# Minimum rank, maximum nullity and zero forcing number for selected graph families*

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**Abstract**

The minimum rank of a simple graph $G$ is defined to be the smallest possible rank over all symmetric real matrices whose $ij$th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in $G$ and is zero otherwise. Maximum nullity is taken over the same set of matrices, and the sum of maximum nullity and minimum rank is the order of the graph. The zero forcing number is the minimum size of a zero forcing set of vertices and bounds the maximum nullity from above. This paper defines the graph families *ciclos* and *estrellas* and establishes the minimum rank and zero forcing number of several of these families. In particular, these families provide the examples showing that the maximum nullity of a graph and its dual may differ, and similarly for zero forcing number.

**Keywords** minimum rank, maximum nullity, zero forcing number, dual, ciclo, estrella

**AMS Classification**: 05C50, 15A03, 15A18

## 1 Introduction

All matrices discussed are real and symmetric; the set of $n \times n$ real symmetric matrices will be denoted by $S_n(\mathbb{R})$. A *graph* $G = (V_G, E_G)$ means a simple undirected graph (no loops, no multiple edges) with a finite nonempty set of vertices $V_G$ and edge set $E_G$ (an edge is a two-element subset of vertices). For $A \in S_n(\mathbb{R})$, the *graph* of $A$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of $A$ is ignored in determining $\mathcal{G}(A)$.

Let $G$ be a graph. The *set of symmetric matrices described by* $G$ is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *maximum nullity of* $G$ is

$$\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\},$$

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and the *minimum rank* of $G$ is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}.$$

Clearly $\mathrm{mr}(G)+\mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices of $G$. Extensive work has been done on the problem of determining minimum rank/maximum nullity of graphs. A variety of techniques have been developed to determine the minimum rank, and the minimum rank of numerous families of graphs has been determined, but in general the problem remains open. See [8] for a survey of results and discussion of the motivation for the minimum rank problem.

The zero forcing number was introduced in [1] and the associated terminology was extended in [2, 3, 7, 10, 11]. Let $G$ be a graph with each vertex colored either white or black. Vertices change color according to the *color-change rule*: If $u$ is a black vertex and exactly one neighbor $w$ of $u$ is white, then change the color of $w$ to black. When the color-change rule is applied to $u$ to change the color of $w$, we say $u$ *forces* $w$ and write $u \to w$. Given a coloring of $G$, the *derived set* is the set of black vertices obtained by applying the color-change rule until no more changes are possible. A *zero forcing set* for $G$ is a subset of vertices $Z$ such that if initially the vertices in $Z$ are colored black and the remaining vertices are colored white, then the derived set is all the vertices of $G$. The *zero forcing number* $\mathrm{Z}(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

**Theorem 1.1.** [1, Proposition 2.4] *For any graph $G$,* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

Let $G = (V_G, E_G)$ be a graph and $W \subseteq V_G$. The *induced subgraph* $G[W]$ is the graph with vertex set $W$ and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ is also denoted by $G - W$, or in the case $W$ is a single vertex $\{v\}$, by $G - v$. Minimum rank is monotone on induced subgraphs, i.e., for any $W \subseteq V_G$, $\mathrm{mr}(G[W]) \leq \mathrm{mr}(G)$. If $e$ is an edge of $G = (V_G, E_G)$, the subgraph $(V_G, E_G \setminus \{e\})$ is denoted by $G - e$. We denote the complete graph on $n$ vertices by $K_n$, the cycle on $n$ vertices by $C_n$ and the path on $n$ vertices by $P_n$. The *union* of $G_i = (V_i, E_i), i = 1, \ldots, h$ is $\cup_{i=1}^h G_i = (\cup_{i=1}^h V_i, \cup_{i=1}^h E_i)$. An (edge) *covering* of a graph $G$ is a set of subgraphs $\{G_i, i = 1, \ldots, h\}$ such that $G = \cup_{i=1}^h G_i$. The following observation is useful when bounding minimum rank from above by using a covering to exhibit a low rank matrix.

**Observation 1.2.** [8] *If $G = \cup_{i=1}^h G_i$, then* $\mathrm{mr}(G) \leq \sum_{i=1}^h \mathrm{mr}(G_i)$.

The *path cover number* $\mathrm{P}(G)$ of $G$ is the smallest positive integer $m$ such that there are $m$ vertex-disjoint induced paths in $G$ such that every vertex of $G$ is a vertex of one of the paths. Path cover number was first used in the study of minimum rank and maximum eigenvalue multiplicity in [12] (since the matrices in $\mathcal{S}(G)$ are symmetric, algebraic and geometric multiplicities of eigenvalues are the same, and since the diagonal is free, maximum eigenvalue multiplicity is the same as maximum nullity). In [12] it was shown that for a tree $T$, $\mathrm{P}(T) = \mathrm{M}(T)$; however, in [4] it was shown that $\mathrm{P}(G)$ and $\mathrm{M}(G)$ are not comparable for graphs unless some restriction is imposed on the type of graph. A graph is *planar* if it can be drawn in the plane with no edge crossings. A graph is *outerplanar* if it has a drawing in the plane without crossing edges such that one face contains all vertices. Recently Sinkovic established the following relationship between $\mathrm{P}(G)$ and $\mathrm{M}(G)$ for outerplanar graphs.

**Theorem 1.3.** [14] *If $G$ is an outerplanar graph, then* $\mathrm{P}(G) \geq \mathrm{M}(G)$.

A connected graph $G$ is *$k$-connected* if for any set of vertices $S$ such that $G - S$ is disconnected, $|S| \geq k$. The *dual* $G^d$ of a 3-connected planar graph $G$ is the graph obtained by putting a dual vertex in each region of a plane drawing of $G$ and a dual edge between two dual vertices whenever the original regions share an original edge (we assume the graph is 3-connected to ensure that the dual is determined by the graph rather than a particular plane embedding). At a research meeting devoted to minimum rank at the American Institute of Mathematics, the following questions were asked:

**Question 1.4.** *If $G$ is a 3-connected planar graph, is it true that* $\mathrm{M}(G^d) = \mathrm{M}(G)$*?*

**Question 1.5.** *If $G$ is a 3-connected planar graph, is it true that* $\mathrm{Z}(G^d) = \mathrm{Z}(G)$*?*

In Section 3 we give examples of graphs $G$ such that $\mathrm{M}(G^d) \neq \mathrm{M}(G)$ and $\mathrm{Z}(G^d) \neq \mathrm{Z}(G)$. The examples are taken from the family of estrellas. This family and the related family of ciclos are defined in Section 2, and the minimum ranks, maximum nullities, and zero forcing numbers of some members of these families are established. In Section 4 we determine the vertex spreads and edge spreads of select members of the ciclo and estrella families, thereby computing the minimum ranks, maximum nullities, and zero forcing numbers of additional families of graphs (spreads are defined in Section 4).

# 2 Ciclo and estrella graph families

**Definition 2.1.** Let $G$ be a graph and let $e$ be an edge of $G$. A *$t$-ciclo of $G$ with $e$*, denoted $C_t(G, e)$, is constructed from a $t$-cycle $C_t$ and $t$ copies of $G$ by identifying each edge of $C_t$ with the edge $e$ in one copy of $G$. If a symbol for the graph identifies a specific edge, or if $G$ is edge transitive (so it is not necessary to specify edge $e$), then the notation $C_t(G)$ is used. A vertex on $C_t$ is called a *cycle* vertex.

The ciclo $C_4(K_4)$ is shown in Figure 1. Ciclos of complete graphs are discussed in Section 2.1. The order of $C_t(G)$ is $(|G|-1)t$. Note that although $C_t(G, e)$ is defined as a union of a $t$-cycle $C_t$ and $t$ copies of $G$ to explain the construction, in fact $C_t(G, e)$ is a union of just the $t$ copies of $G$.

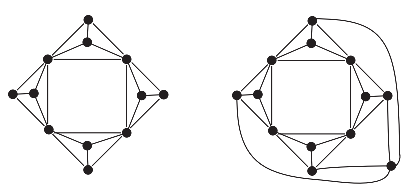

Figure 1: The complete ciclo $C_4(K_4)$ and the complete estrella $S_4(K_4)$.

**Definition 2.2.** Let $G$ be a graph, let $e$ be an edge of $G$, and let $v$ be a vertex of $G$ that is not an endpoint of $e$. A *$t$-estrella of $G$ with $e$ and $v$*, denoted $S_t(G, e, v)$, is the union of a $t$-ciclo $C_t(G, e)$ and the complete bipartite graph $K_{1,t}$ with each degree one vertex of $K_{1,t}$ identified with one copy of $v$. If a symbol for the graph identifies a specific edge and vertex, or if $G$ is vertex and edge transitive (so it is not necessary to specify $e$ and $v$), then the notation $S_t(G)$ is used. The degree $t$ vertex of the $K_{1,t}$ used to construct the estrella is called the *star* vertex of the estrella, and every neighbor of the star vertex is called a *starneighbor* vertex. A cycle vertex in the ciclo that is used to construct the estrella is also called a *cycle* vertex in the estrella.

The estrella $S_4(K_4)$ is shown in Figure 1. The order of $S_t(G)$ is $(|G|-1)t+1$. Estrellas of complete graphs are discussed in Section 2.2. The families of ciclos and estrellas formed from house graphs (see Sections 2.3 and 2.4) are introduced because of their importance as examples answering the duality questions (see Questions 1.4 and 1.5 above). Related families of ciclos are studied in Sections 2.5 and 2.6. Another natural family of ciclos are the cycle ciclos, discussed in Section 2.7.

## 2.1 Complete ciclo $C_t(K_r)$

**Definition 2.3.** The *complete ciclo*, denoted $C_t(K_r)$, is the ciclo of the complete graph $K_r$, with $t, r \geq 3$ (note that $K_r$ is edge transitive). A vertex not on $C_t$ is called a *noncycle* vertex.

The order of $C_t(K_r)$ is $(r-1)t$.

**Theorem 2.4.** *For $t \geq 3, r \geq 3$, $\mathrm{M}(C_t(K_r)) = \mathrm{Z}(C_t(K_r)) = (r-2)t$ and $\mathrm{mr}(C_t(K_r)) = t$.*

*Proof.* First, we will derive a lower bound for the maximum nullity. We know from Observation 1.2 that the minimum rank of a graph will be less than or equal to the sum of the minimum ranks of the subgraphs in a covering of it. Since every $C_t(K_r)$ can be covered by $t$ copies of $K_r$ graphs, each of minimum rank 1, $\mathrm{mr}(C_t(K_r)) \leq t$ and $(r-2)t \leq \mathrm{M}(C_t(K_r))$.

The zero forcing number can be used to bound the maximum nullity from above. There are many possible zero forcing sets of minimum cardinality, but it suffices to exhibit one for each of the two cases $r = 3$ and $r \geq 4$ (see Figure 2).

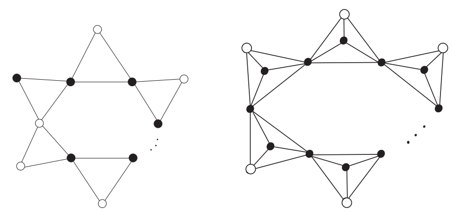

Figure 2: The zero forcing sets for the complete ciclos $C_t(K_3)$ and $C_t(K_r)$.

**Case $r = 3$:** A set $Z$ consisting of $t-1$ cycle vertices and one noncycle vertex adjacent to the cycle vertex that is not in $Z$ is a zero forcing set of $t$ vertices.

**Case $r \geq 4$:** Let $Z$ consist of all the cycle vertices and for each $K_r$, all but one of the noncycle vertices. Then $Z$ is a zero forcing set because there will always be at least one black noncycle vertex in each $K_r$ that will force the one white noncycle vertex, coloring the entire graph. Note that $|Z| = (r-2)t$.

In either case, $\mathrm{M}(C_t(K_r)) \leq \mathrm{Z}(C_t(K_r)) \leq (r-2)t$. □

## 2.2 Complete estrella $S_t(K_r)$

**Definition 2.5.** The *complete estrella*, denoted $S_t(K_r)$, is the estrella of the complete graph $K_r$, with $t, r \geq 3$ (note that $K_r$ is vertex and edge transitive). A vertex in $S_t(K_r)$ that is not the star vertex, not a starneighbor vertex, and not a cycle vertex is called a *standard* vertex.

The order of $S_t(K_r)$ is $(r-1)t+1$, and $S_t(K_4)$ is planar and 3-connected.

**Theorem 2.6.** *For $t \geq 3$ and $r \geq 4$, $\mathrm{mr}(S_t(K_r)) = t+2$ and $\mathrm{M}(S_t(K_r)) = \mathrm{Z}(S_t(K_r)) = (r-2)t-1$.*

*Proof.* Note that $|S_t(K_r)| = (r-1)t+1$. Since $S_t(K_r)$ can be covered by $t$ copies of $K_r$ (each of minimum rank 1) and one $K_{1,t}$ (of minimum rank 2), $\mathrm{mr}(S_t(K_r)) \leq t+2$ and $(r-2)t-1 \leq \mathrm{M}(S_t(K_r))$.

Define a set $Z$ consisting of all cycle vertices and all but one standard vertices; note $|Z| = (r-2)t-1$. We claim $Z$ is a zero forcing set. In each of the complete graphs that has all its standard vertices in $Z$, any black standard vertex can force the one white starneighbor vertex. Then any one of the (now) black starneighbor vertices can force the star vertex. Then the star vertex forces the one remaining white starneighbor vertex, and any black neighbor forces the last white vertex. So the entire graph is black, establishing the claim. Thus,

$$\mathrm{M}(S_t(K_r)) \leq \mathrm{Z}(S_t(K_r)) \leq (r-2)t-1.$$ □

**Theorem 2.7.** *For $t \geq 3$, $\mathrm{mr}(S_t(K_3)) = t$ and $\mathrm{M}(S_t(K_3)) = \mathrm{Z}(S_t(K_3)) = t + 1$.*

*Proof.* By Theorem 2.4, $\mathrm{mr}(C_t(K_3)) = t$, and since $C_t(K_3)$ is an induced subgraph of $S_t(K_3)$,

$$t = \mathrm{mr}(C_t(K_3)) \leq \mathrm{mr}(S_t(K_3)).$$

To show that $\mathrm{mr}(S_t(K_3)) \leq t$, we construct a matrix of rank $t$ in $\mathcal{S}(C_t(K_3))$ and extend it to a matrix in $\mathcal{S}(S_t(K_3))$ without changing the rank of the matrix. Number the vertices of $C_t(K_3)$ as in Figure 3.

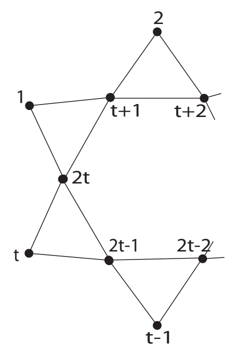


Figure 3: The numbering for $C_t(K_3)$.

Define the $t \times t$ matrix $B$ to be

$$B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

Note that the sum of each of the rows and the sum of each of the columns equal zero. Then the $2t \times 2t$ matrix

$$A = \begin{bmatrix} I & B \\ B^T & B^T B \end{bmatrix} \in \mathcal{S}(C_t(K_3))$$

has rank $A = t$. Extend the matrix $A$ to the $(2t+1) \times (2t+1)$ matrix

$$A' = \begin{bmatrix} I_{t\times t} & B & \mathbf{1}_t \\ B^T & B^T B & 0_t \\ \mathbf{1}_t{}^T & 0_t{}^T & t \end{bmatrix} \in \mathcal{S}(S_t(K_3))$$

Note that $B^T B$ shares properties with $B$ in that for each row and column, the sum is zero as well. Thus the entries of the new column $2t + 1$ of $A'$ is the sum of the columns of $A$, and, similarly for the rows. Thus rank $A' = t$, and

$$\mathrm{mr}(S_t(K_3)) \leq t.$$

□

### 2.3 House ciclo $C_t(H_0)$

**Definition 2.8.** A *house* $H_0$ (also called an *empty house*) is the union of a 3-cycle and a 4-cycle with one edge in common, shown on the left in Figure 4. The symbol $H_0$ also designates the specific edge $e$ and vertex $v$ shown in the figure (this figure also includes numbering that will be used later). A *house ciclo* is $C_t(H_0) = C_t(H_0, e)$.

The house ciclo $C_4(H_0)$ is shown on the right in Figure 4. Note that the order of $C_t(H_0)$ is $4t$ and $C_t(H_0)$ is outerplanar.

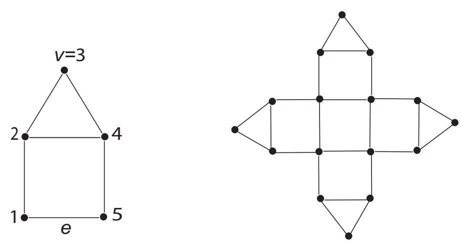


Figure 4: The house $H_0$ and the house ciclo $C_4(H_0)$.

**Observation 2.9.** *For $t \geq 3$, $\mathrm{P}(C_t(H_0)) \leq t$, because Figure 5 shows how to create a covering with $t$ paths.*

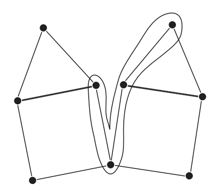

Figure 5: The method for creating a covering of $C_t(H_0)$ by $t$ paths.

**Theorem 2.10.** *For $t \geq 3$, $\mathrm{M}(C_t(H_0)) = t$ and $\mathrm{mr}(C_t(H_0)) = 3t$.*

*Proof.* Because house ciclos are outerplanar, Theorem 1.3 and Observation 2.9 give the following upper bound for the maximum nullity of $C_t(H_0)$: $\mathrm{M}(C_t(H_0)) \leq \mathrm{P}(C_t(H_0)) \leq t$.

Using the obvious covering of the house ciclo $C_t(H_0)$ by the set of $t$ houses $H_0$, and the fact that $\mathrm{mr}(H_0) = 3$, we have the same lower bound on maximum nullity: $\mathrm{M}(C_t(H_0)) = |C_t(H_0)| - \mathrm{mr}(C_t(H_0)) \geq |C_t(H_0)| - 3t \geq t$. Therefore, $\mathrm{M}(C_t(H_0)) = t$ and $\mathrm{mr}(C_t(H_0)) = 3t$. □

**Theorem 2.11.** *For even $t \geq 4$,*

$$\mathrm{Z}(C_t(H_0)) = t.$$

*Proof.* Since $t = \mathrm{M}(C_t(H_0)) \leq \mathrm{Z}(C_t(H_0))$, it suffices to exhibit a zero forcing set $Z$ with $|Z| = t$. Let $Z$ consist of pairs chosen in alternate houses of $C_t(H_0)$ going around the cycle (2 vertices in the first house, skip the second house, 2 vertices in the third house, skip the fourth house, etc.), where each pair of vertices consists of the peak vertex $v = 3$ and its neighbor 2, labeled as in Figure 4. Because $t$ is even, $|Z| = t$. Within each house that contains two black vertices, the remaining three vertices are forced to turn black. Then, the remaining three white vertices in a house in between two houses having all vertices black will be forced. So $Z$ is a zero forcing set. □

In the case $t$ is odd, the method used in the proof of Theorem 2.11 will produce a zero forcing set of order $t+1$, so for $t$ odd, $\mathrm{Z}(C_t(H_0)) \le t+1$. For odd $t \le 9$, it has been established by use of the software [6] that $\mathrm{Z}(C_t(H_0)) = t+1$.

## 2.4 House estrella $S_t(H_0)$

**Definition 2.12.** A *house estrella* is $S_t(H_0) = S_t(H_0, e, v)$ (where $v$ and $e$ are as shown in Figure 4).

The house estrella $S_4(H_0)$ is shown in Figure 6. Note that the order of $S_t(H_0)$ is $4t+1$ and $S_t(H_0)$ is planar and 3-connected.

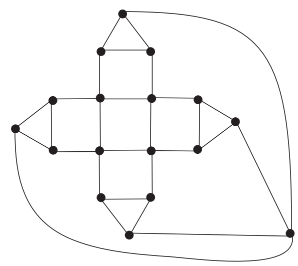

Figure 6: The house estrella $S_4(H_0)$.

We adopt the following convention for numbering the vertices of $S_t(H_0)$. We start the numbering on one of the houses from the lower left corner, starting with 1, and complete the numbering clockwise around the house, as in Figure 4. When that house is done, continue to the clockwise-adjacent house. The star vertex is numbered $4t+1$.

**Theorem 2.13.** *For* $t \ge 3$, $\mathrm{mr}(S_t(H_0)) = 3t$ *and* $\mathrm{M}(S_t(H_0)) = t+1$.

*Proof.* In Theorem 2.10, it was shown that $\mathrm{mr}(C_t(H_0)) = 3t$, and since $C_t(H_0)$ is an induced subgraph of $S_t(H_0)$,

$$3t = \mathrm{mr}(C_t(H_0)) \le \mathrm{mr}(S_t(H_0)).$$

Next, we will construct a specific matrix $A \in \mathcal{S}(C_t(H_0))$ having $\operatorname{rank} A = 3t$ that we can extend to a matrix $A'$ such that $\mathcal{G}(A') = S_t(H_0)$ and $\operatorname{rank} A' = 3t$, thus showing that the minimum rank of $S_t(H_0)$ is also $3t$.

Define the following submatrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \; W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The sum of the adjacency matrix of $C_t(H_0)$ and the $4t \times 4t$ identity matrix is the $4t \times 4t$ matrix

$$A = \begin{bmatrix} V & W & 0 & 0 & \dots & 0 & 0 & 0 & U \\ U & V & W & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & U & V & W & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & U & V & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & V & W & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & U & V & W & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & U & V & W \\ W & 0 & 0 & 0 & \dots & 0 & 0 & U & V \end{bmatrix}. \tag{1}$$

Note that $V$ is the submatrix corresponding to the adjacencies between the vertices numbered $4s+1, 4s+2, 4s+3, 4s+4$, and $V$ lies on the diagonal.

Let $\mathbf{b}$ be the $0,1$-vector describing the adjacencies of the star vertex. If $\mathbf{b} \in \operatorname{range} A$, then there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ and for

$$A' = \begin{bmatrix} A & A\mathbf{x} \\ \mathbf{x}^T A & \mathbf{x}^T A\mathbf{x} \end{bmatrix},$$

$\operatorname{rank} A' = \operatorname{rank} A$ and $\mathcal{G}(A') = S_t(H_0)$. Thus it suffices to show that $\mathbf{b}$ is in the range of $A$.

To prove $\mathbf{b} \in \operatorname{range} A$, we show that $\mathbf{b} \in (\ker A)^\perp$ and apply the fact that for any real symmetric matrix $A$, $(\ker A)^\perp = \operatorname{range} A$ [9, Fact 5.2.15]. Establishing $\mathbf{b} \in (\ker A)^\perp$ can be done by finding a basis for the kernel of $A$ and showing that $\mathbf{b}$ is orthogonal to the vectors in the basis of the kernel. To construct the basis, we construct $t$ linearly independent null vectors (and note that $\operatorname{null} A \leq \mathrm{M}(C_t(H_0)) = t$ by Theorem 2.10).

Let $\alpha = [0, 0, -1, 1]$, $\beta = [0, -1, 0, 1]$, $\omega = [0, -1, 1, 0]$, $0 = [0, 0, 0, 0]$. Then construct the vectors in the following manner

$$\begin{aligned}
\mathbf{v}_1 &= [\underbrace{\beta, \beta, \ldots, \beta, \beta, \beta}_{t}]^T \\
\mathbf{v}_2 &= [\alpha, \underbrace{\beta, \ldots, \beta, \beta}_{t-2}, \omega]^T \\
\mathbf{v}_3 &= [\alpha, \underbrace{\beta, \ldots, \beta}_{t-3}, \omega, 0]^T \\
&\vdots \\
\mathbf{v}_r &= [\alpha, \underbrace{\beta, \ldots, \beta}_{t-r}, \omega, \underbrace{0, \ldots, 0}_{r-2}]^T \\
&\vdots \\
\mathbf{v}_t &= [\alpha, \omega, \underbrace{0, \ldots, 0}_{t-2}]^T
\end{aligned}$$

To show that the vectors $\mathbf{v}_i, i = 1, \ldots, t$ are null vectors of $A$ it is sufficient to observe that

$$\begin{bmatrix} U & V & W \end{bmatrix}_{4\times 12} \begin{bmatrix} \beta^T & \omega^T & 0^T & 0^T & \alpha^T & \alpha^T & \alpha^T & \beta^T & \beta^T & \omega^T \\ \beta^T & \alpha^T & \alpha^T & \alpha^T & \beta^T & \beta^T & \omega^T & \beta^T & \omega^T & 0^T \\ \beta^T & \beta^T & \beta^T & \omega^T & \beta^T & \omega^T & 0^T & \omega^T & 0^T & 0^T \end{bmatrix}_{12\times 10} = 0_{4\times 10}.$$

Next, we show that the vectors $\mathbf{v}_i, i = 1, \ldots, t$ are linearly independent, viewing these vectors as block vectors (as constructed). Suppose $\sum_{i=1}^{t} \gamma_i \mathbf{v}_i = 0$. The vector $\mathbf{v}_1$ has $\beta^T = [0, -1, 0, 1]^T$ as the last block of the vector, so the last coordinate is 1. The vector $\mathbf{v}_2$ has $\omega^T = [0, -1, 1, 0]^T$ as the last block of the vector, so the last coordinate is 0, and the last coordinate of $\mathbf{v}_i, i \geq 3$ is also 0. Thus $\gamma_1 = 0$. Assuming $\gamma_k = 0$, by examining block $t - k + 1$ of $\sum_{i=k+1}^{t} \gamma_i \mathbf{v}_i = 0$, we see that $\gamma_{k+1} = 0$. Thus the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_t$ are linearly independent.

To complete the proof it suffices to show that $0,1$-vector $\mathbf{b}$ describing the adjacencies of the star vertex is orthogonal to $\ker A$. Let $\varphi = [0, 0, 1, 0]$; then $\mathbf{b} = [\varphi, \ldots, \varphi]^T$. Note that

$$\varphi \cdot \alpha = -1 \;, \; \varphi \cdot \beta = 0 \;, \; \varphi \cdot \omega = 1.$$

Then

$$\mathbf{b} \cdot \mathbf{v}_1 = [\varphi, \ldots, \varphi]^T \cdot [\beta, \ldots, \beta]^T = \sum_{i=1}^{t} \varphi \cdot \beta = 0$$

and for $2 \le r \le t$,

$$\mathbf{b} \cdot \mathbf{v}_r = [\varphi, \ldots, \varphi]^T \cdot [\alpha, \underbrace{\beta, \ldots, \beta}_{t-r}, \omega, \underbrace{0, \ldots, 0}_{r-2}]^T = \varphi \cdot \alpha + \sum_{i=1}^{t-r} \varphi \cdot \beta + \varphi \cdot \omega + \sum_{i=1}^{r-2} \varphi \cdot 0 = -1 + 0 + 1 + 0 = 0$$

Therefore, $\mathbf{b} \in (\ker A)^{\perp}$. □

**Corollary 2.14.** *For even $t \ge 4$,*

$$\mathrm{Z}(S_t(H_0)) = t + 1.$$

*Proof.* The zero forcing set $Z$ of Theorem 2.11 together with the star vertex is a zero forcing set of order $t + 1$ and the result then follows from Theorem 2.13. □

For $t$ odd, there is a zero forcing set of order $t + 2$, so for $t$ odd, $\mathrm{Z}(S_t(H_0)) \le t + 2$. For odd $t \le 9$, it has been established by use of the software [6] that $\mathrm{Z}(S_t(H_0)) = t + 2$.

## 2.5 Half-house ciclo $C_t(H_1)$

**Definition 2.15.** A *half-full house* or *half-house* $H_1$ is a house with one diagonal in the square, as shown on the left in Figure 7. The symbol $H_1$ also designates the specific edge $e$ and vertex $v$, as shown in this figure. A *half-house ciclo* is a ciclo of half-houses $C_t(H_1) = C_t(H_1, e)$.

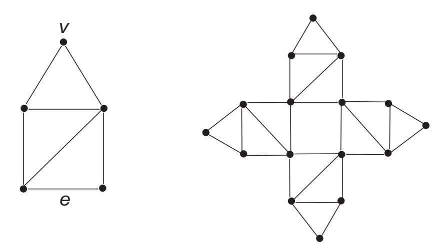


Figure 7: The half-house $H_1$ and half-house ciclo $C_4(H_1)$.

The half-house ciclo $C_4(H_1)$ is also shown in Figure 7. Note that $\mathrm{mr}(H_1) = 3 = |H_1| - 2$. The order of $C_t(H_1)$ is $4t$ and $C_t(H_1)$ is outerplanar. Half-full house ciclos have many properties in common with house ciclos. The proofs of the results below are analogous to the proofs of the corresponding results for house ciclos, and are omitted.

**Observation 2.16.** *For $t \ge 3$, $\mathrm{P}(C_t(H_1)) \le t$.*

**Theorem 2.17.** *For $t \ge 3$, $\mathrm{M}(C_t(H_1)) = t$ and $\mathrm{mr}(C_t(H_1)) = 3t$.*

**Theorem 2.18.** *For even $t$,*

$$\mathrm{Z}(C_t(H_1)) = t.$$

In the case $t$ is odd, $\mathrm{Z}(C_t(H_1)) \le t + 1$.

## 2.6 Full house ciclo $C_t(H_2)$

**Definition 2.19.** A *full house* $H_2$ is the union of $K_4$ and $K_3$ with one edge in common, or equivalently, a house with both diagonals in the square, as shown on the left in Figure 8. The symbol $H_2$ also designates the specific edge $e$ and vertex $v$, as shown in this figure (this figure also includes numbering that will be used later). A *full house ciclo* is a ciclo of full houses $C_t(H_2) = C_t(H_2, e)$.

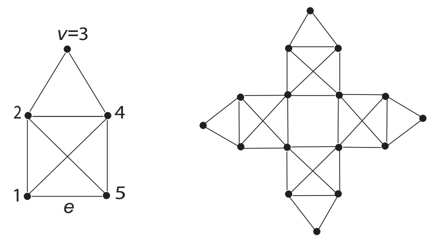


Figure 8: The full house $H_2$ and full house ciclo $C_4(H_2)$.

The full house ciclo $C_4(H_2)$ is also shown in Figure 8. Note that the order of $C_t(H_2)$ is $4t$ and $\mathrm{mr}(H_2) = 2$. We adopt the following convention for numbering the vertices of $C_t(H_2)$. We start the numbering on one of the houses from the lower left corner, starting with 1, and complete the numbering clockwise around the house, as in Figure 8. When that house is done, continue with the clockwise-adjacent house.

**Theorem 2.20.** *For* $t \geq 3$*,* $\mathrm{M}(C_t(H_2)) = \mathrm{Z}(C_t(H_2)) = 2t$ *and* $\mathrm{mr}(C_t(H_2)) = 2t$.

*Proof.* We can bound the maximum nullity from below by bounding the minimum rank from above using a covering of $C_t(H_2)$ with $t$ copies of the full house. Since a full house has minimum rank 2 and $|C_t(H_2)| = 4t$, $2t \leq \mathrm{M}(C_t(H_2))$.

Next, we can derive an upper bound for the maximum nullity by showing that the set

$$Z = \{1, 2, 3, 6, 7, 10, 11, \ldots, 4k + 2, 4k + 3, \ldots, 4(t - 2) + 2, 4(t - 2) + 3, 4(t - 1) + 2\}$$

is a zero forcing set. There are three black vertices of the four vertices in the first house, one in the last, and two in every other house (where the first four of the five vertices actually in a house are associated with that house to avoid duplication). To see that $Z$ is a zero forcing set, examine the first full house. Since vertices 1, 2, and 3 are black, the other two vertices in house 1 are forced, which means the next house already has its first vertex $5 = 4(2-1)+1$ black, in addition to 6 and 7. This process will continue around the ciclo until we reach the last full house, house $t$, which now has vertices $4(t-1)+1, 4(t-1)+2$, and 1 colored, so the remaining 2 vertices in this house can be forced. Since $|Z| = 2t$,

$$2t \leq \mathrm{M}(C_t(H_2)) \leq \mathrm{Z}(C_t(H_2)) \leq |Z| = 2t,$$

and we have equality throughout. □

## 2.7 Cycle ciclo $C_t(C_r)$

**Definition 2.21.** A *cycle ciclo* is a ciclo of cycles $C_t(C_r)$, $r \geq 4$.

The cycle ciclo $C_4(C_6)$ is shown in Figure 9. The order of $C_t(C_r)$ is $(r-1)t$ and $C_t(C_r)$ is outerplanar. Cycle ciclos have many properties in common with house ciclos. Note that $\mathrm{mr}(C_r) = r - 2 = |C_r| - 2$ and $\mathrm{mr}(H_0) = 3 = |H_0| - 2$, and $\mathrm{Z}(C_r) = 2 = \mathrm{Z}(H_0)$. The proof of the next result is similar to the proof of the corresponding result for $C_t(K_3)$.

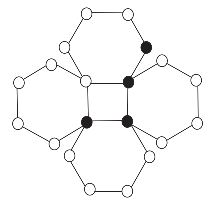

Figure 9: The cycle ciclo $C_4(C_6)$ with zero forcing set.

**Theorem 2.22.** *For $t \geq 3$, $\mathrm{Z}(C_t(C_r)) = \mathrm{M}(C_t(C_r)) = t$ and $\mathrm{mr}(C_t(C_r)) = (r-2)t$.*

*Proof.* Since every $C_t(C_r)$ can be covered by $t$ copies of $C_r$ graphs, each of minimum rank (r-2), $\mathrm{mr}(C_t(C_r)) \leq (r-2)t$ and $t \leq \mathrm{M}(C_t(K_r))$. A set $Z$ consisting of $t-1$ cycle vertices and one noncycle vertex that is in the $C_r$ containing the cycle vertex that is not in $Z$ and is adjacent to the other cycle vertex of that cycle (see Figure 9) is a zero forcing set of $t$ vertices, so $\mathrm{M}(C_t(K_r)) \leq \mathrm{Z}(C_t(K_r)) \leq (r-2)t$. □

### 2.8 Summary

Table 1 summarizes the results established in this section for certain families of ciclos and estrellas.

Table 1: Properties of the Ciclo and Estrella graph families

| Graph $G$ | $\|G\|$ | $\mathrm{mr}(G)$ | $\mathrm{M}(G)$ | $\mathrm{Z}(G)$ |
|---|---|---|---|---|
| $C_t(K_r)$ | $(r-1)t$ | $t$ | $(r-2)t$ | $(r-2)t$ |
| $C_t(H_0)$ | $4t$ | $3t$ | $t$ | $t$ if $t$ even<br>$\leq t+1$ if $t$ odd |
| $C_t(H_1)$ | $4t$ | $3t$ | $t$ | $t$ if $t$ even<br>$\leq t+1$ if $t$ odd |
| $C_t(H_2)$ | $4t$ | $2t$ | $2t$ | $2t$ |
| $C_t(C_r)$ | $(r-1)t$ | $(r-2)t$ | $t$ | $t$ |
| $S_t(K_r)(r \geq 4)$<br>$S_t(K_3)$ | $(r-1)t+1$<br>$2t+1$ | $t+2$<br>$t$ | $(r-2)t-1$<br>$t+1$ | $(r-2)t-1$<br>$t+1$ |
| $S_t(H_0)$ | $4t+1$ | $3t$ | $t+1$ | $t+1$ if $t$ even<br>$\leq t+2$ if $t$ odd |

## 3 Complete estrellas and house estrellas as duals

The next theorem and our previous results show that complete estrellas and house estrellas provide a negative answer to Questions 1.4 and 1.5.

**Theorem 3.1.** *The dual of the complete estrella $S_t(K_4)$ is the house estrella $S_t(H_0)$.*

*Proof.* Since $S_t(K_4)$ is a 3-connected graph, its dual is independent of how it is drawn in the plane, so we draw $S_t(K_4)$ with the star vertex in the center, as shown in Figure 10. Each $K_4$ together with the star vertex produces a house as its dual, so ignoring the infinite region we obtain the house ciclo $C_t(H_0)$ as the dual. The last step to creating the dual is to add a dual point that represents the infinite region outside the $S_t(K_4)$, and it connects to the vertex numbered 3 of each house (with numbering as in Figure 4), creating the house estrella $S_t(H_0)$ shown in Figure 10. □

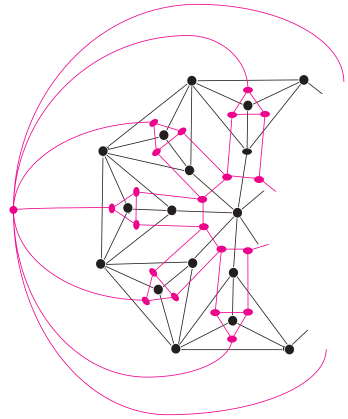

Figure 10: The house estrella $S_t(H_0)$ as the dual of the complete estrella $S_t(K_4)$.

**Corollary 3.2.** *Since* $\mathrm{M}(S_4(K_4)) = \mathrm{Z}(S_4(K_4)) = 7$ *and* $\mathrm{M}(S_4(H_0)) = \mathrm{Z}(S_4(H_0)) = 5$*, this example answers negatively Questions 1.4 and 1.5.*

# 4 Rank spread, null spread, and zero spread

If the minimum rank, maximum nullity, and/or zero forcing number are known for a graph $G$, it is sometimes possible to use this information to determine the same parameter for the graph obtained from $G$ by deleting a vertex or edge. In this section we determine the minimum rank/maximum nullity and zero forcing number of any complete ciclo or complete estrella from which one vertex or one edge has been deleted. Note that a complete ciclo has two types of vertex, a cycle vertex and a noncycle vertex. For a complete estrella there can be four types of vertex: the star vertex, a starneighbor vertex, a cycle vertex, and a standard vertex; note that $S_t(K_3)$ does not have any standard vertices.

## 4.1 Vertex spreads of complete ciclos and estrellas

Let $G$ be a graph and $v$ be a vertex in $G$. The *rank spread* of $v$, defined in [4], is

$$\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v),$$

and it is known [13] that

$$0 \leq \mathrm{r}_v(G) \leq 2.$$

In analogy with the rank spread, the null spread and the zero spread were defined in [7]. The *null spread* of $v$ is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G - v)$. The *zero spread* of $v$ is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$. Clearly, for any graph $G$ and vertex $v$ of $G$,

$$\mathrm{r}_v(G) + \mathrm{n}_v(G) = 1,$$

and thus

$$-1 \leq \mathrm{n}_v(G) \leq 1.$$

**Theorem 4.1.** [11, 7] *For every graph $G$ and vertex $v$ of $G$,*

$$-1 \leq \mathrm{z}_v(G) \leq 1.$$

As might be expected from the loose relationship between zero forcing number and maximum nullity, the parameters $\mathrm{n}_v(G)$ and $\mathrm{z}_v(G)$ are not comparable, and examples of this are given in [7]. However, under certain circumstances we can use one spread to determine the other.

**Observation 4.2.** [5] *Let $G$ be a graph such that $\mathrm{M}(G) = \mathrm{Z}(G)$ and let $v$ be a vertex of $G$. Then $\mathrm{n}_v(G) \geq \mathrm{z}_v(G)$, and so if $\mathrm{z}_v(G) = 1$, then $\mathrm{n}_v(G) = 1$ (equivalently, $\mathrm{r}_v(G) = 0$).*

**Theorem 4.3.** *For any vertex $v$, $\mathrm{M}(C_t(K_r) - v) = \mathrm{Z}(C_t(K_r) - v) = (r-2)t - 1$, or equivalently, $\mathrm{n}_v(C_t(K_r)) = \mathrm{z}_v(C_t(K_r)) = 1$.*

*Proof.* We exhibit a zero forcing set $Z$ for $C_t(K_r) - v$ such that $|Z| = (r-2)t - 1$ (here $r \geq 3$). Since $\mathrm{Z}(C_t(K_r)) = (r-2)t$ and $\mathrm{z}_v(C_t(K_r)) \leq 1$, $\mathrm{z}_v(C_t(K_r)) = 1$ and $\mathrm{Z}(C_t(K_r) - v) = (r-2)t - 1$. Since $\mathrm{M}(C_t(K_r)) = \mathrm{Z}(C_t(K_r))$, by Observation 4.2 $\mathrm{n}_v(C_t(K_r)) = 1$, and thus $\mathrm{M}(C_t(K_r) - v) = (r-2)t - 1$. When exhibiting a zero forcing set, we separate $C_t(K_3)$ from $C_t(K_r)$ with $r \geq 4$. For each of these two cases, there are two types of vertex $v$, a cycle vertex and a noncycle vertex. The zero forcing sets $Z$ are illustrated as black vertices in Figure 11.

**Case $C_t(K_3)$:** For a cycle vertex $v$, let the two noncycle neighbors of $v$ in $C_t(K_3)$ be denoted by $u$ and $w$. Then $Z$ consists of every noncycle vertex except $w$. For a noncycle vertex $v$, let the two neighbors of $v$ (both of which are cycle vertices) be denoted by $u$ and $w$. Then $Z$ consists of $u$ and every noncycle vertex except for the one adjacent to $w$.

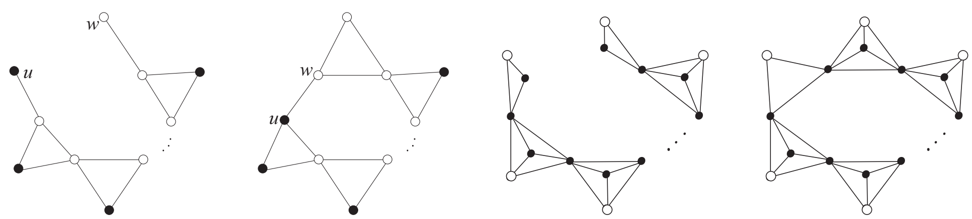


Figure 11: The zero forcing sets for $C_t(K_3)$ ($v$ a cycle vertex and $v$ a noncycle vertex) and $C_t(K_r)$ with $r \geq 4$ ($v$ a cycle vertex and $v$ a noncycle vertex).

**Case $C_t(K_r)$:** Note that each of the one or two copies of $K_r$ in which $v$ was a vertex has now become $K_{r-1}$. For a cycle vertex $v$, $Z$ consists of all the remaining cycle vertices and all but one noncycle vertex in each $K_r$ or $K_{r-1}$. For a noncycle vertex $v$, $Z$ consists of every cycle vertex and all but one noncycle vertex in each $K_r$ or $K_{r-1}$. □

**Theorem 4.4.** *For every vertex $v$, $\mathrm{M}(S_t(K_3) - v) = \mathrm{Z}(S_t(K_3) - v) = t$, or equivalently, $\mathrm{n}_v(S_t(K_3)) = \mathrm{z}_v(S_t(K_3)) = 1$.*

*Proof.* First let $v$ be the star vertex of $S_t(K_3)$. Then $S_t(K_3) - v = C_t(K_3)$, so by Theorems 2.4 and 2.7, $\mathrm{n}_v(S_t(K_3)) = \mathrm{z}_v(S_t(K_3)) = 1$. For any vertex $v$ that is not the star vertex, we exhibit a zero forcing set $Z$ for $S_t(K_3) - v$ such that $|Z| = t$, and as in Theorem 4.3 this establishes the theorem. In addition to the star vertex, there are two types of vertex in $S_t(K_3)$, a cycle vertex and a starneighbor vertex. The zero forcing sets $Z$ are illustrated as black vertices in Figure 12.

For a starneighbor vertex $v$, $Z$ consists of every cycle vertex. For a cycle vertex $v$, let the two starneighbor vertices adjacent to $v$ in $S_t(K_3)$ be denoted by $u$ and $w$. Then $Z$ consists of $u$ and every remaining cycle vertex in $S_t(K_3) - v$. □

**Theorem 4.5.** *Let $r \geq 4$. For every vertex $v$ except the star vertex, $\mathrm{M}(S_t(K_r) - v) = \mathrm{Z}(S_t(K_r) - v) = (r-2)t - 2$, or equivalently, $\mathrm{n}_v(S_t(K_r)) = \mathrm{z}_v(S_t(K_r)) = 1$. If $x$ is the star vertex, then $\mathrm{M}(S_t(K_r) - x) = \mathrm{Z}(S_t(K_r) - x) = (r-2)t$, or equivalently, $\mathrm{n}_x(S_t(K_r)) = \mathrm{z}_x(S_t(K_r)) = -1$.*

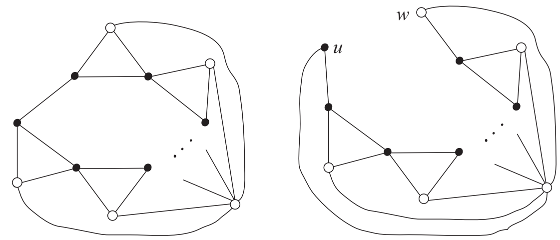


Figure 12: The zero forcing sets for $S_t(K_3) - v$ for $v$ a starneighbor vertex and $v$ a cycle vertex.

*Proof.* First let $x$ be the star vertex of $S_t(K_r)$ with $r \geq 4$. Then $S_t(K_r) - x = C_t(K_r)$, so by Theorems 2.4 and 2.6, $\mathrm{n}_x(S_t(K_r)) = \mathrm{z}_x(S_t(K_r)) = -1$. For any vertex $v$ that is not the star vertex, we exhibit a zero forcing set $Z$ for $S_t(K_r) - v$ of order $(r-2)t - 2$, and as in Theorem 4.3 this establishes the result. The zero forcing sets $Z$ are illustrated as black vertices in Figure 13.

Let $v$ be a cycle vertex, a standard vertex, or a starneighbor vertex, and in $S_t(K_r)$ choose one $K_r$ that does not contain $v$. Note that each of the one or two copies of $K_r$ in which $v$ was a vertex has now become $K_{r-1}$. If $v$ is a cycle vertex or a standard vertex, then $Z$ consists of all remaining cycle vertices, all remaining standard vertices in every $K_{r-1}$ or $K_r$ except the chosen $K_r$, and all but one standard vertices in the chosen $K_r$. If $v$ is a starneighbor vertex, then $Z$ consists of all cycle vertices, all standard vertices in every $K_r$ except the chosen $K_r$, and all but one standard vertices in the chosen $K_r$ and in the $K_{r-1}$. □

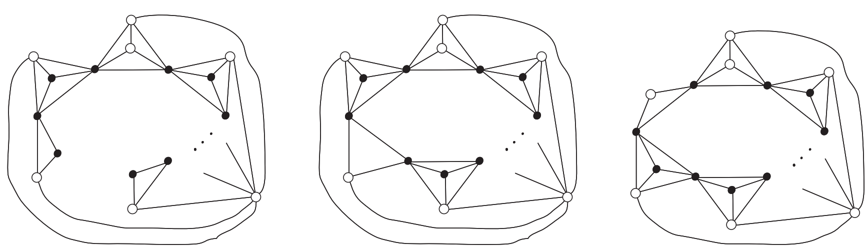

Figure 13: The zero forcing sets for $S_t(K_r)$ for $v$ a cycle vertex, $v$ a standard vertex, and $v$ a starneighbor vertex (with $r \geq 4$).

## 4.2 Edge spreads of complete ciclos and estrellas

In analogy with the rank, null, and zero spreads for vertex deletion, spreads for edge deletion were defined in [7]. Let $G$ be a graph and $e$ be an edge in $G$. The *rank edge spread* of $e$ is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G-e)$. The *null edge spread* of $e$ is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G-e)$. The *zero edge spread* of $e$ is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G-e)$. Clearly, for any graph $G$ and edge $e$ of $G$, $r_e(G) + n_e(G) = 0$ [7].

**Observation 4.6.** [13] *For any graph $G$ and edge $e$ of $G$, $-1 \leq \mathrm{r}_e(G) \leq 1$ and thus $-1 \leq \mathrm{n}_e(G) \leq 1$.*

**Theorem 4.7.** [7] *For every graph $G$ and every edge $e$ of $G$,*

$$-1 \leq \mathrm{z}_e(G) \leq 1.$$

It is known that although the bounds on the zero edge spread are the same as the bounds on the null edge spread, they are not comparable [7]. As with vertex spread, under certain circumstances we can use one spread to determine the other.

**Observation 4.8.** [7] *Let $G$ be a graph such that* $\mathrm{M}(G) = \mathrm{Z}(G)$ *and let $e$ be an edge of $G$. Then* $\mathrm{n}_e(G) \geq \mathrm{z}_e(G)$*, and so if* $\mathrm{z}_e(G) = 1$*, then* $\mathrm{n}_e(G) = 1$ *(equivalently,* $\mathrm{r}_e(G) = 0$*).*

An edge is classified based on its vertices. For a complete ciclo, there can be three types of edge: cycle-cycle, noncycle-cycle, and noncycle-noncycle (if $r \geq 4$). For a complete estrella there can be six types of edge: cycle-cycle, standard-cycle (if $r \geq 4$), cycle-starneighbor, standard-standard (if $r \geq 5$), standard-starneighbor (if $r \geq 4$), and star-starneighbor.

**Theorem 4.9.** *For any edge $e$,* $\mathrm{M}(C_t(K_r) - e) = \mathrm{Z}(C_t(K_r) - e) = (r-2)t - 1$*, or equivalently,* $\mathrm{n}_e(C_t(K_r)) = \mathrm{z}_e(C_t(K_r)) = 1$.

*Proof.* We exhibit a zero forcing set $Z$ for $C_t(K_r) - e$ such that $|Z| = (r-2)t - 1$ (here $r \geq 3$). Since $\mathrm{Z}(C_t(K_r)) = (r-2)t$ and $\mathrm{z}_e(C_t(K_r)) \leq 1$, $\mathrm{z}_e(C_t(K_r)) = 1$ and $\mathrm{Z}(C_t(K_r) - e) = (r-2)t - 1$. Since $\mathrm{M}(C_t(K_r)) = \mathrm{Z}(C_t(K_r))$, by Observation 4.8 $\mathrm{n}_e(C_t(K_r)) = 1$, and thus $\mathrm{M}(C_t(K_r) - e) = (r-2)t - 1$. When exhibiting a zero forcing set, we separate $C_t(K_3)$ from $C_t(K_r)$ with $r \geq 4$. The zero forcing sets $Z$ are illustrated as black vertices in Figure 14.

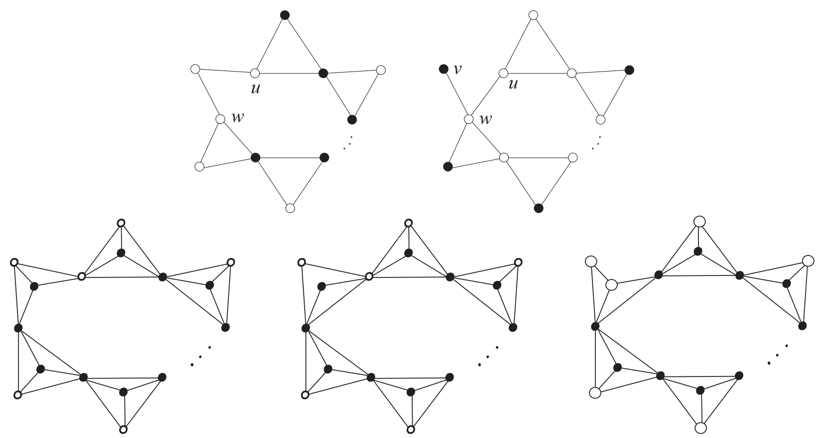


Figure 14: The zero forcing sets for $C_t(K_3) - e$ ($e$ a cycle-cycle edge and $e$ a noncycle-cycle edge) and $C_t(K_r) - e$ with $r \geq 4$ ($e$ a cycle-cycle edge, $e$ a noncycle-noncycle edge, and $e$ a noncycle-cycle edge).

**Case $C_t(K_3)$:** There are two types of edges $e$, a cycle-cycle edge and a noncycle-cycle edge. For a cycle-cycle edge $e = \{u, w\}$, $Z$ consists of the noncycle vertex in a $K_3$ that contains $u$ but not $w$, and every cycle vertex except $u$ and $w$. For a noncycle-cycle edge $e = \{v, u\}$, let $v$ be the noncycle vertex of $e$ and let $u$ be the cycle vertex of $e$. Then $Z$ consists of every noncycle vertex except for the one adjacent to $u$.

**Case $C_t(K_r)$:** There are three types of edges: cycle-cycle, noncycle-noncycle, and noncycle-cycle. For $e$ a cycle-cycle edge or noncycle-noncycle edge, $Z$ consists of all cycle vertices except for one of the two cycle vertices in $K_r - e$ and all but one noncycle vertex in each $K_r$ or $K_r - e$; in the case that $e$ is a noncycle-noncycle edge, the noncycle vertex in $K_r - e$ that is not in $Z$ must be an endpoint of $e$ (this is relevant when $r \geq 5$). For a noncycle-cycle edge, $Z$ consists of all the cycle vertices, all but one noncycle vertex in each $K_r$, and all but two noncycle vertices in the $K_r - e$; one of the two noncycle vertices in $K_r - e$ that is not in $Z$ must be an endpoint of $e$ (this is relevant when $r \geq 5$). □

**Theorem 4.10.** *For every edge $e$,* $\mathrm{M}(S_t(K_3) - e) = \mathrm{Z}(S_t(K_3) - e) = t$*, or equivalently,* $\mathrm{n}_e(S_t(K_3)) = \mathrm{z}_e(S_t(K_3)) = 1$.

*Proof.* We exhibit a zero forcing set $Z$ for $S_t(K_3) - e$ such that $|Z| = t$, and as in Theorem 4.9 this establishes the theorem. The zero forcing sets $Z$ are illustrated as black vertices in Figure 15. There are three types of edges: cycle-cycle, star-starneighbor, and cycle-starneighbor. For a cycle-cycle edge or star-starneighbor edge $e$, let the two cycle vertices of the $K_3$ that contains at least one endpoint of $e$ be denoted by $u$ and $w$. Then $Z$ consists of the starneighbor vertex in the $K_3$ that contains $u$ but not $w$, and all cycle vertices except for $w$. For a cycle-starneighbor edge, $Z$ consists of all cycle vertices. □

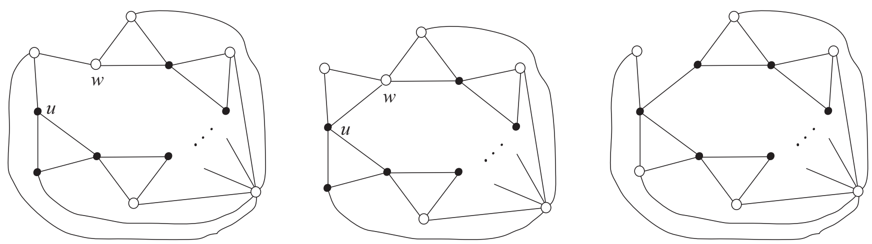


Figure 15: The zero forcing sets for $S_t(K_3) - e$ where $e$ is a cycle-cycle edge, a star-starneighbor edge, and a cycle-starneighbor edge.

**Theorem 4.11.** *Let $r \geq 4$. For every edge $e$ except a star-starneighbor edge,* $\mathrm{M}(S_t(K_r)-e) = \mathrm{Z}(S_t(K_r)-e) = (r-2)t-2$, *or equivalently,* $\mathrm{n}_v(S_t(K_r)) = \mathrm{z}_v(S_t(K_r)) = 1$. *If $d$ is a star-starneighbor edge, then* $\mathrm{M}(S_t(K_r)-d) = \mathrm{Z}(S_t(K_r)-d) = (r-2)t-1$, *or equivalently,* $\mathrm{n}_d(S_t(K_r)) = \mathrm{z}_d(S_t(K_r)) = 0$.

*Proof.* There can be 6 types of edges: cycle-cycle, standard-cycle, cycle-starneighbor, standard-standard (if $r \geq 5$), standard-starneighbor, and star-starneighbor. For any edge $e$ that is not a star-starneighbor edge, we exhibit a zero forcing set $Z$ for $S_t(K_r) - e$ of order $(r-2)t-2$, and as in Theorem 4.9 this establishes the result. The zero forcing sets $Z$ are illustrated as black vertices in Figure 16.

Let $e$ be a standard-cycle edge, a cycle-starneighbor edge, or a standard-standard edge. Let $u$ be a cycle vertex that is not an endpoint of $e$ and is in the $K_r - e$. Then $Z$ consists of all cycle vertices, all standard vertices in each $K_r$ (or $K_r - e$) except those that contain $u$, and all but one of the standard vertices in the $K_r$ and $K_r - e$ that contain $u$. In the case that $e$ is a standard-standard edge, the standard vertex in $K_r - e$ that is not in $Z$ must be an endpoint of $e$ (this is relevant when $r \geq 6$).

For a cycle-cycle edge $e = \{w, u\}$, $Z$ consists of all cycle vertices except $w$ and $u$, all standard vertices in each $K_r$ (or $K_r - e$) except those that contain $u$, all but one of the standard vertices in the $K_r$ and $K_r - e$ that contain $u$, and the starneighbor vertex in the $K_r$ and $K_r - e$ that contain $u$.

For a standard-starneighbor edge, choose one cycle vertex $u$ in the $K_r - e$. Then $Z$ consists of all cycle vertices except for $u$, all standard vertices in each $K_r$ except the $K_r$ that contains $u$, all standard vertices in $K_r - e$, and all but one of the standard vertices in the one $K_r$ that contains $u$.

For a star-starneighbor edge $d$, let $Z$ be the set that consists of all cycle vertices and all standard vertices except one standard vertex in a $K_r$ that does not contain an endpoint of $d$. Then $Z$ is a zero forcing set for $S_t(K_r) - d$. Since $S_t(K_r) - d$ can be covered by $t$ copies of $K_r$ and one $K_{1,t-1}$, $\mathrm{mr}(S_t(K_r) - d) \leq t + 2$. Thus

$$(r-2)t - 1 = |S_t(K_r) - d| - (t+2) \leq \mathrm{M}(S_t(K_r) - d) \leq \mathrm{Z}(S_t(K_r) - d) \leq (r-2)t - 1$$

and we have equality throughout. □

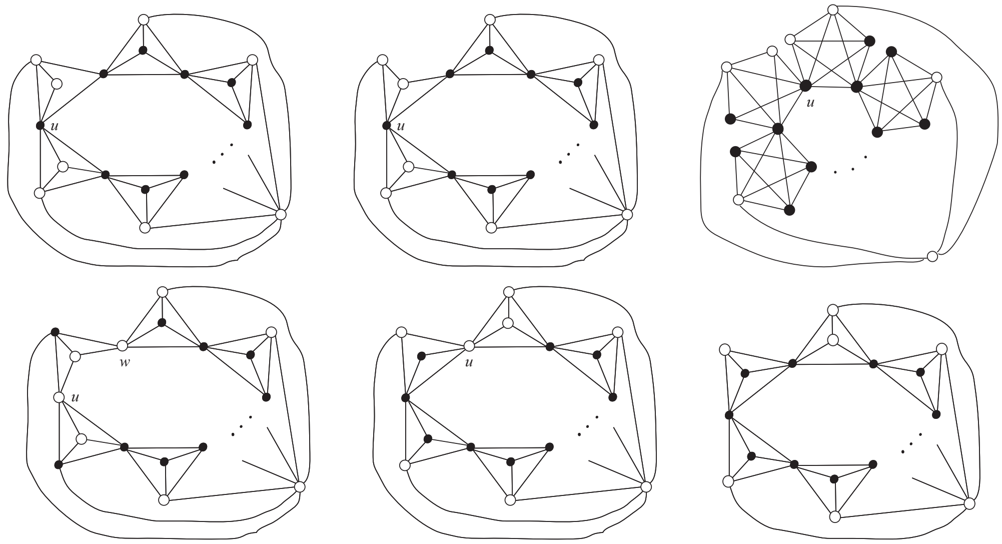


Figure 16: The zero forcing sets for $S_t(K_r) - e$ for $e$ a standard-cycle edge, a cycle-starneighbor edge, a standard-standard edge, a cycle-cycle edge, a standard-starneighbor edge, and a star-starneighbor edge (with $r \geq 4$).

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