The Minimum Rank of Symmetric Matrices Described by a Graph: A Survey *

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Abstract

The minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. This paper surveys the current state of knowledge on the problem of determining the minimum rank of a graph and related issues.

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1 Introduction

The *minimum rank problem for a simple graph* (the *minimum rank problem* for short) asks us to determine the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G; this problem has received considerable attention (see references). Since the maximum rank is always the order of G (e.g., use a diagonally dominant matrix), there is no interest in an analogous maximum rank problem. Furthermore, it is straightforward (by considering rank one diagonal perturbations) that any rank between the minimum and full rank can be achieved.

The zero-nonzero pattern described by the graph has tremendous influence on minimum rank; for example, a matrix associated with a path on n vertices is a symmetric tridiagonal matrix (up to labeling) with nonzero sub- and super-diagonal and thus has minimum rank $n-1$, whereas the complete graph on n vertices has minimum rank 1.

The solution to the minimum rank problem is equivalent to the determination of the maximum multiplicity of an eigenvalue among the same family of matrices. The inverse eigenvalue problem of a graph has been an important motivation for the study of minimum rank. Given a collection of real numbers, $\lambda_1, \ldots, \lambda_n$, the problem of finding a symmetric matrix A that satisfies certain properties and has $\lambda_1, \ldots, \lambda_n$ as its eigenvalues is called an *Inverse Eigenvalue Problem*. The *Inverse Eigenvalue Problem of a Graph (IEPG)* asks us to determine, for a given graph G, what eigenvalues are possible for a real symmetric matrix A having nonzero off-diagonal entries

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determined by the adjacency of G. (For a discussion of connections between IEPG and spectral graph theory, see [30].)

Perhaps the earliest result on multiplicities of eigenvalues of the family of real symmetric matrices described by a graph is Parter's 1960 paper [51] on trees, and most of the work on trees since then has relied on this work and Wiener's 1984 paper [55]. In 1996 Nylen [50] initiated the study of minimum rank of the symmetric matrices described by a graph and gave a method for its computation for a tree, subsequently improved by Johnson and Leal-Duarte [37], Wei and Weng [54], Johnson and Saiago [42], and others, so the minimum rank of a tree is readily computable (see algorithms in Subsection 2.1). In contrast, although the IEPG has been solved for some families of trees, for most trees the IEPG is unsolved. Only limited progress on the minimum rank problem has been made on graphs that are not trees, with most of the progress in the last few years.

The rank or minimum rank of a family of matrices associated with a graph has also played a role in various other problems, including:

- singular graphs, nullity of the adjacency matrix;
- biclique decompositions and the biclique partition number (Graham-Pollack Theorem);
- Hermitian rank and inertia;
- eigensharp graphs;
- Lovász ϑ function and orthonormal labellings of graphs.

In spectral graph theory one of the most important tools is the (0,1) adjacency matrix (A_G), and a long-standing open problem is to characterize the graphs G whose adjacency matrix is singular; such graphs are often called *singular graphs* (see also [17]). In addition, many people have studied the nullspace of the adjacency matrix, paying particular attention to the nullity (or the dimension of the nullspace). The nullity of a bipartite graph seems to be of interest in chemistry (see [19]). The nullity of a tree has long been known (see [18]) and investigations about the nullity of various more general graphs continues currently.

Partitioning the edges of a graph has been an important research area historically in graph theory (see [25, 45, 49, 52]). One particular partition is a *biclique partition* — that is, a partition of the edges of a graph into subgraphs in such a way that each subgraph is a complete bipartite graph (or a biclique). The size of the smallest such partition is called the biclique partition number of G and is denoted by $bp(G)$ (see also [29]). In the seminal paper of Graham and Pollack [25] it is proved that

$$bp(G) \geq \max\{i_+(A_G), i_-(A_G)\}, \tag{1}$$

where A_G is the (0,1) adjacency matrix of G and i_+ (i_-) is the number of positive (negative) eigenvalues of a symmetric matrix. In the special case when G is a bipartite graph it follows easily that $bp(G) \geq \frac{1}{2}\text{rank}(A_G)$.

When equality holds in (1), the graph is called *eigensharp* (see [45]). Many graphs are known to be eigensharp (e.g. trees, C_n, K_n), although a complete description of all eigensharp graphs is still unknown.

Another connection to rank is seen through the variant notion of Hermitian rank. For any $n \times n$ Hermitian matrix B, the *Hermitian rank* of B denoted by $h(B)$, is the smallest k such that $B = XY^* + YX^*$ for $n \times k$ complex matrices X, Y. In [26] it is shown that $h(B) = \max\{i_+(B), i_-(B)\}$. So in particular, $bp(G) \geq h(A_G)$.

Along similar lines, graph labellings is another prominent subject in graph theory that also has ties to minimum rank. Suppose $G = (V, E)$ is a graph. Then an *orthonormal labeling of G of dimension d* is a function $f : V \longrightarrow \mathbb{R}^d$ such that $f(u) \cdot f(v) = 0$ whenever vertices u and v are not

adjacent, and $|f(u)| = 1$ for all $u \in V$ (see [22]). Let $d(G)$ denote the smallest dimension d over all orthonormal labellings of G. Then it is easy to see that $d(G)$ is equal to the minimum rank of a symmetric positive semidefinite matrix whose graph is given by G (see definition of $\mathcal{G}(B)$ below). Furthermore, Lovász [47] has shown that

$$\alpha(G) \leq \vartheta(G) \leq d(G) \leq \chi(\bar{G}),$$

where $\alpha(G)$ is the size of the largest independent set, $\vartheta(G)$ is the so-called Lovász ϑ function, and $\chi(\bar{G})$ is the chromatic number of the complement of G (i.e., the clique cover number of G). It is noted in [22], and is easy to verify, that $d(G)$ is equal to the minimum rank of a matrix lying in the set

$$\{X : X \in S_n^+, I - A_G \leq X \leq I + A_G\},$$

where the ordering is entry-wise and S_n^+ denotes the set of real $n \times n$ positive semidefinite matrices.

Definitions, notation, and elementary results on the minimum rank problem are described in this section. Section 2 summarizes the current state of knowledge of the minimum rank problem. Section 3 surveys minimum rank problems for other families of matrices described by a graph. Except when explicitly stated otherwise (primarily in Section 3), all matrices discussed are real and symmetric and all graphs are simple (in fact, we will define "graph" to mean a simple graph).

We begin with the basic definitions and the association of matrices and graphs. A *graph* is a pair $G = (V, E)$, where V is the set of vertices (usually $\{1, \ldots, n\}$ or a subset thereof) and E is the set of edges (an edge is a two-element subset of vertices). A *general graph* allows multiple edges and/or loops. Every graph or general graph is finite (finite number of vertices and finite number of edges) and has nonempty vertex set. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

A *path* is a graph $P_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1, \ldots, n-1\}$. A *cycle* is a graph $C_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1, \ldots, n-1\} \cup \{\{v_n, v_1\}\}$. The *length* of a path or cycle is the number of edges. A *complete graph* is a graph $K_n = (\{v_1, \ldots, v_n\}, E)$ such that $E = \{\{v_i, v_j\} : i \neq j, i, j = 1, \ldots, n\}$. A graph (V, E) is *bipartite* if the vertex set V can be partitioned into two parts U, W, such that no edge of E has both endpoints in one part. A *complete bipartite graph* is a graph $K_{p,q} = (U \cup W, E)$ such that $|U| = p, |W| = q$ and $E = \{\{u, w\} : u \in U, w \in W\}$.

Let S_n denote the set of real symmetric $n \times n$ matrices. For a matrix $A \in S_n$, the *spectrum* of A is the multiset of the n roots of the characteristic polynomial, and is denoted by $\sigma(A)$. For $B \in S_n$, let $\text{mult}_B(\lambda)$ denote the multiplicity of λ as a root of the characteristic polynomial of B (i.e., the multiplicity of λ if λ is an eigenvalue of B and 0 otherwise).

For $B \in S_n$, the *graph* of B, denoted $\mathcal{G}(B)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} | \, b_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of B is ignored in determining $\mathcal{G}(B)$. In addition, we let $\mathcal{S}(G) = \{B \in S_n : \mathcal{G}(B) = G\}$. The *adjacency matrix* A_G of a graph $G = (\{1, \ldots, n\}, E)$ is the 0,1-matrix defined by $(A_G)_{ij} = 1$ if and only if $i \neq j$ and $\{i, j\} \in E$. The *Laplacian matrix* of G is $L_G = \text{diag}(\deg(1), \ldots, \deg(n)) - A_G$ and the *signless Laplacian matrix* of G is $|L_G| = \text{diag}(\deg(1), \ldots, \deg(n)) + A_G$ (here $\deg(v)$ denotes the degree, or the number of edges incident with v). Clearly $\mathcal{G}(A_G) = \mathcal{G}(L_G) = \mathcal{G}(|L_G|) = G$.

Example 1.1. For the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, $\mathcal{G}(B)$ is shown in Figure 1.

The *minimum rank* of G is defined to be

$$\text{mr}(G) = \min\{\text{rank}(B) : \; B \in S_n \text{ and } \mathcal{G}(B) = G\}.$$

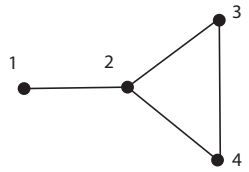


Figure 1: The graph $\mathcal{G}(B)$ for B in Example 1.1

The *maximum multiplicity* of G is given as

$$M(G) = \max\{\text{mult}_B(\lambda) : \ \lambda \in \mathbb{R}, B \in S_n \text{ and } \mathcal{G}(B) = G\}.$$

Since maximum multiplicity of any eigenvalue is the same (by translation by a scalar matrix) as maximum multiplicity of eigenvalue 0, maximum multiplicity is sometimes called *maximum nullity* or *maximum corank*.

Observation 1.2. *The following results are well-known and straightforward.*

1. $M(G) + \text{mr}(G) = |G|$.
2. $\text{mr}(G) \leq |G| - 1$.
3. $\text{mr}(P_n) = n - 1$.
4. *For* $n \geq 2$, $\text{mr}(K_n) = 1$. *If* G *is connected,* $mr(G) = 1$ *implies* $G = K_{|G|}$.
5. $\text{mr}(K_{p,q}) = 2$.

Example 1.3. Let G be the graph in Figure 1 and let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Since $\mathcal{G}(A) = G$ and $G \neq K_4$, $\text{mr}(G) = 2$. (The matrix A also illustrates Observation 1.8 below).

The fact that a path is the only graph having $\text{mr}(G) = |G| - 1$ (cf. Observation 1.2.3), although well-known, is nontrivial. It is a consequence of the following theorem.

Theorem 1.4. [23] (Fiedler's Tridiagonal Matrix Theorem) *If* A *is a real symmetric* $n \times n$ *matrix such that for all real diagonal matrices* D, $\text{rank}(A + D) \geq n - 1$, *then* A *is irreducible and there is a permutation matrix* P *such that* $P^T AP$ *is tridiagonal.*

Corollary 1.5. $\text{mr}(G) = |G| - 1$ *if and only if* $G = P_{|G|}$.

If $R \subseteq \{1, 2, \ldots, n\}$ and $B \in S_n$, then $B[R]$ denotes the *principal submatrix* of B whose rows and columns are indexed by R, and $B(R)$ is the complementary principal submatrix obtained from B by deleting the rows and columns indexed by R. In the special case when $R = \{k\}$, we use $B(k)$ to denote $B(R)$.

A graph $G' = (V', E')$ is a *subgraph* of graph $G = (V, E)$ if $V' \subseteq V, E' \subseteq E$. The subgraph $G[R]$ of $G = (V, E)$ *induced* by $R \subseteq V$ is the subgraph with vertex set R and edge set $\{\{i, j\} \in E \mid i, j \in R\}$; $G(R)$ is used to denote $G[V \setminus R]$. The result $G(\{v\})$ of deleting a vertex v is also denoted by $G - v$. A graph G is *connected* if for any two vertices v, w, G contains as a subgraph a path from v to w. Otherwise it is *disconnected*. (A graph of order one is connected by

definition.) A *(connected) component* of a graph is a maximal connected subgraph. The *distance* between two vertices in a graph G is the number of edges in a shortest path between them. The *diameter* of G, $\operatorname{diam}(G)$, is the maximum distance between any two vertices of G.

Observation 1.6. *Let $B \in S_n$. The following results are well-known and straightforward.*

1. *If the connected components of G are $G_1, \ldots, G_t$, then*
$$\operatorname{mr}(G) = \sum_{i=1}^{t} \operatorname{mr}(G_i) \qquad \text{and} \qquad M(G) = \sum_{i=1}^{t} M(G_i).$$
2. *If G' is an induced subgraph of G then $mr(G') \leq mr(G)$.*
3. *For the cycle on n vertices, $\operatorname{mr}(C_n) = n - 2$ (see below).*
4. *For a connected graph G, $\operatorname{diam}(G) \leq \operatorname{mr}(G)$ (G contains $P_{\operatorname{diam}(G)+1}$ as an induced subgraph.).*
5. $\operatorname{rank}(B) - 2 \leq \operatorname{rank}(B(k)) \leq \operatorname{rank}(B)$.
6. [50] *For any vertex v of G, $0 \leq \operatorname{mr}(G) - \operatorname{mr}(G - v) \leq 2$.*
7. [50] *Adding or removing an edge from a graph G can change minimum rank by at most 1.*
8. *If $R \subseteq \{1, \ldots, n\}, k \in \{1, \ldots, n\}$, then $\mathcal{G}(B[R]) = \mathcal{G}(B)[R]$ and $\mathcal{G}(B(k)) = \mathcal{G}(B) - k$. (There is a slight abuse of notation here- technically we need to index the entries of $B[R]$ by R.)*

For item 3 above, certainly $\operatorname{mr}(C_n) < n - 1$ since C_n is not a path. However, C_n contains the path on $n - 1$ vertices as an induced subgraph, so $\operatorname{mr}(C_n) \geq n - 2$.

To determine the minimum rank of a graph, it is sufficient to be able to determine minimum rank of its connected components (cf. Observation 1.6.1). Hence it is customary to restrict consideration to connected graphs.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets from V that are not in E. The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^{h} G_i = (\cup_{i=1}^{h} V_i, \cup_{i=1}^{h} E_i)$. The *intersection* of $G_i = (V_i, E_i)$ is $\cap_{i=1}^{h} G_i = (\cap_{i=1}^{h} V_i, \cap_{i=1}^{h} E_i)$ (provided the intersection of the vertices is nonempty).

If $G = \cup_{i=1}^{h} G_i$, a matrix A of rank at most $\sum_{i=1}^{h} \operatorname{mr}(G_i)$ having $\mathcal{G}(A) = G$ can be obtained by choosing (for each $i = 1, \ldots, h$) a matrix A_i that realizes $\operatorname{mr}(G_i)$, embedding A_i in a matrix $\widetilde{A_i}$ of size $|G|$, choosing $a_i \in \mathbb{R}$ such that no cancellation of nonzero entries occurs, and letting $A = \sum_{i=1}^{h} a_i \widetilde{A_i}$.

Observation 1.7. *If $G = \cup_{i=1}^{h} G_i$ then $\operatorname{mr}(G) \leq \sum_{i=1}^{h} \operatorname{mr}(G_i)$.*

A subgraph G' of a graph G is a *clique* if G' has an edge between every pair of vertices of G' (i.e., G' is isomorphic to $K_{|G'|}$). A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a *clique covering* of G. The *clique covering number* of G, denoted by $\operatorname{cc}(G)$, is the smallest number of cliques in a clique covering of G. The next corollary, which is well-known, is a special case of Observation 1.7.

Observation 1.8. *If G is a graph, $\operatorname{mr}(G) \leq \operatorname{cc}(G)$.*

A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. More generally, v is a cut-vertex of a graph G if v is a cut-vertex of a component of G. A graph is 2-*connected* if its order is at least 3 and it has no cut-vertex. A subset S of vertices of a connected graph G is a *cut-set* if $G(S)$ is disconnected. More generally, S is a cut-set of a graph G if S is contained in one component of G and S a cut-set of that component of G.

2 Minimum Rank of Symmetric Matrices described by a Simple Graph

The minimum rank of a tree or unicyclic graph is easy to compute, and in Subsections 2.1 and 2.4 we include algorithms for computation of the minimum rank of a tree or unicyclic graph. Progress on the minimum rank problem for graphs that are not trees has come through several different approaches. Graphs having relatively extreme minimal rank have been characterized (see Subsection 2.2). Minimum rank can sometimes be computed by a reduction operation that allows computation of minimum rank from the minimum ranks of proper subgraphs. This has been done by deleting cut-vertices (Subsection 2.3) or viewing the graph as a join (Subsection 2.5). The Colin de Verdière type parameter ξ (see Subsection 2.6) can be used to bound minimum rank from above (and induced subgraphs can be used to bound minimum rank from below).

At the AIM workshop [1], the minimum ranks of several families of graphs were computed, and subsequently the minimum rank of all graphs of order less than or equal to six was computed. This information is available on-line in the form of a minimum rank graph catalog [2], and will be updated regularly.

2.1 Trees

A graph is *acyclic* (also called a *forest*) if it does not contain a cycle. A *tree* is a connected acyclic graph. Throughout this subsection, T will denote a tree.

A simple consequence of the Cauchy interlacing inequalities [28, Fact 8.2.5] is that for any real symmetric matrix B,

$$\text{mult}_B(\lambda) - 1 \leq \text{mult}_{B(k)}(\lambda) \leq \text{mult}_B(\lambda) + 1.$$

One might expect that it is most common for multiplicity to decrease or perhaps remain the same, but for trees, the work of Parter [51] and Wiener [55] shows that for a matrix whose pattern of nonzero off-diagonal entries is described by a tree, for multiplicity greater than one there must always be a vertex whose deletion *raises* the multiplicity; such a vertex is called a *Parter-Wiener (PW)* vertex. Vertex k is a *strong PW vertex* of B for λ if k is a PW vertex of B for λ and λ is an eigenvalue of at least three components of $\mathcal{G}(B) - k$. What we refer to as a strong PW vertex has also been called a *Parter vertex* or a *Wiener vertex* in the literature, and the term *weak* has sometimes been applied to what is designated here as a PW vertex. The structure of the eigenspace is analyzed in terms of PW vertices in [43].

Theorem 2.1. [39, 51, 55] (Parter-Wiener Theorem) *If T is a tree, $\mathcal{G}(B) = T$ and $\text{mult}_B(\lambda) \geq 2$, then there is a strong PW vertex of B for λ.*

The minimum rank problem is a small part of the Inverse Eigenvalue Problem of a Graph. With the exception of results about maximum multiplicity of an eigenvalue, most of the progress on the IEPG has been limited to specific families of trees. Much of this work is based on determination of possible ordered list of multiplicities for the tree.

If the distinct eigenvalues of $B \in S_n$ are $\breve{\beta}_1 < \cdots < \breve{\beta}_q$ with multiplicities $m_1, \ldots, m_q$, then $(m_1, \ldots, m_q)$ is called the *ordered multiplicity list* of B and the number of distinct eigenvalues of $B \in S_n$ is denoted $q(B)$. In addition, the minimum number of distinct eigenvalues of a graph G is $q(G) = \min\{q(B) : B \in \mathcal{S}(G)\}$.

In the works [4], [23], [38], and [40] the possible ordered multiplicity lists of matrices in $\mathcal{S}(T)$ have been determined for the following families of trees:

- paths
- double paths

- stars
- generalized stars
- double generalized stars

Furthermore, for T in any of these families above, if there is a matrix $B \in \mathcal{S}(T)$ with distinct eigenvalues $\breve{\beta}_1 < \cdots < \breve{\beta}_r$ having multiplicities $m_1, \ldots, m_r$, then for any real numbers $\gamma_1 < \cdots < \gamma_r$, there is a matrix in $\mathcal{S}(T)$ having eigenvalues $\gamma_1, \ldots, \gamma_r$ with multiplicities $m_1, \ldots, m_r$. Thus for any of these trees, determination of the possible ordered multiplicity lists of the graph is equivalent to the solution of the Inverse Eigenvalue Problem of the graph, and that problem has been solved in these cases.

In the paper [3] it was shown that there can be restrictions on which real numbers may appear as the eigenvalues for an attainable ordered multiplicity list. That is, there exist trees for which the determination of the ordered multiplicity list of the tree is not equivalent to the solution of the Inverse Eigenvalue Problem of that tree.

In [46] it was shown that $q(T) \geq \operatorname{diam}(T)+1$. Later Barioli and Fallat [3] gave the first example showing that $q(t) > \operatorname{diam}(T) + 1$ is possible (the example is what is now called the (3,2)-whirl). Kim and Shader [44] generalized this class and noted that the gap can be worse (as a function of n). In their work, [44] they introduced the use of Smith Normal Form as a tool to study this aspect of the IEPG.

The minimum rank of a tree T can be computed easily by computing one the parameters $\Delta(T)$ or $P(T)$. For a graph G,

$$\Delta(G) = \max\{p - q : \text{there is a set of } q \text{ vertices whose deletion leaves } p \text{ paths}\}.$$

(Note that an isolated vertex is a path of order 1.) The *path cover number* of G, $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G; such a set of paths realizing $P(G)$ is called a *minimal path cover*. Note that some authors do not require the paths to be induced; this distinction is irrelevant for trees, but relevant for graphs that are not trees.

Theorem 2.2. [37] $M(T) = P(T) = \Delta(T) = |T| - \operatorname{mr}(T)$.

The parameters $\Delta(T)$, or $P(T)$, and hence $M(T)$ and $\operatorname{mr}(T)$, can be computed by any of a number of algorithms, including Algorithm 2.3, due to Johnson and Saiago [42], or its variant Algorithm 2.5 below (see also [50] for an algorithm using appropriate vertices). A vertex of degree at least 3 is called a *high degree vertex*. The set of high degree vertices of G is denoted $H(G)$.

Algorithm 2.3. [28, Algorithm 34.1] *Computation of* $\operatorname{mr}(T)$ *and* $\Delta(T)$.
Input: A tree T.
Output: $\operatorname{mr}(T), \Delta(T)$, *and a set* Q *of vertices of* T *whose deletion realizes* $\Delta(T)$.

1. *Set* $Q = \emptyset$ *and* $T' = T$.
2. *While* $H(T') \neq \emptyset$:
   (a) *Remove from* T' *the set* Q' *of all vertices* $v \in H(T')$ *such that*
   $\deg_{T'}(v) - \deg_{T'[H(T')]}(v) \geq 2$
   ($T'[H(T')]$ *is the subgraph of* T' *induced by the high degree vertices of* T').
   (b) $Q = Q \cup Q'$.

3. $\Delta(T) = p - |Q|$ *where* p *is the number of components (all of which are paths) in* $T(Q)$.

4. $\mathrm{mr}(T) = |T| - \Delta(T)$.

Algorithm 2.5 below is sometimes more useful than Algorithm 2.3. Algorithm 2.5 can be easily adapted to compute path cover number and the minimum rank of a unicyclic graph (see Algorithm 2.16 in Subsection 2.4). Algorithm 2.5 can also be generalized to compute the minimum rank of a tree that allows loops (see Section 3).

The method of computation of $\Delta(T)$ in Algorithm 2.5 uses pendent generalized stars and works from the outside in. A path P in G is a *pendent path* of vertex v if P is a component of $G - v$ and (in G) P is connected to v by one of its end-points. A tree is a *generalized star* if it has at most one high degree vertex; if a generalized star has a high degree vertex, this vertex is called the *center*. A *pendent generalized star* of G is a connected induced subgraph S of G such that: there is exactly one high degree vertex v of G in S (v is called the *center* of S); $\deg_G(v) = k + 1$ and exactly k of the components of $G - v$ are pendent paths of v; S is induced by the vertices of the k pendent paths and v. The following result is known, but we are not aware of a source (it does follow from [50] where it is shown that trees have appropriate vertices, since choosing an outermost appropriate vertex will yield a pendent generalized star).

Lemma 2.4. *Any tree* T *contains a pendent generalized star or is a generalized star.*

Proof. If T has no high degree vertices, then T is a path, which is a generalized star (without center). If T has at least one high degree vertex, define T_1 to be the tree obtained from T by removing all pendent paths. If $|T_1| = 1$ then T is a generalized star. If $|T_1| > 1$, let v be a vertex of degree 1 in T_1 (such a vertex must exist since T_1 is a tree). Then v together with its pendent paths forms a pendent generalized star. □

Algorithm 2.5. *Computation of* $\mathrm{mr}(T)$ *and* $\Delta(T)$ *for a tree* T.
Input: A tree T.
Output: $\mathrm{mr}(T), \Delta(T)$, *and a set* Q *of vertices of* T *whose deletion realizes* $\Delta(T)$.

1. *Set* $Q = \emptyset$ *and* $T' = T$.
2. *While* T' *has a pendent generalized star:*
   (a) *Let* Q' *be the the centers all pendent generalized stars of* T'.
   (b) *Remove the vertices in* Q' *from* T'.
   (c) $Q = Q \cup Q'$.
3. *If* T' *is a generalized star with center* v, $Q = Q \cup \{v\}$.
4. $\Delta(T) = p - |Q|$ *where* p *is the number of components in* $T(Q)$.
5. $\mathrm{mr}(T) = |T| - \Delta(T)$.

Example 2.6. We use Algorithm 2.5 to compute $\Delta(T) = M(T)$ and $\mathrm{mr}(T)$ for the tree T shown in Figure 2.
Step 2:
First iteration: $Q = Q' = \{1, 3, 5, 9\}$.
Second iteration: $Q' = \{4, 7\}$ and $Q = \{1, 3, 4, 5, 7, 9\}$.
Step 3: The tree $T(Q)$ is shown in Figure 3. $\Delta(T) = M(T) = 19 - 6 = 13$.
Step 4: $\mathrm{mr}(T) = 32 - 13 = 19$.

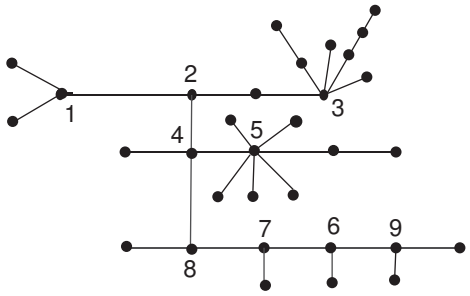


Figure 2: The tree T in Example 2.6

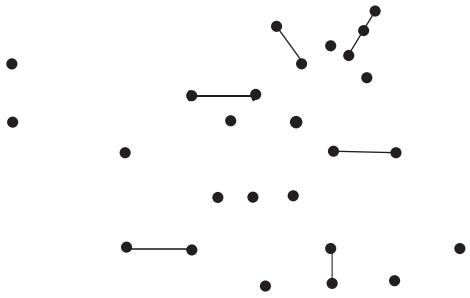


Figure 3: The forest $T(Q)$ in Example 2.6

Unfortunately, the results of this subsection fail for graphs that are not trees. For example, the cycle has $\mathrm{mr}(C_n) = n - 2$ so there is a matrix B having $\mathcal{G}(B) = C_n$ and $\mathrm{mult}_B(0) = 2$, but there is no PW vertex, since the deletion of any vertex leaves P_{n-1}. For graphs that are not trees, Theorem 2.2 need not be true either. In fact, P and M are non-comparable (see Example 2.7). From the Cauchy interlacing theorem, for any simple graph G, $\Delta(G) \leq M(G)$, and it is shown in [6] that for any graph G, $\Delta(G) \leq P(G)$. See [6] and [7] for more information about the relationship between the parameters M, P, Δ.

Example 2.7. The wheel on 5 vertices, W_5, shown in Figure 4, has $P(W_5) = 2$ by inspection, and $\mathrm{mr}(W_5) = 2$ (by Theorem 2.8 below), so $M(W_5) = 3 > P(W_5)$. The penta-sun, H_5, which is

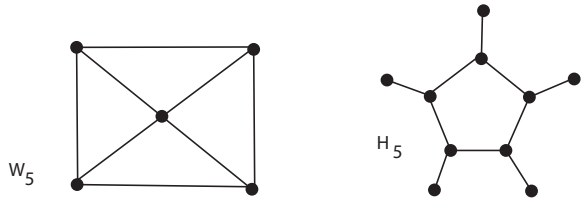


Figure 4: The wheel W_5 and the penta-sun H_5

shown in Figure 4, has $P(H_5) = 3$ by inspection. By repeated application of Theorem 2.11 below, it is shown in [6] that $\mathrm{mr}(H_5) = 8$ and thus that $P(H_5) > M(H_5) = 2$.

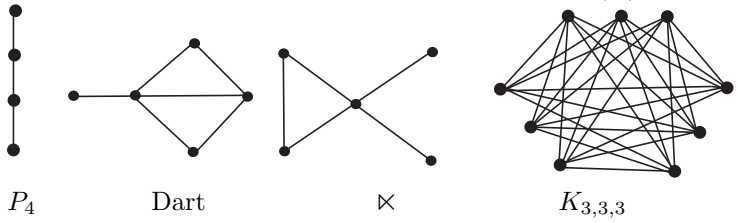
2.2 Graphs having extreme minimum rank

As noted in Section 1, a graph G has minimum rank $|G|-1$ if and only if $G = P_{|G|}$, and a connected graph G has minimum rank 1 if and only if $G = K_{|G|}$. In the past few years the graphs having minimum rank 2 and $|G|-2$ have been characterized. Unfortunately, the proofs are difficult and do not appear to easily generalize (although progress along related lines continues currently).

Barrett, van der Holst and Loewy [9] characterized graphs having minimum rank 2 (and initiated the study of minimum rank over fields other than the real numbers - see Subsection 3.1). They gave characterizations using forbidden induced subgraphs (Theorem 2.8) and the graph complement (Theorem 2.9).

Theorem 2.8. [9] *A connected graph G has $mr(G) \leq 2$ if and only if G does not contain as an induced subgraph any of P_4, Dart, $\ltimes$, or $K_{3,3,3}$ (the complete tripartite graph), all shown in Figure 5.*

Figure 5: Forbidden induced subgraphs for $mr(G) \leq 2$



Theorem 2.9. [9] *A graph G has* $\mathrm{mr}(G) \leq 2$ *if and only if the complement of G is of the form*

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

for appropriate nonnegative integers $k, s_1, s_2, p_1, q_1, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$ for all $i = 1, \ldots, k$.

The graph operation denoted by $\vee$ in Theorem 2.9 is referred to as the join and is formally defined in Subsection 2.5 below.

Graphs G having minimum rank $|G|-2$ have been characterized by Hogben and van der Holst [31] using ξ (see Theorems 2.10 and 2.26) and independently (and for fields other than the real numbers) by Johnson, Loewy, and Smith [41].

The *dual* of a plane embedding of a planar graph G is obtained as follows: Place a new vertex in each face of the embedding; these are the vertices of the dual. Two dual vertices are adjacent if and only if the two faces of G share an edge of G. A *linear 2-tree* is a 2-connected graph G that can be embedded in the plane such that the graph obtained from the dual of G after deleting the vertex corresponding to the infinite face is a path. Equivalently, a linear 2-tree is a "path" of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle. An example of a linear 2-tree is shown in Figure 6.

Theorem 2.10. [31] *Let G be a 2-connected graph. Then* $\mathrm{mr}(G) = |G| - 2$ *if and only if G is a linear 2-tree.*

As an application of these results (and Theorem 2.11 below), the minimum rank of any graph of order six or less has been determined [2], because graphs having minimum rank 1, 2, 4, 5 are characterized.

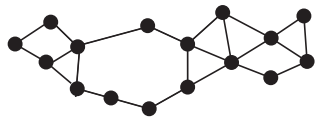


Figure 6: A linear 2-tree

2.3 Graphs with cut-vertices

The *rank-spread* of G at vertex v is defined to be

$$r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v).$$

As noted in [50] (cf. Observation 1.6.6), for any vertex v of G, $0 \leq r_v(G) \leq 2$. The following theorem can be used to reduce the problem of determining minimum rank of a graph that has a cut-vertex to determining the minimum rank of several smaller order graphs.

Theorem 2.11. [6], [35] (cut-vertex reduction) *If G has a cut-vertex, the problem of computing the minimum rank of G can be reduced to computing minimum ranks of certain subgraphs. Specifically, let v be a cut-vertex of G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then*

$$r_v(G) = \min\left\{\sum_1^h r_v(G_i),\ 2\right\}$$

and thus

$$\mathrm{mr}(G) = \sum_1^h \mathrm{mr}(G_i - v) + \min\left\{\sum_1^h r_v(G_i),\ 2\right\}.$$

The following corollary is an immediate consequence of Theorem 2.11, and tends to be a handy fact.

Corollary 2.12. *If $r_v(G_i) = 0$ for all but at most one of the G_i, then*

$$\mathrm{mr}(G) = \sum_{i=1}^h \mathrm{mr}(G_i).$$

Example 2.13. This example illustrates the use of Corollary 2.12. The graph G shown in Figure 7 has cut-vertex 4 and the induced subgraphs $G[\{1,2,3,4\}]$ and $G_1 = G[\{4,5,6,7,8,9,10,11,12\}]$ associated with the two components.

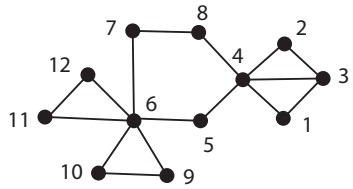


Figure 7: A graph to which Corollary 2.12 can be applied to compute minimum rank

By Corollary 1.5 and Observation 1.2.4, $\mathrm{mr}(G[\{1,2,3,4\}]) = 2 = \mathrm{mr}(G[\{1,2,3\}])$, so $r_4(G[\{1,2,3,4\}]) = 0$. Thus

$$\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G[\{1,2,3,4\}]) = mr(G_1) + 2.$$

The graph G_1 has the cut-vertex 6 and induced subgraphs $G[\{4,5,6,7,8\}]$, $G[\{6,9,10\}]$, and $G[\{6,11,12\}]$. Since complete graphs have rank 1, $\mathrm{mr}(G[\{6,9,10\}]) = \mathrm{mr}(G[\{9,10\}]) = 1$, so $r_6(G[\{6,9,10\}]) = 0$, and similarly $r_6(G[\{6,11,12\}]) = 0$. Thus,

$$\mathrm{mr}(G_1) = \mathrm{mr}(G[\{4,5,6,7,8\}]) + \mathrm{mr}(G[\{6,9,10\}]) + \mathrm{mr}(G[\{6,11,12\}]) = 3 + 1 + 1 = 5.$$

Thus, $\mathrm{mr}(G) = \mathrm{mr}(G_1) + 2 = 5 + 2 = 7$.

Analogous cut-set reduction results for cut-sets of size two have been observed recently by van der Holst [32], although many more cases have to be considered and so its utility is unclear.

2.4 Unicyclic graphs

As noted in Subsection 2.1, for trees on n vertices, the minimum rank is equal to n minus the path cover number; or equivalently the path cover number coincides with the maximum multiplicity. In Example 2.7 it is demonstrated that this relation no longer persists for graphs in general. However, for unicyclic graphs there is a still a weak relationship between the minimum rank and path cover number, and this can be used to compute minimum rank easily (see Algorithm 2.16). Just to be clear, for a graph G, the path cover number of G is defined to be the smallest number of vertex disjoint induced paths that cover the vertices of G.

For the 5-sun in Example 2.7, $\mathrm{mr}(H_5) = 8$ and thus $P(H_5) > M(H_5) = 2$, and in some sense, this is the worst case for unicyclic graphs.

Let C_n be an n-cycle. The graph H_n obtained from C_n by appending a leaf to each vertex on C_n is called an *n-sun*. Let $U \subseteq V(C_n)$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun* (see Figure 8).

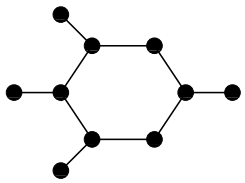


Figure 8: Partial 6-sun

In a effort to calculate the minimum rank of a unicyclic graph, in [7] a new parameter was considered and defined as $\eta(G) = P(G) - M(G)$. Of course, for a tree T, $\eta(T) = 0$. Since $M(G) = |G| - \mathrm{mr}(G)$, it follows that $\eta(G) = P(G) - |G| + \mathrm{mr}(G)$, or

$$\mathrm{mr}(G) = |G| - P(G) + \eta(G).$$

A type of graph surgery on unicyclic graphs was considered, which was referred to as "trimming branches."

A vertex v is said to be *appropriate* if there exist at least two pendent paths from v [50]. In [6] it was shown that any appropriate vertex of a graph has rank-spread 2. A vertex v is called a *peripheral leaf* if $\deg(v) = 1$ and $\deg(u) \leqslant 2$ where u is the only neighbour of v. We consider (induced) subgraphs of a given graph G, obtained by the following "trimming" procedures.

1. *Deletion of an appropriate vertex.* If v is an appropriate vertex, $G' = G - v$ is said to be *obtained from G by deletion of an appropriate vertex.*

2. *Deletion of an isolated path.* If one component of G is a path P, the graph $G' = G - V(P)$ is said to be *obtained from G by deletion of an isolated path.* In general, this process is required after the deletion of an appropriate vertex, which leaves two or more isolated paths.

3. *Deletion of a peripheral leaf.* If v is a peripheral leaf, $G' = G - v$ is said to be *obtained from G by deletion of a peripheral leaf.*

A *trimmed form* $\breve{G}$ of a graph G is an induced subgraph, obtained by a sequence of the above mentioned trimming operations, that does not contain peripheral leaves, isolated paths and/or appropriate vertices. In [7] it was shown that the trimmed form of a graph is unique, and that the parameter η is preserved under trimming. Consequently, a formula for both η and mr of a unicyclic graph followed.

Corollary 2.14. [7] *Let G be a unicyclic graph. Then*

$$\eta(G) = \begin{cases} 1 & \text{if } \breve{G} \text{ is an } n\text{-sun, } n > 3, \text{ odd;} \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.15. *Let G be a unicyclic graph on n vertices. Then*

$$\mathrm{mr}(G) = \begin{cases} n - P(G) + 1 & \text{if } \breve{G} \text{ is an } n\text{-sun, } n > 3, \text{ odd} \\ n - P(G) & \text{otherwise.} \end{cases}$$

Algorithm 2.5 can be adapted to find a minimal path cover and thus to compute path cover number and minimum rank of a tree or unicyclic graph.

Algorithm 2.16. *Computation of* $\mathrm{mr}(G)$ *and* $P(G)$ *for a tree or unicyclic graph G.*
Input: A tree or unicyclic graph G.
Output: $\mathrm{mr}(G), P(G), \eta(G)$ *and a minimal path cover* $\mathcal{P}$.

1. *Set* $\mathcal{P} = \emptyset$, $\eta(G) = 0$, *and* $G' = G$.

2. *While G' has a pendent generalized star:*

   (a) *Let v be the the center of a pendent generalized star of G'; let the pendent paths be denoted $P_1, P_2, \ldots, P_k$.*

   (b) *Let P_0 be the path obtained by joining P_1 and P_2 to v by edges of G.* $\mathcal{P} = \mathcal{P} \cup \{P_0, P_3, \ldots, P_k\}$.

   (c) *Remove v from G'.*

3. *If G' has a cycle C, then*

   (a) *If C has a vertex v of degree ≥ 4 in G':*

      i. *Let the pendent paths of v be denoted $P_1, P_2, \ldots, P_k$. Let P_0 be the path obtained by joining P_1 and P_2 to v by edges of T where $P_1, P_2, \ldots, P_k$ are the pendent paths.* $\mathcal{P} = \mathcal{P} \cup \{P_0, P_3, \ldots, P_k\}$.

      ii. *Remove v from G'.*

      iii. *Go to Step 2.*

   (b) *Else C does not have a vertex v of degree ≥ 4 in G':*

i. Select a minimal path cover for C and its pendent paths.

ii. If every vertex of C has a pendent path and the length of C is odd and greater than 3 then $\eta(G) = 1$.

4. *If G' is a generalized star with center v, then let P_0 be the path obtained by joining P_1 and P_2 to v by edges of T where $P_1, P_2, \ldots, P_k$ are the pendent paths. $\mathcal{P} = \mathcal{P} \cup \{P_0, P_3, \ldots, P_k\}$.*

5. *$P(G)$ is the number of paths in $\mathcal{P}$.*

6. $\mathrm{mr}(G) = |G| - \mathcal{P}(G) + \eta(G)$.

Example 2.17. We use Algorithm 2.16 to compute $P(G) = M(G)$ and $\mathrm{mr}(G)$ for the unicyclic graph G shown in Figure 9.

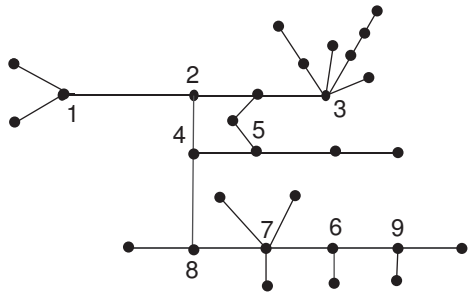


Figure 9: The graph G in Example 2.17

Initially we can choose any of $\{1, 3, 9\}$ as the center of a pendent generalized star, and we choose all of these before choosing any other center. (Although not required in the form of the algorithm given here, the intent is to select all pendent generalized stars of the original graph first and work from the outside in.) Then vertex 7 is the center of a pendent generalized star. All of these vertices are removed from G' and the paths shown in Figure 10 are in the path cover $\mathcal{P}$.

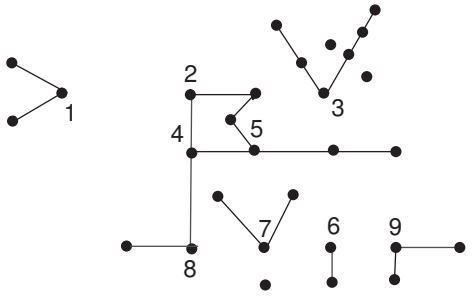


Figure 10: Partial path cover for the graph G in Example 2.17

Since the component containing the cycle has a vertex without a pendent path, step 3(b)ii does not happen and $\eta = 0$. Finally we cover the cycle and its pendent paths as shown in Figure 11. Thus $P(G) = 10$ and $\mathrm{mr}(G) = 29 - 10 + 0 = 19$.

In [7] the trimming procedure was also applied to determine the minimum rank of any graph in the family of so-called block-cycle graphs, that generalizes unicyclic graphs.

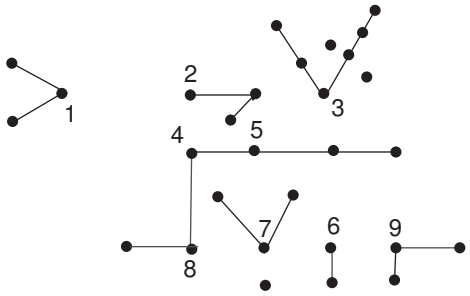


Figure 11: Path cover for the graph G in Example 2.17

2.5 Joins of graphs

Considering Theorem 2.9 above it is evident that the graph operation "join" is both useful and plays a role in the minimum rank of graphs.

The *join* $G \vee G'$ of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the union of $G \cup G'$ and the complete bipartite graph with with vertex set $V \cup V'$ and partition $\{V, V'\}$.

In [5] the minimum rank of the join of graphs was considered explicitly under the assumption of a new property known as *balanced inertia*. A symmetric matrix A has *balanced inertia* if $i_-(A) \leq i_+(A) \leq i_-(A) + 1$, where $i_-(A)$ ($i_+(A)$) is the number of negative (positive) eigenvalues of A. In an analogous fashion we say that a graph G is *inertia-balanced* if there exists $A \in \mathcal{S}(G)$ that satisfies

- $\operatorname{rank}(A) = \operatorname{mr}(G)$, and
- A has balanced inertia.

Many graphs are known to be inertia-balanced such as trees, K_n, and C_n. Before we state the main results in [5] on the minimum rank of the join of graphs, we need to lay out some prerequisite terminology.

There were two items that played a key role in the analysis in [5]: isolated vertices; and $K_{3,3,3}$. We define the *join minimum rank* of G to be $\operatorname{jmr}(G) = \operatorname{mr}(G \vee K_1)$, and note that $\operatorname{mr}(G) \leq \operatorname{jmr}(G) \leq \operatorname{mr}(G) + 2$ (cf. Observation 1.6.6).

Let $G = G = \bigvee_{i=1}^{r} G_i$ be a join of r graphs. Then G is said to be *anomalous* if

1. for each i, $\operatorname{jmr}(G_i) \leqslant 2$; and
2. $K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$ is a subgraph of G.

Such a definition is essential, since, for the inertia-balanced graph $G = K_{3,3,3} = \overline{K}_3 \vee \overline{K}_3 \vee \overline{K}_3$, we have $3 = \operatorname{mr}(G) \neq \max\{\operatorname{jmr}(G_i)\} = 2$.

One of the main results in [5] is the following

Theorem 2.18. [5] *Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, where each G_i is inertia-balanced. Then G is inertia-balanced, and*

$$\operatorname{mr}(G) = \begin{cases} \max_i\{\operatorname{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

Example 2.19. (Complete multipartite graph) Let $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. Then we let $G = K_{n_1,n_2,\ldots,n_k} := \overline{K}_{n_1} \vee \overline{K}_{n_2} \vee \cdots \vee \overline{K}_{n_k}$. Using the result above and [5] it follows that

$$\mathrm{mr}(G) = \begin{cases} 0 & \text{if } k = 1; \\ 1 & \text{if } n_1 = n_2 = 1, n_3 = 0; \\ 2 & \text{if } n_1 > 1, n_3 < 3; \\ 3 & \text{if } n_3 \geq 3; \end{cases}$$

Looking more closely at Theorem 2.8 we observe that P_4 is a forbidden induced subgraph, so any graph G that satisfies $\mathrm{mr}(G) \leq 2$ must not contain P_4. The class of graphs that do not contain P_4 as an induced subgraph are well-studied and are known as *decomposable graphs* (or *cographs*). An equivalent formulation of a decomposable graph is that it can be expressed as a sequence of unions and joins of isolated vertices. In [5] the minimum rank of all decomposable graphs was completely worked out and is given by

Theorem 2.20. [5] *Let $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, be a connected decomposable graph. Then G is inertia-balanced, and*

$$\mathrm{mr}(G) = \begin{cases} \max_i\{\mathrm{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

As a consequence, another characterization of $\mathrm{mr}(G) = 2$ graphs can be stated in the following manner.

Corollary 2.21. [5] *A connected graph G has minimum rank 2 if and only if $G = \bigvee_{i=1}^{r} G_i$, $r > 1$, where either*

1. *$G_i = K_{m_i} \cup K_{n_i}$, for suitable $m_i \geqslant 1$, $n_i \geqslant 0$, or*
2. *$G_i = \overline{K}_{m_i}$, for a suitable $m_i \geqslant 3$;*

and option (2) occurs at most twice.

2.6 Colin de Verdière-type parameters

A graph parameter ζ is *monotone on induced subgraphs* if for any induced subgraph H of G, $\zeta(H) \leq \zeta(G)$. Recall that minimum rank is monotone on induced subgraphs (cf. Observation 1.6.2). This property can be useful in bounding $\mathrm{mr}(G)$ from below; for example, the length of the longest induced path in G is a lower bound for $\mathrm{mr}(G)$. Unfortunately, maximum multiplicity of an eigenvalue is not monotone on induced subgraphs (the deletion of any vertex with rank-spread equal to 2 increases M).

In 1990 Colin de Verdière ([15] in English) introduced the graph parameter μ which is equal to the maximum multiplicity of eigenvalue 0 among all generalized Laplacian matrices having a given graph and also satisfying the so-called Strong Arnold Hypothesis (defined below). Reference [34] provides an excellent introduction to the parameter μ from a linear algebra perspective. The parameter μ was the first of several related parameters that are minor monotone and bound the maximum eigenvalue multiplicity from below. In this subsection we discuss several of these Colin de Verdière-type parameters and their use for computing the maximum eigenvalue multiplicity (or equivalently, the minimum rank) of a graph. These parameters are most useful when the graph has a large number of edges (since a matrix with many nonzero entries is more likely to satisfy the Strong Arnold Hypothesis), and least useful for trees, where a convenient method already exists for evaluation of maximum multiplicity and minimum rank (see Algorithms 2.3 and 2.5 in Subsection 2.1).

The computation of Colin de Verdière-type parameters is generally difficult, but once the value of at least one of these parameters is known for a specific graph G, this information can be used to obtain desirable bounds for any graph that has G as a minor. For the purpose of computing minimum rank or maximum eigenvalue multiplicity, these parameters should be used only for connected graphs (each component should be analyzed separately).

A symmetric real matrix M is said to satisfy the *Strong Arnold Hypothesis* provided there does not exist a nonzero symmetric matrix X satisfying:

- $MX = 0$.
- $M \circ X = 0$.
- $I \circ X = 0$,

where $\circ$ denotes the Hadamard (entrywise) product and I is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [15], [34]). An interesting short note on testing for the Strong Arnold Hypothesis can be found at [24].

The *Colin de Verdière number* $\mu(G)$ (or μ for short) [15] is the maximum multiplicity of 0 as an eigenvalue among matrices L that satisfy:

- L is a *generalized Laplacian matrix* of G, i.e., $L \in S_n, \mathcal{G}(L) = G$ and all off-diagonal entries of L are non-positive (L is a Z-matrix).
- L has exactly one negative eigenvalue (of multiplicity 1).
- L satisfies the Strong Arnold Hypothesis.

The parameter μ is used to characterize planarity (see [34] for precise definitions).

Theorem 2.22. [15, 34, 48]

- $\mu(G) \leq 1$ *if and only if* G *is a disjoint union of paths,*
- $\mu(G) \leq 2$ *if and only if* G *is outerplanar,*
- $\mu(G) \leq 3$ *if and only if* G *is planar,*
- $\mu(G) \leq 4$ *if and only if* G *is linklessly embeddable.*

The planarity characterizations can be useful in determining μ, and hence in bounding minimum rank. If G is not outerplanar, then $\mu(G) \geq 3$, so $M(G) \geq 3$, and thus $\mathrm{mr}(G) \leq |G| - 3$. If G is not planar, then $\mu(G) \geq 4$, so $M(G) \geq 4$, and hence $\mathrm{mr}(G) \leq |G| - 4$.

The parameter $\nu(G)$ [16] is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A \in S_n$ that satisfy:

- $\mathcal{G}(A) = G$.
- A is positive semidefinite.
- A satisfies the Strong Arnold Hypothesis.

See Subsection 3.2 for more information on applications of $\nu(G)$ to positive semidefinite minimum rank.

In [8] the parameter $\xi(G)$ was introduced as the Colin de Verdière-type parameter specifically designed for use in computing minimum rank, by removing any unnecessary restrictions (the Strong Arnold Hypothesis seems to be necessary to obtain minor monotonicity). We define $\xi(G)$ to be the maximum multiplicity of 0 as an eigenvalue among matrices $A \in S_n$ that satisfy:

- $\mathcal{G}(A) = G$.
- A satisfies the Strong Arnold Hypothesis.

Clearly, $\mu(G) \leq \xi(G) \leq M(G)$ and $\nu(G) \leq \xi(G) \leq M(G)$. An example in which both $\mu(G) < \xi(G)$ and $\nu(G) < \xi(G)$ is given in [8]. Complete bipartite graphs provide immediate examples of graphs G having $\xi(G) < M(G)$ (cf. Theorem 2.23 below).

Theorem 2.23. [8] *The values of $\xi(G)$ are known for the following graphs.*

1. $\xi(K_p) = p - 1$
2. $\xi(K_{p,q}) = p + 1$ *if* $p \leq q$ *and* $3 \leq q$.
3. $\xi(P_n) = 1$
4. *If T is a tree that is not a path, then* $\xi(T) = 2$.

A *contraction* of G is obtained by identifying two adjacent vertices of G, suppressing any loops or multiple edges that arise in this process. A *minor* of G arises by performing a series of deletions of edges, deletions of isolated vertices, and/or contraction of edges. A graph parameter ζ is *minor monotone* if for any minor G' of G, $\zeta(G') \leq \zeta(G)$.

Theorem 2.24. [8] *The parameter $\xi(G)$ is minor monotone.*

Corollary 2.25.

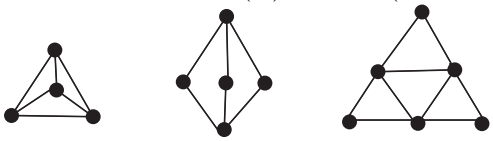
1. *If K_p is a minor of G, then* $\mathrm{mr}(G) \leq |G| - p + 1$.
2. *If $p \leq q, 3 \leq q$ and $K_{p,q}$ is a minor of G, then* $\mathrm{mr}(G) \leq |G| - p - 1$.

The Strong Arnold Hypothesis seems to be essential to minor-monotonicity: any matrix realizing $M(K_{1,4}) = 3$ does not satisfy the Strong Arnold Hypothesis. Obviously, a minor monotone graph parameter is monotone on subgraphs and thus on induced subgraphs.

Theorem 2.26. [31] *Let G be a 2-connected graph of order n. The following are equivalent:*

1. $\xi(G) = 2$.
2. $M(G) = 2$.
3. $\mathrm{mr}(G) = n - 2$.
4. *G has no K_4-, $K_{2,3}$-, or T_3-minor (see Figure 12).*
5. *G is a linear 2-tree.*

Figure 12: Forbidden minors for $mr(G) = n - 2$ (for 2-connected graphs)

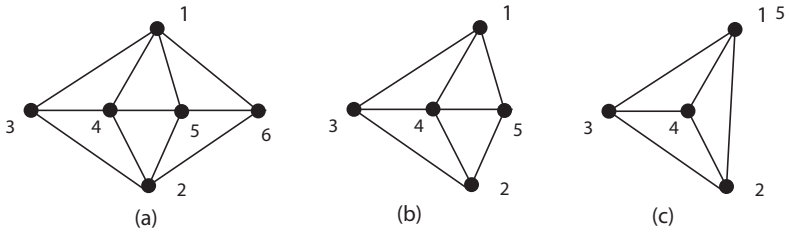


K_4 $K_{2,3}$ T_3

Example 2.27. This example shows how we can use monotonicity to compute minimum rank of the graph G shown in Figure 13(a). Note that the vertices 3, 4, 5, 6 induce a path of length four, so $3 = \mathrm{mr}(P_4) \leq \mathrm{mr}(G)$.

Figures 13(b),(c) show that K_4 is a minor of G (delete vertex 6 in (b) and then contract on edge $\{1, 5\}$ in (c)). So $3 = \xi(K_4) \leq \xi(G) \leq M(G)$. Thus $\mathrm{mr}(G) \leq 6 - 3 = 3$, and so $\mathrm{mr}(G) = 3$.

Figure 13: Using ξ to compute minimum rank



3 Minimum Rank Problems for Other Families of Matrices

In this section we briefly survey the literature on other families of matrices described by a graph for which the minimum rank question has been studied. Many basic facts are the same for all the various types of minimum rank problem (although the value of minimum rank varies with the family), so in this section we adopt the perspective that we are finding the minimum of the ranks of the matrices in a given family $\mathcal{F}$ of matrices, and naturally define

$$\mathrm{mr}(\mathcal{F}) = \min\{\mathrm{rank}(A) : \ A \in \mathcal{F}\}.$$

We begin by defining several families of matrices associated with a graph G, and give examples showing that the minimum rank for these families may be different from the conventional minimum rank of G, and observing basic facts that remain true for each of these classes.

One important generalization of the minimum rank problem is to consider symmetric matrices over a field F other than the real numbers. Study of this problem was initiated by Barrett, van der Holst and Loewy in [9]. The *set of symmetric matrices of graph* G over field F is given by

$$\mathcal{S}_G^F = \{A \in S_n(F) : \ \mathcal{G}(A) = G\},$$

where $\mathcal{G}(A)$ is defined in the same way as for real symmetric matrices and $S_n(F)$ denotes the class of symmetric $n \times n$ matrices over F. Thus $\mathrm{mr}(G) = \mathrm{mr}(\mathcal{S}_G^{\mathbb{R}})$. Minimum rank of a graph can vary with the field, as the next example illustrates.

Example 3.1. When $F = \mathbb{Z}_2$ we have that $\mathrm{mr}(\mathcal{S}_{K_{3,3,3}}^{\mathbb{Z}_2}) = 2$ by choosing all diagonal elements 0 (and hence $\mathrm{mr}(\mathcal{S}_{K_{3,3,3}}^{F}) = 2$ for any field of characteristic 2), but $\mathrm{mr}(K_{3,3,3}) = 3$ by Example 2.19.

Over the complex numbers it is perhaps more natural to consider Hermitian matrices rather than symmetric matrices; we denote the set of $n \times n$ Hermitian matrices by H_n. The *set of Hermitian matrices of graph* G over $\mathbb{C}$ is denoted by

$$\mathcal{H}_G = \{A \in H_n : \ \mathcal{G}(A) = G\},$$

where $\mathcal{G}(A)$ is defined in the same way as for real symmetric matrices.

Example 3.2. From Example 2.19 we know that $\mathrm{mr}(K_{3,3,3}) = 3$, but $\mathrm{mr}(\mathcal{H}_{K_{3,3,3}}) = 2$, since it is straightforward to verify that

$$\mathrm{rank}\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\ 0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\ 0 & 0 & 0 & 1 & 1 & 1 & i & i & i \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0 \\ -i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0 \\ -i & -i & -i & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = 2.$$

Two other families of matrices associated with a graph are subsets of the real $n \times n$ positive semidefinite matrices, which we denote by S_n^+, and the complex $n \times n$ positive semidefinite matrices, which we denote by H_n^+. The *set of symmetric positive semidefinite matrices of graph G* is

$$\mathcal{S}_G^+ = \{A \in S_{|G|}^+ : \ \mathcal{G}(A) = G\},$$

and the *set of Hermitian positive semidefinite matrices of graph G* is

$$\mathcal{H}_G^+ = \{A \in H_{|G|}^+ : \ \mathcal{G}(A) = G\}.$$

Note that translation by a diagonal matrix may not be possible when considering positive semidefinite matrices.

Example 3.3. Since no diagonal element can be 0 in a positive semidefinite matrix of a connected graph, it follows that for any $B \in H_{K_{1,3}}^+$, $B(1)$ is a 3×3 invertible diagonal submatrix, so $\mathrm{mr}(H_{K_{1,3}}^+) \geq 3$. Let $L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$ be the Laplacian of $K_{1,3}$; since $\mathrm{rank}(L) = 3$, we have $\mathrm{mr}(H_{K_{1,3}}^+) = 3$. Note $\mathrm{mr}(K_{1,3}) = 2$.

Observation 3.4. *Let G be a graph. The following statements are well-known and straightforward.*

1. $\mathrm{mr}(\mathcal{H}_G) \leq \mathrm{mr}(G) \leq \mathrm{mr}(\mathcal{S}_G^+)$.
2. $\mathrm{mr}(\mathcal{H}_G) \leq \mathrm{mr}(\mathcal{H}_G^+) \leq \mathrm{mr}(\mathcal{S}_G^+)$.

Whereas the first three inequalities in Observation 3.4 can be strict (see Examples 3.2 and 3.3), no example is known of a graph G in which $\mathrm{mr}(\mathcal{H}_G^+) < \mathrm{mr}(\mathcal{S}_G^+)$.

Observation 3.5. *Let G be a graph and let $\mathcal{F}_G$ be one of $\mathcal{S}_G^F, \mathcal{H}_G, \mathcal{H}_G^+, \mathcal{S}_G^+$. The following results are well-known and straightforward.*

1. $\mathrm{mr}(\mathcal{F}) \leq |G| - 1$.
2. *If $G = P_n$, then* $\mathrm{mr}(\mathcal{F}_G) = n - 1$.
3. *If $G = K_n$, then* $\mathrm{mr}(\mathcal{F}_G) = 1$.
4. *If G is connected and* $\mathrm{mr}(\mathcal{F}_G) = 1$, *then $G = K_{|G|}$.*

5. *If the connected components of G are $G_1, \ldots, G_t$ and $\mathcal{F}_{G_i}$ denotes for G_i the same family as $\mathcal{F}_G$, then*
$$\mathrm{mr}(\mathcal{F}_G) = \sum_{i=1}^{t} \mathrm{mr}(\mathcal{F}_{G_i}).$$

6. *$\mathrm{mr}(\mathcal{F}_G) \leq |G| - c$, where c is the number of connected components of G.*

7. *If G' is an induced subgraph of G and $\mathcal{F}_{G'}$ denotes for G' the same family as $\mathcal{F}_G$, then $\mathrm{mr}(\mathcal{F}_{G'}) \leq \mathrm{mr}(\mathcal{F}_G)$.*

The next two subsections survey results on the minimum rank over arbitrary fields and over positive semidefinite matrices.

If G is a general graph that allows loops but not multiple edges, the minimum rank of the family of matrices described by G was studied in [20], where Algorithm 2.5 was adapted to allow computation of the minimum rank of a tree (with loops).

3.1 Minimum rank of matrices over other fields

The two distinctions that become immediately apparent as sources of variation in minimum rank over arbitrary fields are the field characteristic (characteristic 2 versus everything else) and field size (especially finite versus infinite).

Example 3.6. For the graph G shown in Figure 14, $\mathrm{mr}(G) = 2$, as can be seen by considering the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$, whereas $\mathrm{mr}(\mathcal{S}_G^{\mathbb{Z}_2}) = 3$ by computation with all possible diagonal elements or by [9].

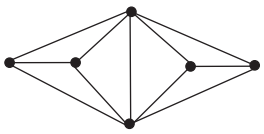


Figure 14: A graph G with $\mathrm{mr}(\mathcal{S}_G^{\mathbb{Z}_2}) > \mathrm{mr}(G)$.

Example 3.7. In [9], [10] it is established that for $G = K_{2,2,2} \vee K_1$ given in Figure 15, $\mathrm{mr}(\mathcal{S}_G^{\mathbb{Z}_3}) = 3$ and $\mathrm{mr}(\mathcal{S}_G^F) = 2$ where F is an infinite field of characteristic 3 (see also Theorems 3.11 and 2.8).

Observation 3.8. *Let G be a graph and let F be a field. The following results are well-known and straightforward.*

1. *If F is a subfield of a field K, then $\mathrm{mr}^F(G) \geq \mathrm{mr}^K(G)$.*

2. *If F is infinite, $\mathrm{mr}^F(G) \leq \mathrm{cc}(G)$, and more generally, if $G = \cup_{i=1}^{h} G_i$, then $\mathrm{mr}^F(G) \leq \sum_{i=1}^{h} \mathrm{mr}^F(G_i)$.*

3. *$\mathrm{mr}(\mathcal{S}_{K_{p,q}}^F) = 2$.*

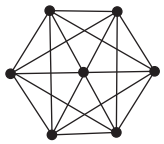


Figure 15: A graph for which minimum rank over a subfield is strictly greater.

Fiedler's Tridiagonal Matrix Theorem has been extended to arbitrary fields (with a few exceptional matrices each having the graph K_n) by Rheinboldt, Shepherd, Bento and Leal-Duarte, thereby establishing the following corollary.

Corollary 3.9. [11], [53] $\mathrm{mr}(\mathcal{S}_G^F) = |G| - 1$ *if and only if* $G = P_{|G|}$.

As a consequence, $\mathrm{mr}(\mathcal{S}_{C_n}^F) = n - 2$. For a tree T, the minimum rank is independent of field, and it can be computed by Algorithm 2.3 or 2.5.

Theorem 3.10. [14] *For a tree* T *and any field* F, $\mathrm{mr}(\mathcal{S}_T^F) = \mathrm{mr}(T)$.

Graphs having minimum rank at most 2 are characterized by forbidden subgraphs and also by graph complements in [9] and [10] for every field (with the characterization depending on field characteristic and size). It was shown in [9] that the graphs P_4, Dart, $\ltimes$ (see Figure 5) and $P_3 \cup K_2$ and $K_2 \cup K_2 \cup K_2$ all have minimum rank 3 over any field. The rank of $K_{3,3,3}$ (see Figure 5) depends on the field.

Theorem 3.11. [9] *Let* F *be any infinite field of characteristic not two. For any graph* G, $\mathrm{mr}(\mathcal{S}_G^F) \leq 2$ *if and only if* $\mathrm{mr}(G) \leq 2$.

See Theorems 2.8 and 2.9 for the characterizations of minimum rank two over $\mathbb{R}$. In [10] it was shown that Example 3.7 is typical, in the sense that for any finite field F and infinite field of the same characteristic, there is a graph G such that $2 = \mathrm{mr}(\mathcal{S}_G^K) < \mathrm{mr}(\mathcal{S}_G^F)$.

Wayne Barrett has observed that the proof of Theorem 2.11 remains valid over any field, so cut-vertex reduction (or its corollary) can be applied. The *rank-spread* of $\mathcal{S}_G^F$ at vertex v is defined as $r_v(\mathcal{S}_G^F) = \mathrm{mr}(\mathcal{S}_G^F) - \mathrm{mr}(\mathcal{S}_{G-v}^F)$.

Theorem 3.12. (cut-vertex reduction) *If* G *has a cut-vertex, the problem of computing the minimum rank of* G *can be reduced to computing minimum ranks of certain subgraphs. Specifically, let* v *be a cut-vertex of* G. *For* $i = 1, \ldots, h$, *let* $W_i \subseteq V(G)$ *be the vertices of the* i*th component of* $G - v$ *and let* G_i *be the subgraph induced by* $\{v\} \cup W_i$. *Then*

$$\mathrm{mr}(\mathcal{S}_G^F) = \sum_1^h \mathrm{mr}(\mathcal{S}_{G_i - v}^F) + \min\left\{ \sum_1^h r_v(\mathcal{S}_{G_i}^F),\ 2 \right\}.$$

Corollary 3.13. *If* $r_v(\mathcal{S}_{G_i}^F) = 0$ *for all but at most one of the* G_i, *then*

$$\mathrm{mr}(\mathcal{S}_G^F) = \sum_{i=1}^h \mathrm{mr}(\mathcal{S}_{G_i}^F).$$

In [21] is is shown that for a finite field, the graphs having minimum rank $\leq k$ can always be characterized by a finite set of forbidden induced subgraphs. In [41] all graphs having minimum rank $|G| - 2$ over an infinite field F are characterized.

3.2 Positive semidefinite minimum rank

The following result essentially follows from Observations 1.8 and 3.5.3 and the fact that positive semidefinite matrices are closed under addition.

Observation 3.14. $\mathrm{mr}(\mathcal{S}_G^+) \leq \mathrm{cc}(G)$, *and more generally, if* $G = \cup_{i=1}^h G_i$ *then* $\mathrm{mr}(\mathcal{S}_G^+) \leq \sum_{i=1}^h \mathrm{mr}(\mathcal{S}_{G_i}^+)$ *and* $\mathrm{mr}(\mathcal{H}_G^+) \leq \sum_{i=1}^h \mathrm{mr}(\mathcal{H}_{G_i}^+)$.

As noted in Example 3.3, the positive definite or Hermitian positive definite minimum rank can be larger than the minimum rank, and $\mathrm{mr}(\mathcal{H}_G^+) = |G| - 1$ does not imply that G is a path.

Theorem 3.15. [33] *The following statements are equivalent.*

1. *G is a tree;*
2. $\mathrm{mr}(\mathcal{H}_G^+) = |G| - 1$*;*
3. $\mathrm{mr}(\mathcal{S}_G^+) = |G| - 1$.

This theorem also follows from results and techniques of Colin de Verdière about the parameter ν (and a version of ν for complex matrices). Just as $\mathrm{mr}(G) \leq |G| - \xi(G)$, $\mathrm{mr}(\mathcal{S}_G^+) \leq |G| - \nu(G)$. Thus $\mathrm{mr}(\mathcal{S}_G^+) = |G| - 1$ implies $\nu(G) = 1$ and G is connected. In [16] it is shown that for a connected graph G, $\nu(G) = 1$ if and only if G is a tree. If T is a tree, then $\mathrm{mr}(\mathcal{S}_T^+) = |T| - 1$ since the least (and greatest) eigenvalue of $A \in \mathcal{S}_T^+$ must be simple (see [38]).

In the series of papers [12], [13], [27], [36], several interesting results are compiled on the (Hermitian) positive definite minimum rank. Some important observations include: a myriad of notions of linearly independent vertices; Hermitian positive definite minimum rank is reduced to the bipartite case; positive definite minimum rank is computed for chordal graphs and for joins of graphs.

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