a block-clique mardigraph, then D has ss P_0 -completion [FJTU, Theorem 4.1], and if D has all vertices unspecified then D has ss P_0 -completion by Theorem 4.7. ■

The class of P_0 -matrices seems to behave more like the (weakly) sign symmetric classes than like the class of P -matrices.

Example 9.6 The pattern $Q = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ does not have P_0 -completion. It is not possible to choose x such that

the partial P_0 -matrix $A = \begin{matrix} & & 0 & -1 & x \\ & & 0 & 0 & -1 \\ & & -1 & 0 & 0 \end{matrix}$ is a P_0 -matrix because $\det A = -1$.

Example 9.7 The pattern $Q = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,1), (4,3), (4,4)\}$, whose mardigraph is a symmetric 4-cycle with specified vertices, does not have P_0 -completion for any of the classes P_0 -, non-negative P_0 -, wss P_0 - matrices. The partial matrix

$B = \begin{matrix} & & 0 & 1 & x & 0 \\ & & 0 & 0 & 1 & y \\ & & z & 0 & 0 & 1 \\ & & 1 & w & 0 & 0 \end{matrix}$ is a partial P_0 -matrix specifying Q . But values cannot be

assigned to x, y, z, w so as to complete B to a P_0 -matrix, because $\det B[1,2,3] = z$, $\det B[1,3,4] = x$, and $\det B[1,3] = -xz$, forcing $xz = 0$, but $\det B = -1 + xzyw$.

Example 9.8 The pattern $Q = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,1), (4,4)\}$, whose mardigraph is a 4-cycle with specified vertices, does not have P_0 -completion for either of the classes non-negative P_0 - or wss

P_0 - matrices. The partial matrix $A = \begin{matrix} & & 0 & 1 & ? & ? \\ & & ? & 0 & 1 & ? \\ & & ? & ? & 0 & 1 \\ & & 1 & ? & ? & 0 \end{matrix}$ is a partial P_0 -matrix

specifying Q . Suppose A were completed to a P_0 -matrix B . By examining 2×2 minors we see that b_{14}, b_{21}, b_{32} , and b_{43} must all be 0. Thus

$B = \begin{matrix} & & 0 & 1 & x & 0 \\ & & 0 & 0 & 1 & y \\ & & z & 0 & 0 & 1 \\ & & 1 & w & 0 & 0 \end{matrix}$, which cannot be a P_0 -matrix for any x, y, z, w by the

previous example.

10. Summary of known results

In this section we summarize the current state of knowledge for the classes of matrices in Table 1. From the point of view of analyzing effective strategies for matrix completion problems, it is interesting to look at whether certain building and reduction techniques work for each class (Table 2) and whether a particular mardigraph has \mathcal{R} -completion (Table 3). If one desires information about a particular class, it is useful to have it summarized (Table 4).

In Section 4 we found that some classes have the property that a pattern has \mathcal{R} -completion if and only if the principal subpattern defined by the diagonal entries has \mathcal{R} -completion. In this case we say the problem reduces to the principal subpattern defined by the diagonal positions. These results are listed in column 2 of Table 2. In Section 3 we saw that for some classes we can create new patterns having \mathcal{R} -completion by building a block triangular pattern, each of whose blocks had \mathcal{R} -completion, or, equivalently, we can reduce the problem of \mathcal{R} -completion to the study of irreducible patterns (strongly connected mardigraphs). These results are summarized in column 3. In Section 5, we saw that for many classes, an \mathcal{R} -tree built from mardigraphs that have \mathcal{R} -completion will have \mathcal{R} -completion. Equivalently, for these classes we can reduce the \mathcal{R} -completion problem to the study of nonseparable mardigraphs. These results are summarized in column 4. For a class that has both the latter two properties ("Yes" in column 3 and "Yes" in column 4), the problem is thus reduced to the study of strongly connected nonseparable mardigraphs.

In Table 3, "Yes" means every partial \mathcal{R} -matrix specifying the mardigraph can be completed to a \mathcal{R} -matrix; "No" means there is a partial \mathcal{R} -matrix specifying the graph that cannot be completed to a \mathcal{R} -matrix. In Table 4 "completion" means \mathcal{R} -completion for the class being discussed, and a pattern for 3×3 matrices is called a 3×3 pattern, etc.

In Tables 2 and 3, "NA" means not applicable, i.e., positionally symmetric patterns are required when matrices must be symmetric. An asterisk (*) by a "Yes" means it is true for all positionally symmetric patterns (i.e., symmetric mardigraphs) of the given type. The notation $\$n$ at the head of a column in Tables 2 and 3 means that this property or mardigraph is discussed in Section n of this paper. In Table 4 the original sources of the results are listed.

As can be seen from the tables, the question of completion is completely answered for the classes of positive (semi)definite matrices, M -matrices, M_0 -matrices and $ss P_0$ -matrices. For some of the remaining classes considerable progress has been made, while for others very little is known.

Table 2: Results organized by pattern and strategy, including patterns omitting diagonal positions

Class	Reduces to principal subpattern determined by diagonal positions (§4)	Reduces to irreducible principal subpatterns (strongly connected mardigraphs) (§3)	Reduces to nonseparable mardigraphs (§5, §7)
P_0 -matrices	No	Yes	Yes
$P_{0,1}$ -matrices	No	Yes	Yes
P-matrices	Yes	Yes	Yes
weakly sign symmetric P_0 -matrices	No	Yes	Yes
weakly sign symmetric $P_{0,1}$ -matrices	No	Yes	Yes
M_0 -matrices	No	Yes	No
weakly sign symmetric P-matrices	Yes	Yes	Yes
M-matrices	Yes	Yes	No
Fischer matrices	No	Yes	Yes
Koteljanskii matrices	No	Yes	Yes for mardigraphs with all vertices specified.
ss P_0 -matrices	No	No	Yes
ss $P_{0,1}$ -matrices	No	No	Yes
ss P-matrices	Yes	Yes	Yes
non-negative P_0 -matrices	No	Yes	Yes
non-negative $P_{0,1}$ -matrices	No	Yes	Yes
non-negative P-matrices	Yes	Yes	Yes
inverse M-matrices	No	No	Yes
positive $P_{0,1}$ -matrices	No		
positive P-matrices	Yes	Yes	Yes
positive semidefinite	No	NA	Yes*
positive definite	Yes*	NA	Yes*
symmetric M_0 -matrices	No	NA	No
symmetric M-matrices	Yes*	NA	No
symmetric inverse M-matrices	No	NA	Yes*

Table 3: Results organized by mardigraph

Class	 (§4)	All mardigraphs of order 2 (§4)	All digraphs of order 3 (all vertices specified) (§4, §6, §7, §8, §9)	Any mardigraph with all vertices unspecified (§4)
P_0 -matrices	No	No	No	Yes
$P_{0,1}$ -matrices	Yes	Yes	Yes	Yes
P-matrices	Yes	Yes	Yes	Yes
weakly sign symmetric P_0 -matrices	No	No	No	Yes
weakly sign symmetric $P_{0,1}$ -matrices	Yes	Yes	No	Yes
M_0 -matrices	No	No	No	Yes
weakly sign symmetric P-matrices	Yes	Yes	No	Yes
M-matrices	Yes	Yes	No	Yes
Fischer matrices	Yes	Yes		No
Koteljanskii matrices	Yes	Yes		No
ss P_0 -matrices	No	No	No	Yes
ss $P_{0,1}$ -matrices	Yes	Yes	No	Yes
ss P-matrices	Yes	Yes	No	Yes
non-negative P_0 -matrices	No	No		Yes
non-negative $P_{0,1}$ -matrices	Yes	Yes	Yes	Yes
non-negative P-matrices	Yes	Yes	Yes	Yes
inverse M-matrices	Yes	Yes	No	No
positive $P_{0,1}$ -matrices	Yes	Yes		Yes
positive P-matrices	Yes	Yes	Yes	Yes
positive semidefinite	No	No	Yes*	Yes*
positive definite	Yes	Yes*	Yes*	Yes*
symmetric M_0 -matrices	No	No	No	Yes*
symmetric M-matrices	Yes	Yes*	No	Yes*
symmetric inverse M-matrices	Yes	Yes*		No

Table 3 (continued)

Class	Symmetric double triangle with all vertices specified (§9)	symmetric n-cycle with all vertices specified (§6, §7, §8, §9)	n-cycle with all vertices specified (§6, §7, §8, §9)
P_0 -matrices	No	No for $n=4$	
$P_{0,1}$ -matrices		Yes	Yes
P-matrices	Yes	Yes	Yes
weakly sign symmetric P_0 -matrices	No	No	No for $n=3,4$
weakly sign symmetric $P_{0,1}$ -matrices	No	No	No for $n=3$
M_0 -matrices	No	No	No
weakly sign symmetric P-matrices	No	No for $n=4$	No for $n=3$
M-matrices	No	No	No
Fischer matrices	No	No for $n=4$	
Koteljanskii matrices	No	No for $n=4$	
ss P_0 -matrices		No	No
ss $P_{0,1}$ -matrices	No	No	No for $n=3$
ss P-matrices	No	No for $n=4$	No for $n=3$
non-negative P_0 -matrices	No	No for $n=4$	No for $n=4$
non-negative $P_{0,1}$ -matrices	No	Yes	Yes
non-negative P-matrices	No	Yes	Yes
inverse M-matrices	No	No	No
positive $P_{0,1}$ -matrices	No		
positive P-matrices	No	Yes	Yes
positive semidefinite	Yes	No	NA
positive definite	Yes	No	NA
symmetric M_0 -matrices	No		NA
symmetric M-matrices	No		NA
symmetric inverse M-matrices	No	No	NA

Table 4: Summary of Results for each Class

Class	Summary of Results
P_0 -matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected nonseparable mardigraphs. Any mardigraph with all vertices unspecified has completion. Some 2×2 patterns lack completion. [FJTU],[JK], §4, §5, §9.
$P_{0,1}$ -matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected nonseparable mardigraphs. Every 2×2 pattern has completion. Some 3×3 patterns lack completion, but any 3×3 that omits an off-diagonal position has completion. Any mardigraph with all vertices unspecified has completion. (Symmetric) cycles with all vertices specified have completion. [FJTU], §4, §5, §8.
P-matrices	Every positionally symmetric pattern has completion. Every 3×3 pattern has completion. Family of digraphs that do not have completion. Reduces to principal subpattern determined by diagonal positions. Reduces to strongly connected nonseparable digraphs. Cycles with all vertices specified have completion. [JK], [DH], §5, §8.
weakly sign symmetric P_0 -matrices	Does not reduce to principal subpattern determined by the diagonal. Positionally symmetric pattern that includes the diagonal has completion iff its graph is block-clique. Reduces to strongly connected nonseparable mardigraphs. Any mardigraph with all vertices unspecified has completion. Some 2×2 patterns lack completion. [FJTU], §4, §5, §9.
weakly sign symmetric $P_{0,1}$ -matrices	Does not reduce to principal subpattern determined by the diagonal. Positionally symmetric pattern that includes the diagonal has completion iff its graph is block-clique. Reduces to strongly connected nonseparable mardigraphs. Any mardigraph with all vertices unspecified has completion. Every 2×2 pattern has completion. Some 3×3 patterns lack completion. [FJTU], §4, §5, §9.
M_0 -matrices	Done. A pattern has completion if and only if it is permutation similar to a block triangular pattern in which each diagonal block is complete or omits all diagonal positions (if and only if every strongly connected induced submardigraph is a clique or has all vertices unspecified). Does not reduce to principal subpattern determined by the diagonal. §7.
weakly sign symmetric P-matrices	Reduces to principal subpattern determined by diagonal positions. Reduces to strongly connected nonseparable mardigraphs. Every 2×2 pattern has completion. Some 3×3 patterns lack completion. [FJTU], §4, §5, §9.
M-matrices	Done. Reduces to principal subpattern determined by diagonal positions. A pattern that includes the diagonal has completion if and only if it is permutation similar to a block triangular with complete diagonal blocks (if and only if every strongly connected induced subdigraph is a clique). [H2].
Fischer matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected nonseparable mardigraphs. Every 2×2 pattern has completion. [FJTU], §4, §5.
Koteljanskii matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected mardigraphs. Every block-clique graph has completion. Every 2×2 pattern has completion. [FJTU], §4.
sign symmetric P_0 -matrices	Done. A pattern has completion iff each connected component of its mardigraph has all vertices unspecified or is block-clique. Does not reduce to principal subpattern determined by the diagonal. [FJTU], §4, §9.
sign symmetric $P_{0,1}$ -matrices	A positionally symmetric pattern that includes the diagonal has completion iff its graph is block-clique. Does not reduce to principal subpattern determined by the diagonal. Does not reduce to strongly connected mardigraphs. Reduces to nonseparable mardigraphs. Every 2×2 pattern has completion. Some 3×3 patterns lack completion. [FJTU], §4, §5.

sign symmetric P-matrices	Reduces to principal subpattern determined by diagonal positions. Reduces to strongly connected nonseparable mardigraphs. Every 2x2 pattern has completion. Some 3x3 patterns lack completion. [FJTU], §4, §5.
non-negative P ₀ -matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected nonseparable mardigraphs. Any mardigraph with all vertices unspecified has completion. One 2x2 pattern lacks completion. Some 2x2 patterns lack completion. [FJTU], §4, §5, §9.
non-negative P _{0,1} -matrices	Does not reduce to principal subpattern determined by the diagonal. Reduces to strongly connected nonseparable mardigraphs. Every 2x2 pattern has completion. Some 3x3 patterns lack completion, but any 3x3 that omits an off-diagonal position has completion. Any mardigraph with all vertices unspecified has completion. (Symmetric) cycles with all vertices specified have completion. [FJTU], §4, §5, §8.
non-negative P-matrices	Reduces to principal subpattern determined by diagonal positions. Reduces to strongly connected nonseparable digraphs. Every 3x3 pattern has completion. (Symmetric) cycles with all vertices specified have completion. [FJTU], §4, §5, §8.
inverse M-matrices	A positionally symmetric pattern that includes the diagonal has completion iff its graph is block-clique. A pattern has completion iff in the pattern-mardigraph the induced subdigraph of every alternate path to a single arc is complete and every nonseparable strongly connected induced submardigraph has completion. A nonseparable strongly connected induced submardigraph that has completion must be homogeneous. A cycle has completion iff the induced submardigraph of the cycle is complete or is the cycle itself and at least one of the vertices of the cycle is not specified. [JS1], [H1], [H3], §6.
positive P _{0,1} -matrices	Because questions remain regarding induced subdigraphs and disconnected graphs, the study of this class is problematic.
positive P-matrices	Reduces to principal subpattern determined by diagonal positions. Reduces to strongly connected nonseparable digraphs. Every 3x3 pattern has completion. (Symmetric) cycles with all vertices specified have completion. [FJTU], §4, §5, §8.
positive semidefinite	Done. A (positionally symmetric) pattern has completion iff each connected component of its mardigraph has all vertices unspecified or is chordal with all vertices specified. Does not reduce to principal subpattern determined by the diagonal. [GJSW], [JS3], §4.
positive definite	Done. Reduces to principal subpattern determined by diagonal positions. A (positionally symmetric) pattern that includes the diagonal has completion iff its graph is chordal [GJSW].
symmetric M ₀ -matrices	Done. A (positionally symmetric) pattern has completion if and only if it is permutation similar to a block diagonal pattern in which each diagonal block is complete or omits all diagonal positions (if and only if each component of the mardigraph is either a clique or has all vertices are unspecified). Does not reduce to principal subpattern determined by the diagonal. §4, [H5].
symmetric M-matrices	Reduces to principal subpattern determined by diagonal positions. A (positionally pattern) that includes the diagonal has completion if and only if it is permutation similar to a block diagonal pattern in which each diagonal block is complete (if and only if every component of the graph is a clique). §4, [H5].
symmetric inverse M-matrices	Done. A (positionally symmetric) pattern has completion iff its graph is block-clique and no diagonal position is omitted that corresponds to a vertex in a block of order > 2. Does not reduce to principal subpattern determined by the diagonal. [JS2], §4, [H4].

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