THE ENHANCED PRINCIPAL RANK CHARACTERISTIC SEQUENCE FOR HERMITIAN MATRICES

STEVE BUTLER*, MINERVA CATRAL†, H. TRACY HALL‡, LESLIE HOGBEN§, XAVIER MARTÍNEZ-RIVERA*, BRYAN SHADER¶, AND P. VAN DEN DRIESSCHE‖

Abstract. The enhanced principal rank characteristic sequence (epr-sequence) of an $n \times n$ matrix is a sequence $\ell_1\ell_2 \cdots \ell_n$, where ℓ_k is A, S, or N according as all, some, or none of its principal minors of order k are nonzero. There has been substantial work on epr sequences of symmetric matrices (especially real symmetric matrices) and real skew-symmetric matrices, and incidental remarks have been made about results extending (or not extending) to (complex) Hermitian matrices. We undertake a systematic study of epr-sequences of Hermitian matrices; the differences with symmetric matrices are quite striking. Various results are established regarding the attainability by Hermitian matrices of epr-sequences that contain two Ns with a gap in between. Hermitian adjacency matrices of mixed graphs that begin NAN are characterized. All attainable epr-sequences of Hermitian matrices of orders 2, 3, 4, and 5 are listed with justifications.

Keywords. Principal rank characteristic sequence, enhanced principal rank characteristic sequence, mixed graph, Hermitian adjacency matrix, minor.

AMS subject classifications. 15B57, 15A15, 15A03, 05C50.

1. Introduction.

For a given $n \times n$ matrix that is symmetric over a field F or complex Hermitian, and for a fixed $k \in \{0, \ldots, n\}$, the question of determining the existence or nonexistence of a principal submatrix of rank k was addressed in Brualdi et al. [2] and Barrett et al. [1]. This information was presented in the principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. The enhanced principal rank characteristic

*Department of Mathematics, Iowa State University, Ames, IA 50011, USA (butler@iastate.edu, xaviermr@iastate.edu).

†Department of Mathematics, Xavier University, Cincinnati, OH 45207, USA (catralm@xavier.edu).

‡Department of Mathematics, Brigham Young University, Provo UT 84602, USA (H.Tracy@gmail.com).

§Department of Mathematics, Iowa State University, Ames, IA 50011, USA (hogben@iastate.edu) and American Institute of Mathematics, 600 E. Brokaw Road, San Jose, CA 95112, USA (hogben@aimath.org).

¶Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA (bshader@uwyo.edu).

‖Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 2Y2, Canada (pvdd@math.uvic.ca). Research supported in part by an NSERC Discovery grant.

sequence is a refinement of this sequence and was introduced in Butler et al. [4] to illuminate further the property of full rank principal submatrices of given dimension.

Throughout this paper, $\mathbb{R}_n$ (respectively, $\mathbb{C}_n$, $\mathbb{H}_n$, $\mathbb{K}_n$) denotes the set of $n \times n$ real symmetric (respectively, complex symmetric, Hermitian, skew-Hermitian) matrices and $\mathbb{F}_n$ denotes one of $\mathbb{R}_n, \mathbb{C}_n, \mathbb{H}_n, \mathbb{K}_n$.

Definition 1.1. [2, Definition 1.1] The *principal rank characteristic sequence* of $B \in \mathbb{F}_n$ is the sequence (pr-sequence) $\text{pr}(B) = r_0]r_1r_2 \cdots r_n$ where for $k = 1, \ldots, n$,

$$r_k = \begin{cases} 1 & \text{if } B \text{ has a nonzero order-}k \text{ principal minor;} \\ 0 & \text{otherwise,} \end{cases}$$

and $r_0 = 1$ if and only if B has a 0 diagonal entry.

Definition 1.2. [4, Definition 1.1] The *enhanced principal rank characteristic sequence* of $B \in \mathbb{F}_n$ is the sequence (epr-sequence) $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all order-}k \text{ principal minors of the given order are nonzero;} \\ \mathtt{S} & \text{if some but not all order-}k \text{ principal minors are nonzero;} \\ \mathtt{N} & \text{if none of the order-}k \text{ principal minors is nonzero, i.e., all are zero.} \end{cases}$$

A (pr- or epr-) sequence is *attainable* over $\mathbb{F}_n$ if there exists a matrix $B \in \mathbb{F}_n$ that realizes the sequence and is *forbidden* over $\mathbb{F}_n$ if no such matrix exists. The set of all epr-sequences attainable by matrices in $\mathbb{F}_n$ is denoted by $\text{attain}(\mathbb{F}_n)$.

The principal rank characteristic sequence was introduced in [2] where the focus was on pr-sequences of real symmetric matrices and with a simplification of the principal minor assignment problem [7] as a motivation. The study was continued in [1] where results over $\mathbb{R}_n$ were extended and where the problem was investigated over various fields. The enhanced principal rank characteristic sequence was introduced in [4] where results over symmetric matrices, including constructions of attainable epr-sequences and forbidden epr-(sub)sequences over various fields, were presented. In [5], Fallat et al. considered the problem over skew-symmetric matrices and gave a complete characterization of the attainable epr-sequences for real skew-symmetric matrices. Further results on attainable pr- and epr- sequences, including classifications of some families of attainable sequences, were given by Martínez-Rivera in [9].

In this paper, we focus our study on the epr-sequences of Hermitian matrices. In Section 2, we identify certain subsequences forbidden over $\mathbb{H}_n$. In Section 3, we establish results regarding sequences in $\text{attain}(\mathbb{H}_n)$ that contain two `N`s with a gap in between, and in particular those that have the subsequence `NAN`. Section 4 discusses epr-sequences attainable by Hermitian adjacency matrices. Probabilistic techniques are used in Section 5 to construct Hermitian matrices attaining a family of epr-sequences. In Section 6 we identify all epr-sequences attainable over $\mathbb{H}_n$ but not over $\mathbb{R}_n$ for $n \leq 5$. Finally in Section 7 we discuss relationships between sets of epr-sequences attained by the various classes of matrices that we consider.

For $B \in \mathbb{F}_n$, $\alpha, \beta \subseteq [n] := \{1, 2, \ldots, n\}$, the submatrix of B lying in rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Further, the complementary submatrix obtained from B by deleting the rows indexed by α and columns indexed by β is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while the complementary principal submatrix is denoted by $B(\alpha)$. The all-ones vector of size n is denoted by $\mathbb{1}_n$.

Following the notation in [1], we let $\overline{\ell_i \cdots \ell_j}$ indicate that the (complete) sequence may be repeated as many times as desired (or may be omitted entirely).

1.1. Results used. The purpose of this section is to list results from the literature that we cite frequently and/or simple extensions to Hermitian matrices of results for real symmetric matrices. In many cases we give the results names. Note that some of the results cited are true more generally, e.g. for symmetric matrices over other fields, but here we specialize to the complex Hermitian case.

Observation 1.3. [4, Observation 2.2] An epr-sequence of a complex Hermitian matrix B must end in $\mathtt{N}$ or $\mathtt{A}$.

Theorem 1.4. [4, Theorem 2.3] ($\mathtt{NN}$ Theorem) *Suppose* $B \in \mathbb{H}_n$*,* $\operatorname{epr}(B) = \ell_1 \cdots \ell_n$*, and* $\ell_k = \ell_{k+1} = \mathtt{N}$ *for some* k*. Then* $\ell_i = \mathtt{N}$ *for all* $i \geq k$*.*

Theorem 1.5. [4, Theorem 2.4] (Inverse Theorem) *If* $B \in \mathbb{H}_n$ *and* $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_{n-1}\mathtt{A}$*, then* $\operatorname{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2} \cdots \ell_1 \mathtt{A}$.

Proposition 1.6. [4, Proposition 2.5] *The epr-sequence* $\mathtt{SN}\cdots\mathtt{A}\cdots$ *is forbidden for Hermitian matrices.*

Corollary 1.7. [4, Corollary 2.7] ($\mathtt{NSA}$ Theorem) *No Hermitian matrix can have* $\mathtt{NSA}$ *in its epr-sequence. Further, no Hermitian matrix can have the epr-sequence* $\cdots\mathtt{ASN}\cdots\mathtt{A}\cdots$.

Theorem 1.8. [4, Theorem 2.6] (Inheritance Theorem) *Suppose that* $B \in \mathbb{H}_n$*,* $m \leq n$*, and* $1 \leq i \leq m$*.*

1. *If* $[\operatorname{epr}(B)]_i = \mathtt{N}$*, then* $[\operatorname{epr}(C)]_i = \mathtt{N}$ *for all* $m \times m$ *principal submatrices* C*.*
2. *If* $[\operatorname{epr}(B)]_i = \mathtt{A}$*, then* $[\operatorname{epr}(C)]_i = \mathtt{A}$ *for all* $m \times m$ *principal submatrices* C*.*
3. *If* $[\operatorname{epr}(B)]_m = \mathtt{S}$*, then there exist* $m \times m$ *principal submatrices* C_A *and* C_N *of* B *such that* $[\operatorname{epr}(C_A)]_m = \mathtt{A}$ *and* $[\operatorname{epr}(C_N)]_m = \mathtt{N}$*.*
4. *If* $i < m$ *and* $[\operatorname{epr}(B)]_i = \mathtt{S}$*, then there exists an* $m \times m$ *principal submatrix* C_S *such that* $[\operatorname{epr}(C_S)]_i = \mathtt{S}$*.*

THEOREM 1.9. (Real Skew Theorem) [5, Theorem 3.3] *An epr-sequence* $\ell_1\ell_2\cdots\ell_n$ *is attainable by a real skew-symmetric matrix if and only if the following conditions hold.*

1. $\ell_k = \mathtt{N}$ *for* k *odd;*
2. *If* $\ell_k = \ell_{k+1} = \mathtt{N}$, *then* $\ell_j = \mathtt{N}$ *for all* $j \geq k$*;*
3. $\ell_n \neq \mathtt{S}$.

The next result is stated in [4] for symmetric matrices over a field of characteristic not two, but the proof remains valid for Hermitian matrices.

THEOREM 1.10. [4, Proposition 2.13] (Schur Complement Theorem) *Suppose* $B \in \mathbb{H}_n$ *with* $\operatorname{rank} B = m$. *Let* $B[\alpha]$ *be a nonsingular principal submatrix of* B *with* $|\alpha| = k \leq m$ *and let* $C = B/B[\alpha]$ *be the Schur complement of* $B[\alpha]$ *in* B. *Then the following results hold.*

1. $C \in \mathbb{H}_{n-k}$.
2. *Assuming the indexing of* C *is inherited from* B, *any principal minor of* C *is given by*

$$\det C[\gamma] = \det B[\gamma \cup \alpha] / \det B[\alpha].$$

3. $\operatorname{rank} C = m - k$.
4. *Any nonsingular principal submatrix of* B *of order at most* m *is contained in a nonsingular principal submatrix of order* m.

We state an immediate corollary of Theorem 1.10 in a form we will use.

COROLLARY 1.11. *Suppose* $B \in \mathbb{H}_n$, $\operatorname{epr}(B) = \ell_1 \cdots \ell_n$, *and let* $B[\alpha]$ *be a nonsingular principal submatrix of* B *with* $|\alpha| = k \leq \operatorname{rank} B$. *Let* $C = B/B[\alpha]$ *be the Schur complement of* $B[\alpha]$ *in* B *and let* $\operatorname{epr}(C) = \ell'_1 \cdots \ell'_{n-k}$. *Then* $\ell'_j = \ell_{j+k}$ *for* $\ell_j \in \{\mathtt{A}, \mathtt{N}\}$ *and* $j = 1, ..., n-k$.

It was established in [4, Theorem 5.1] that, for real symmetric matrices, any attainable epr-sequence starting $\mathtt{AN}\cdots$ is attainable by a real symmetric matrix with every entry equal to 1 or -1. In Section 6, we demonstrate that the epr-sequence $\mathtt{ANAAN}$ is attainable by a Hermitian matrix; however, this sequence is not attainable by a real symmetric matrix (see [4, Table 1]), revealing that the result of [4, Theorem 5.1] cannot be extended to Hermitian matrices. There is, however, a natural extension, which we now present.

PROPOSITION 1.12. *Over* $\mathbb{H}_n$, *any attainable epr-sequence starting* $\mathtt{AN}\cdots$ *is attainable by a Hermitian matrix with each entry having modulus* 1 *and all entries in the first row, first column and diagonal equal to* 1.

Proof. Let $B = [b_{jk}]$ be a Hermitian matrix with $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose

$\ell_1\ell_2 = \texttt{AN}$. Observe that each entry of B is nonzero. The matrix B and the matrix $\frac{1}{b_{11}}B$ have the same epr-sequence. Hence, we may assume that $b_{11} = 1$. Let $D = \operatorname{diag}(1, \frac{1}{b_{12}}, \dots, \frac{1}{b_{1n}})$. Then D^*BD is a Hermitian matrix with the same epr-sequence as B with all entries in the first row (and hence first column) equal to 1. Since each principal submatrix of D^*BD of order 2 including the $(1,1)$-entry is singular, each diagonal entry of D^*BD is 1. Since each principal submatrix of D^*BD of order 2 is singular, and because each diagonal entry is 1, each entry of D^*BD has modulus 1. □

2. Forbidden (sub)sequences. In this section we establish that epr-sequences of matrices in $\mathbb{H}_n$ cannot include certain subsequences, or cannot include them in certain positions.

PROPOSITION 2.1. *No Hermitian matrix has an epr-sequence starting* $\texttt{ANAN}\cdots$ *or* $\texttt{ANAS}\cdots$.

Proof. Suppose to the contrary that there exists a Hermitian matrix B with epr-sequence starting $\texttt{ANAN}\cdots$ or $\texttt{ANAS}\cdots$. By the Inheritance Theorem there exists a 4×4 principal submatrix C of B with epr-sequence $\texttt{ANAN}$; by Proposition 1.12, we may assume that

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & a & \bar{b} \\ 1 & \bar{a} & 1 & c \\ 1 & b & \bar{c} & 1 \end{bmatrix},$$

where a, b and c have modulus 1. Subtracting the first row of C from rows 2, 3 and 4, we see that

$$\begin{aligned} \det C &= \det \begin{bmatrix} 0 & a-1 & \bar{b}-1 \\ \bar{a}-1 & 0 & c-1 \\ b-1 & \bar{c}-1 & 0 \end{bmatrix} \\ &= (a-1)(b-1)(c-1) + (\bar{a}-1)(\bar{b}-1)(\bar{c}-1) \\ &= (a-1)(b-1)(c-1) + \frac{1}{a}(1-a)\frac{1}{b}(1-b)\frac{1}{c}(1-c) \\ &= (a-1)(b-1)(c-1)(1-\frac{1}{abc}). \end{aligned}$$

with the third equality coming from the fact that each of a, b and c has modulus 1. Since C is singular, we conclude that either $a = 1$, $b = 1$, $c = 1$ or $abc = 1$. This contradicts the fact that $0 \neq \det C(\{4\}) = a + \bar{a} - 2$, $0 \neq \det C(\{3\}) = b + \bar{b} - 2$, $0 \neq \det C(\{2\}) = c + \bar{c} - 2$, and $0 \neq \det C(\{1\}) = abc + \bar{a}\bar{b}\bar{c} - 2$. □

COROLLARY 2.2. *If the epr-sequence* $\texttt{XYNAN}$ *occurs as a subsequence of the epr-sequence of a Hermitian matrix, then* $\texttt{X} = \texttt{N}$ *and* $\texttt{Y} \neq \texttt{N}$. *In particular, the subsequences* $\texttt{AYNAN}$ *and* $\texttt{SYNAN}$ *are forbidden for* $\texttt{Y} \in \{\texttt{A}, \texttt{S}, \texttt{N}\}$.

Proof. Suppose $B \in \mathbb{H}_n$ has an epr-sequence containing $\texttt{XYNAN}$. By the $\texttt{NN}$ Theorem, $\texttt{Y} \neq \texttt{N}$. To obtain a contradiction, suppose $\texttt{X} \in \{\texttt{A}, \texttt{S}\}$ and $\texttt{X}$ occurs in position k. Let $B[\alpha]$ be a $k \times k$ nonsingular principal submatrix of B. By Corollary 1.11, $B/B[\alpha]$ has epr-sequence $\texttt{ZNAN}\cdots$, where $\texttt{Z} \in \{\texttt{A}, \texttt{S}, \texttt{N}\}$. By the $\texttt{NN}$ Theorem, $\texttt{Z} \neq \texttt{N}$. By Proposition 2.1, $\texttt{Z} \neq \texttt{A}$. By Proposition 1.6, $\texttt{SN}\cdots\texttt{A}\cdots$ is prohibited, so $\texttt{Z} \neq \texttt{S}$, and we have a contradiction. □

According to [4, Corollary 2.10], the epr-sequence $\texttt{SANA}$ is prohibited in the epr-sequence of a symmetric matrix over a field of characteristic not 2. For Hermitian matrices, however, we demonstrate in Section 6 that $\texttt{ASANA}$ is attainable, revealing that $\texttt{SANA}$ is not prohibited in an attainable sequence. However, there is a necessary condition given in the next result.

PROPOSITION 2.3. *In the epr-sequence of a Hermitian matrix, the epr-sequence* $\texttt{SANA}$ *can occur only as the terminal subsequence.*

Proof. Let $B \in \mathbb{H}_n$ and $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k\cdots\ell_{k+3} = \texttt{SANA}$. For the sake of contradiction, suppose $n > k+3$. By Corollary 2.2, $\texttt{SANAN}$ is prohibited, implying that $\ell_{k+4} \neq \texttt{N}$. Now, suppose that $\ell_{k+4} = \texttt{A}$. By the Inheritance Theorem and the Inverse Theorem, B has a $(k+4)\times(k+4)$ principal submatrix whose inverse has epr-sequence $\texttt{ANAS}\cdots\texttt{A}$, a contradiction to Proposition 2.1. Hence, $\texttt{SANAA}$ cannot occur in the epr-sequence of a Hermitian matrix.

Finally, suppose $\ell_{k+4} = \texttt{S}$. By the Inheritance Theorem, B has a $(k+4)\times(k+4)$ principal submatrix with epr-sequence $\cdots\texttt{SANAX}$, where $\texttt{X}$ is $\texttt{A}$ or $\texttt{N}$, contradicting the assertions above. □

The next result also restricts the location of a subsequence in attainable epr-sequences.

PROPOSITION 2.4. *No Hermitian matrix can have an epr-sequence starting* $\texttt{NSSNA}\cdots$.

Proof. Let $B \in \mathbb{H}_n$ have epr-sequence $\texttt{NSSNA}\cdots$. By the Inheritance Theorem, B has an appropriate principal submatrix C with $\text{epr}(C) = \texttt{NSXNA}$, where $\texttt{X} \in \{\texttt{A}, \texttt{S}, \texttt{N}\}$. By the $\texttt{NN}$ Theorem, and because $\texttt{NSA}$ is prohibited, $\texttt{X} = \texttt{S}$, so that $\text{epr}(C) = \texttt{NSSNA}$. By the Inverse Theorem, $\text{epr}(C^{-1}) = \texttt{NSSNA}$. Since C has a zero minor of order 2, we assume, without loss of generality, that $C[\{1,2\}]$ is singular; as each diagonal entry of C is zero, $C[\{1,2\}] = O_{2\times 2}$. From this and the fact that $CC^{-1} = I_5$,

$$O_{2\times 3} = (CC^{-1})[\{1,2\},\{3,4,5\}] = C[\{1,2\},\{3,4,5\}]C^{-1}[\{3,4,5\}].$$

As C is nonsingular, $C[\{1,2\},\{3,4,5\}]$ has full rank, i.e., it has rank 2; thus, the null space of $C[\{1,2\},\{3,4,5\}]$ has dimension 1. Since the column space of $C^{-1}[\{3,4,5\}]$

is contained in the null space of $C[\{1,2\},\{3,4,5\}]$, $C^{-1}[\{3,4,5\}]$ has at most one linearly independent column; then, as every diagonal entry of $C^{-1}[\{3,4,5\}]$ is zero, the fact that $C^{-1}[\{3,4,5\}]$ is Hermitian implies that $C^{-1}[\{3,4,5\}] = O_{3\times 3}$. It follows that C^{-1} is singular, a contradiction. □

3. Gaps between two Ns.

3. Gaps between two `N`s. Consider the following problem raised in [2, Question 6.6]: Fix some $s \geq 1$. Is it the case that for any $n \times n$ real symmetric matrix B with $\mathrm{pr}(B) = r_0]r_1 \cdots r_n$, if $r_k = r_{k+s} = 0$, then $r_i = 0$ for all i with $k+s \leq i \leq n$? As noted in [2], the 00 theorem (the pr-sequence form of the `NN` Theorem) implies the answer to the question is yes when $s = 1$. It was also shown there that the answer is yes for $s = 3$ but is no for s even and $s = 5$ in [2, Theorem 6.5, Lemmas 3.3, 3.6, and Example 6.7]. The positive answer for $s = 3$ is used in [9] to determine all attainable pr-sequences that have a 0 in each subsequence of length 3 and all attainable epr-sequences that have an `N` in each subsequence of length 3. We translate the question to the language of epr-sequences.

Question 3.1. Let $s \geq 1$ be a fixed integer and let B be a Hermitian matrix. Does $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ with $\ell_k = \ell_{k+s} = \mathtt{N}$ imply that $\ell_q = \mathtt{N}$ for all $q \geq k+s$?

Because of the `NN` Theorem, we know the answer is affirmative when $s = 1$. Section 3.1 answers this question negatively for $s \geq 2$ and Hermitian matrices, which behave quite differently from real symmetric matrices. Section 3.2 discusses in more detail the form of sequences containing a `NAN` subsequence (which has $\ell_k = \ell_{k+2} = \mathtt{N}$).

3.1. Answer to Question 3.1.

3.1. Answer to Question 3.1. Before answering Question 3.1 negatively for all $s \geq 2$ in Theorem 3.3 below, we need the following lemma.

Lemma 3.2. *For $t \neq 0$, let T_n be the $n \times n$ matrix with 0s on the main diagonal, t in every entry above the main diagonal, and $(1/t)$ in every entry below the main diagonal. Then, for $n \geq 1$,*

$$\det T_n = \frac{(-1)^{n+1}}{t^{n-2}} \sum_{j=0}^{n-2} t^{2j}.$$

Thus $\det T_n = 0$ if and only if $\sum_{j=0}^{n-2} t^{2j} = 0$

Proof. We proceed by induction. For the case $n = 1$, we have $\det T_1 = 0$, while the right-hand side is an empty sum (which by convention is 0). For the case $n = 2$, we have

$$\det \begin{bmatrix} 0 & t \\ 1/t & 0 \end{bmatrix} = -1 = \frac{(-1)^3}{t^0} \sum_{j=0}^{0} t^{2j}.$$

Now assume the result holds up through some value of k, and consider what happens for the case $k+1$. We have the following:

$$\det(T_{k+1}) = \det \begin{bmatrix} 0 & t & t & \cdots & t & t \\ 1/t & 0 & t & \cdots & t & t \\ 1/t & 1/t & 0 & \cdots & t & t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1/t & 1/t & 1/t & \cdots & 0 & t \\ 1/t & 1/t & 1/t & \cdots & 1/t & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & t & t & \cdots & t & t \\ 1/t & -t & 0 & \cdots & 0 & 0 \\ 0 & 1/t & -t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -t & 0 \\ 0 & 0 & 0 & \cdots & 1/t & -t \end{bmatrix} = (-1)^{k+2} t \ (1/t)^k - t \det(T_k).$$

The second equality is obtained by starting with the last row and subtracting the previous row, and then repeating this process going up a row at a time. The third equality is obtained by expanding the determinant along the last column.

We can now conclude

$$\begin{aligned} (-1)^{k+2} t^{k-1} \det T_{k+1} &= (-1)^{k+2} t^{k-1} \big((-1)^{k+2}/t^{k-1} - t \det T_k \big) \\ &= 1 + (-1)^{k+1} t^k \det T_k \\ &= 1 + t^2 \sum_{j=0}^{k-2} t^{2j} = \sum_{j=0}^{k-1} t^{2j}. \end{aligned}$$

This establishes the formula for the determinant of T_{k+1}. □

Theorem 3.3. *Let $s \geq 2$ and $1 \leq k \leq s-1$. Then the epr-sequence of order n having $\ell_i =$* `N` *for $i \equiv k \pmod{s}$ and* `A`*s in all other positions, is attainable by a Hermitian matrix.*

Proof. It will suffice to establish this for $k = 1$. To see this, suppose $2 \leq k \leq s-1$, choose n' with $n' > n$ and $n' \equiv k+1 \pmod{s}$, and consider the matrix B realizing the epr-sequence of order n' where there are `N`s in positions $\equiv 1 \pmod{s}$ and `A`s in all other positions. By assumption, the last letter will be `A` (since $n' \not\equiv 1 \pmod{s}$). Thus the matrix B is invertible, and the epr-sequence of B^{-1} will have `N`s in positions $\equiv k \pmod{s}$ and `A`s in all other locations by the Inverse Theorem. Finally, any principal submatrix of B^{-1} of order n gives the desired realization by the Inheritance Theorem.

For the case $k = 1$, we claim that the matrix T_n with $t = e^{\pi i/s}$ from Lemma 3.2 is a realization. In particular, since a principal submatrix of order m for such a matrix is T_m, with $t = e^{\pi i/s}$, then it will suffice to show that T_m has zero determinant if and only if $m \equiv 1 \pmod{s}$. By Lemma 3.2, we have

$$\det T_m = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{m-2} \left(e^{2\pi i/s}\right)^j = 0.$$

Finally, we note that sum is found by repeatedly adding s-th roots of unity. The sum of all s of the s-th roots of unity is 0 and the sum of any q consecutive sth roots of unity is nonzero for $q < s$, so the sum is nonzero if and only if the number of terms in the sum (i.e., $m-1$) is not a multiple of s. That is, $\det T_m \neq 0$ if and only if $m \not\equiv 1 \pmod{s}$. □

This naturally raises the question of what happens when we want the `N`s to occur in the positions $\equiv s \pmod{s}$ and all other values to be `A`. This leads to the following question, which has an affirmative answer when $s = 2$ (see Proposition 2.1).

QUESTION 3.4. For $s \geq 2$, is the sequence of order $2s$ with `N`s in positions s and $2s$, and with `A`s in all other positions, unattainable by a Hermitian matrix?

3.2. `NAN` and real skew-like sequences.

The next remark relates the epr-sequences of Hermitian matrices and skew-Hermitian matrices.

REMARK 3.5. If K is a skew-Hermitian matrix, then iK is Hermitian, and if H is a Hermitian matrix then iH is skew-Hermitian. Thus $\text{attain}(\mathbb{H}_n) = \text{attain}(\mathbb{K}_n)$, so by Theorem 1.9 every epr-sequence $\ell_1 \cdots \ell_n$ that has $\ell_k = \texttt{N}$ for every odd k, obeys the `NN` Theorem, and has $\ell_n \neq \texttt{S}$ is attained by a Hermitian matrix.

Motivated by Theorem 1.9, we make the following definition.

DEFINITION 3.6. The sequence $\ell_1\ell_2 \cdots \ell_n$ is *real skew-like* if $\ell_1\ell_2 \cdots \ell_n \in \text{attain}(\mathbb{H}_n)$ and $\ell_j = \texttt{N}$ for every odd j with $1 \leq j \leq n$. For an odd integer p, a subsequence $\ell_p \cdots \ell_q$ of an attainable epr-sequence $\ell_1\ell_2 \cdots \ell_n$ is *real skew-like* if $\ell_j = \texttt{N}$ for every odd j with $p \leq j \leq q$.

Observe that the epr-sequence of a real skew-symmetric matrix is real skew-like (hence the name).

PROPOSITION 3.7. *Over $\mathbb{H}_n$, any attainable epr-sequence starting* `NAN`$\cdots$ *is real skew-like.*

Proof. Let $B \in \mathbb{H}_n$ and $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose $\ell_1\ell_2\ell_3 = \texttt{NAN}$. Since B has zero diagonal, the condition $\ell_2 = \texttt{A}$ implies that every off-diagonal entry of B is nonzero. Let $D = \text{diag}(1, \frac{1}{b_{12}}, \ldots, \frac{1}{b_{1n}})$, $B' = D^*BD$ and $B' = [b'_{kj}]$. Now, observe that

$$B' = \begin{bmatrix} 0 & \mathbb{1}^T \\ \mathbb{1} & B'(\{1\}) \end{bmatrix}.$$

Since $\operatorname{epr}(B') = \operatorname{epr}(B)$, and because $\ell_3 = \mathtt{N}$,

$$0 = \det B'[\{1,k,j\}] = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & x \\ 1 & \overline{x} & 0 \end{bmatrix} = x + \overline{x} = 2\operatorname{Re}(x),$$

for some x. Thus, $B'(\{1\}) = iK'$, where K' is an $(n-1)\times(n-1)$ real skew-symmetric matrix. Let $D_2 = \operatorname{diag}(1, i, \dots, i)$, $B'' = D_2^* B' D_2$ and $B'' = [b''_{kj}]$. For $k, j > 1$, observe that $b''_{kj} = b'_{kj}$, $b''_{1j} = i$ and $b''_{j1} = -i$. Thus, $B'' = iK''$, where K'' is real and skew-symmetric. Since $\operatorname{epr}(K'')$ is real skew-like, observing that $\operatorname{epr}(B) = \operatorname{epr}(B') = \operatorname{epr}(B'') = \operatorname{epr}(K'')$ leads to the desired conclusion. □

Corollary 3.8. *Let $B \in \mathbb{H}_n$ and $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k\ell_{k+1}\ell_{k+2} =$ $\mathtt{NAN}$. Then $\ell_{k+2j} = \mathtt{N}$ for $k + 2j \le n$.*

Proof. Since the case with $k = 1$ is covered by Proposition 3.7, assume $k \ge 2$. Suppose to the contrary that $\ell_{k+2j} \ne \mathtt{N}$ for some $j \ge 2$. By the Inheritance Theorem, B has a $(k+2j)\times(k+2j)$ principal submatrix B' with $\operatorname{epr}(B') = \ell'_1\ell'_2\cdots\ell'_{k+2j}$ having $\ell'_k\ell'_{k+1}\ell'_{k+2} = \mathtt{NAN}$ and $\ell'_{k+2j} = \mathtt{A}$. By the $\mathtt{NN}$ Theorem, $\ell'_{k-1} \ne \mathtt{N}$, implying that B' has a nonsingular $(k-1)\times(k-1)$ principal submatrix, say $B'[\alpha]$. It follows from Corollary 1.11 that $B'/B'[\alpha]$ is a (Hermitian) matrix of order $(k+2j)-(k-1) = 2j+1$, with $\operatorname{epr}(B'/B'[\alpha]) = \mathtt{NAN}\cdots\mathtt{A}$; since $\operatorname{epr}(B'/B'[\alpha])$ does not contain $\mathtt{N}$ in the odd position $2j+1$, $\operatorname{epr}(B'/B'[\alpha])$ is not real skew-like, a contradiction to Proposition 3.7. □

Unlike for symmetric matrices over a field of characteristic not 2 (see [4, Theorem 2.14]), it is shown in Section 6 that the epr-sequence $\mathtt{NAS}$ is not prohibited in the epr-sequence of a Hermitian matrix; however, $\mathtt{NAS}$ is prohibited if it occurs in the subsequence $\mathtt{ANAS}$.

Proposition 3.9. *The epr-sequence $\mathtt{ANAS}$ cannot occur as a subsequence of the epr-sequence of a Hermitian matrix.*

Proof. Let $B \in \mathbb{H}_n$ and $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$. Suppose $\ell_k\cdots\ell_{k+3} = \mathtt{ANAS}$. By Proposition 2.1, $k \ge 2$. Then, by Proposition 2.3 and Corollary 3.8, $\ell_{k-1} = \mathtt{A}$. By the Inheritance Theorem, B has a $(k+3)\times(k+3)$ principal submatrix C with epr-sequence $\cdots\mathtt{AANAN}$. By Corollary 1.11, the epr-sequence of the Schur complement in C of a (necessarily nonsingular) $(k-1)\times(k-1)$ principal submatrix, has epr-sequence $\mathtt{ANAN}$, contradicting Proposition 2.1. □

Proposition 3.10. *Let $B \in \mathbb{H}_n$ and $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose $\ell_k\ell_{k+1}\ell_{k+2} =$ $\texttt{NAN}$, where k is even. Then $\ell_j = \texttt{N}$ for all $j \geq k+2$.*

Proof. By the $\texttt{NN}$ Theorem, it suffices to show that $\ell_{k+3} = \texttt{N}$. Suppose to the contrary that $\ell_{k+3} \neq \texttt{N}$. By the Inheritance Theorem and the Inverse Theorem, B has a nonsingular $(k+3) \times (k+3)$ principal submatrix whose inverse has epr-sequence $\texttt{NAN} \cdots \texttt{A}$. This contradicts Proposition 3.7, since $k+3$ is odd. □

Theorem 3.11. *Over $\mathbb{H}_n$, every attainable epr-sequence containing $\texttt{NANA}$ is real skew-like.*

Proof. Let $B \in \mathbb{H}_n$ with $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose $\ell_k\ell_{k+1}\ell_{k+2}\ell_{k+3} = \texttt{NANA}$. By Proposition 3.10, k is odd. By Corollary 3.8, $\ell_k \cdots \ell_n$ is real skew-like. To conclude, we show that $\ell_1 \cdots \ell_{k-1}$ is real skew-like. For the sake of contradiction, suppose $\ell_j \neq \texttt{N}$ for some odd j with $1 \leq j \leq k-1$. By the Inheritance Theorem, B has a nonsingular $(k+3) \times (k+3)$ principal submatrix B' whose epr-sequence $\ell'_1\ell'_2 \cdots \ell'_{k+3}$ has $\ell'_k \cdots \ell'_{k+3} = \texttt{NANA}$ and $\ell'_j \neq \texttt{N}$. By the Inverse Theorem, $\text{epr}((B')^{-1}) = \texttt{NAN} \cdots$ does not have $\texttt{N}$ in position $(k+3)-j$, contradicting Proposition 3.7, because $(k+3)-j$ is odd. □

Corollary 3.12. *Let $B \in \mathbb{H}_n$ and $\text{epr}(B) = \ell_1\ell_2 \cdots \ell_n$. Suppose $\ell_k\ell_{k+1}\ell_{k+2}\ell_{k+3} = \texttt{NANS}$, where $n > k+3$. Then the following hold.*

1. *k is odd;*
2. *$\ell_k \cdots \ell_n$ is real skew-like;*
3. *$\ell_j \neq \texttt{A}$ for odd j.*

Proof. (1): By Proposition 3.10, k is odd. (2): The assertion that $\ell_k \cdots \ell_n$ is real skew-like follows from Corollary 3.8. (3): The conclusion is already established in (2) for odd $j > k-2$. Now, suppose to the contrary that $\ell_j = \texttt{A}$ for some odd $j \leq k-2$. By the Inheritance Theorem, B has a $(k+3) \times (k+3)$ principal submatrix B' with $\text{epr}(B') = \cdots \texttt{A} \cdots \texttt{NANA}$ having $\texttt{A}$ in the odd position j, implying that $\text{epr}(B')$ is not real skew-like, a contradiction to Theorem 3.11. □

Conjecture 3.13. *Over $\mathbb{H}_n$, every attainable epr-sequence containing $\texttt{NAN}$ is real skew-like.*

4. Hermitian adjacency matrices of mixed graphs. Introduced by Liu and Li in [8], and independently by Guo and Mohar in [6], the Hermitian adjacency matrix associates a Hermitian matrix with a (simple) mixed graph or (simple) digraph. The term *simple* means that loops and duplicate edges (directed or undirected) are not allowed; since all our graphs and digraphs are simple we omit the term 'simple' and define graphs and digraphs to prohibit loops and duplicates. Technically a mixed graph may have both undirected edges and directed edges but may not have more than one edge of any kind between a given pair of vertices, whereas in a digraph all edges are directed, it is permitted to have both directed edges (u, v) and (v, u), but more than one copy of any directed edge is prohibited. There is a one-to-one correspondence between mixed graphs and digraphs, by associating an undirected edge $\{u, v\}$ with the pair of directed edges (u, v) and (v, u). We use the the term mixed graph, since that was the original term in [8] and more naturally generalizes the adjacency matrix of an (undirected) graph. We will use uv to denote an edge between u and v, either directed or undirected. The underlying graph G_Γ of a mixed graph Γ is the graph obtained from Γ by replacing every directed edge (u, v) by the undirected edge $\{u, v\}$.

Let Γ be a mixed graph on n vertices. The *Hermitian adjacency matrix* $\mathcal{H}(\Gamma) = [h_{kj}]$ is the $n \times n$ matrix with entries over the complex field given by

$$h_{kj} = \begin{cases} 1 & \text{if } \Gamma \text{ has an undirected edge from } k \text{ to } j; \\ i & \text{if } \Gamma \text{ has an directed edge from } k \text{ to } j; \\ -i & \text{if } \Gamma \text{ has an directed edge from } j \text{ to } k; \\ 0 & \text{otherwise.} \end{cases}$$

This generalizes the usual (0,1) adjacency matrix for an undirected graph.

Example 4.1. *The* 6×6 *Hermitian adjacency matrix corresponding to the mixed graph shown in Figure 4.1 is*

$$M_{\texttt{NSNASA}} = \begin{bmatrix} 0 & 0 & i & -i & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -i \\ -i & 0 & 0 & 0 & 1 & 1 \\ i & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & 1 & 1 & 0 & 0 \end{bmatrix},$$

which has epr-sequence `NSNASA`*.*

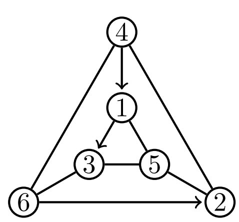

Fig. 4.1. *The mixed graph for Example 4.1*

We now consider the epr-sequences attainable by Hermitian adjacency matrices, which must start with N. There are additional restrictions on any Hermitian adjacency matrix with epr-sequence starting NA.

PROPOSITION 4.2. *Suppose* Γ *is a mixed graph of order* n*, let* $\mathcal{H}(\Gamma) = H = [h_{kj}]$*, and* $\operatorname{epr}(H) = \ell_1 \cdots \ell_n$.

1. $\ell_1 = \mathtt{N}$.
2. *For* $n \geq 2$, $\ell_2 = \mathtt{A}$ *if and only if* G_Γ *is a complete graph.*
3. *For* $n \geq 4$, $\ell_2 = \mathtt{A}$ *implies* $\ell_4 = \mathtt{A}$.

Proof. The first two statements are clear. For the third, suppose H has a 4×4 principal submatrix $H' = [h'_{kj}]$. By the Inheritance Theorem, $\operatorname{epr}(H')$ starts with NA, which implies that every off-diagonal entry of H' is nonzero. With $D = \operatorname{diag}(1, \frac{1}{h'_{12}}, \frac{1}{h'_{13}}, \frac{1}{h'_{14}})$,

$$D^* H' D = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & \overline{b} \\ 1 & \overline{a} & 0 & c \\ 1 & b & \overline{c} & 0 \end{bmatrix},$$

for some $a, b, c \in \{\pm 1, \pm i\}$. It follows that

$$\det(D^* H' D) = |a|^2 + |b|^2 + |c|^2 - 2\operatorname{Re}(ab + ac + bc) = 3 - 2\operatorname{Re}(ab + ac + bc) \neq 0.$$

Thus $D^* H' D$, and therefore H', is nonsingular. □

Theorem 4.3 below characterizes Hermitian adjacency matrices that begin NAN, strengthening Proposition 3.7 for such matrices. A *tournament* is an oriented complete graph, i.e., every edge is directed. Thus, the Hermitian adjacency matrix of a tournament has a 0 diagonal and $\pm i$ off-diagonal entries such that the resulting matrix is Hermitian.

THEOREM 4.3. *Suppose* H *is the Hermitian adjacency matrix of a mixed graph* Γ *of order* n*. The following are equivalent:*

1. $\operatorname{epr}(H) = \mathtt{NAN} \cdots$.
2. G_Γ *is complete and each triangle in* Γ *contains an odd number of directed edges.*
3. G_Γ *is complete and*
   (a) Γ *is a tournament, or*
   (b) *every vertex is incident with an undirected edge and the subgraph of undirected edges is a complete bipartite graph.*

If these conditions hold, then $\operatorname{epr}(H) = \mathtt{NAN}\overline{\mathtt{AN}}$ *if* n *is odd, and* $\operatorname{epr}(H) = \mathtt{NAN}\overline{\mathtt{AN}}\mathtt{A}$ *if* n *is even.*

Proof. For $u, s, t \in V(\Gamma)$, $H[\{u,s,t\}] = \begin{bmatrix} 0 & a & \overline{c} \\ \overline{a} & 0 & b \\ c & \overline{b} & 0 \end{bmatrix}$ with $a, b, c \in \{1, \pm i\}$, and $\det H[\{u,s,t\}] = abc + \overline{abc} = 2\,\mathrm{Re}(abc) = 0$ if and only if abc is purely imaginary.

(1) $\Rightarrow$ (2): Since every 2×2 principal submatrix is nonsingular, every off-diagonal entry is nonzero, and G_Γ is complete. Since every 3×3 principal submatrix is singular, abc is purely imaginary. Since $a, b, c \in \{1, \pm i\}$, exactly one or three of a, b, c are purely imaginary, i.e., one or three of the pairs of vertices taken from u, s, t are directed.

(2) $\Rightarrow$ (3): If Γ has no undirected edges, then Γ is a tournament, because the underlying graph of Γ is complete. So suppose Γ has an undirected edge and v is a vertex incident with an undirected edge. Partition the vertices of Γ as follows: V_1 is v together with the set of vertices adjacent to v by a directed edge and V_2 the set of vertices adjacent to v by an undirected edge. Since the underlying graph is complete, all vertices are in one of these sets and they are clearly disjoint. Let G be the subgraph of Γ having $V(G) = V(\Gamma)$ and $E(G)$ is the set of undirected edges in Γ. We show G is a complete bipartite graph with partite sets V_1 and V_2. Suppose that $x, y \in V_1$ and $w, z \in V_2$. By definition of V_1, vx and vy are directed edges. Since $\Gamma[\{v,x,y\}]$ can't have exactly two directed edges, xy is directed. By definition of V_2, vw and vz are undirected. Since $\Gamma[\{v,w,z\}]$ must have at least one directed edge, zw is directed. Thus all edges of G (undirected edges of Γ) are between V_1 and V_2, so G is bipartite. By definition of V_1 and V_2, vx is directed and vw is undirected. Since $\Gamma[\{v,x,w\}]$ can't have exactly two directed edges, xw is undirected. Thus G is the complete bipartite graph with partite sets V_1 and V_2.

(3) $\Rightarrow$ (1): If Γ has no undirected edges, then Γ is a tournament and the values a, b, and c in $H[\{u,s,t\}]$ are all purely imaginary, so abc is purely imaginary and $H[\{u,s,t\}]$ is singular. So assume every vertex is incident with an undirected edge and the subgraph of undirected edges is complete bipartite. Choose any three vertices. If they are all in the same partite set then the triangle has 3 directed edges and as before $H[\{u,s,t\}]$ is singular. If two are in one partite set and one in the other, then without loss of generality $a = 1, b = 1$ and $c = \pm i$, so $abc = \pm i$ and $H[\{u,s,t\}]$ is singular.

Now assume $n \geq 4$ and $\mathrm{epr}(H) = \mathtt{NAN}\ell_4 \cdots \ell_n$. With $D = \mathrm{diag}(1, \frac{1}{h_{12}}, \dots, \frac{1}{h_{1n}})$, $H' := D^* H D = \begin{bmatrix} 0 & \mathbb{1}^T \\ \mathbb{1} & H'(\{1\}) \end{bmatrix}$, where $H'(\{1\})$ has zero diagonal and off-diagonal entries in $\{\pm 1, \pm i\}$. Since $\mathrm{epr}(H') = \mathrm{epr}(H)$,

$$0 = \det H'[\{1,k,j\}] = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & x \\ 1 & \overline{x} & 0 \end{bmatrix} = x + \overline{x} = 2\,\mathrm{Re}(x),$$

for some x. Since this determinant must be zero, $x \in \{\pm i\}$ and $H'(\{1\}) = iK'$, where K' is an $(n-1) \times (n-1)$ real skew-symmetric matrix with every off-diagonal entry ± 1. Let $D_2 = \operatorname{diag}(1, i, \ldots, i)$ and $H'' = D_2^* H' D_2$. Then H'' is a Hermitian matrix with every diagonal entry equal to zero and with every off-diagonal entry equal to $\pm i$. Observe that $H'' = iK$ where K is a real skew-symmetric matrix with every off-diagonal entry equal to 1 or -1, and every principal submatrix of K is also a skew-symmetric matrix. The determinant of such a skew-symmetric matrix is zero for all odd orders, and is nonzero for all even orders [10, Proposition 1]. Thus, $\operatorname{epr}(H'') = \mathtt{NAN}\overline{\mathtt{AN}}$ if n is odd, $\operatorname{epr}(H'') = \mathtt{NAN}\overline{\mathtt{AN}}\mathtt{A}$ if n is even, and $\operatorname{epr}(H) = \operatorname{epr}(H'')$. □

Note that the epr-sequence $\mathtt{NAN}$ is not attainable by any real symmetric matrix [4, Theorem 2.14], but, by Theorem 4.3, it is attainable by the Hermitian adjacency matrix of a tournament of order 3. The next result gives another restriction for epr-sequences of $(0,1)$ adjacency matrices.

Proposition 4.4. *Let $n \geq 5$, B be a $(0,1)$ adjacency matrix with* $\operatorname{epr}(B) = \mathtt{N}\ell_2\ell_3\mathtt{A}\ell_5 \cdots \ell_n$. *Then* $\ell_5 = \mathtt{A}$.

Proof. Suppose to the contrary that $\ell_5 \neq \mathtt{A}$. By the Inheritance Theorem, B has a 5×5 principal submatrix $B' = [b'_{kj}]$ having $\operatorname{epr}(B') = \mathtt{N}\ell'_2\ell'_3\mathtt{AN}$. By the $\mathtt{NN}$ Theorem, $\ell'_2 \neq \mathtt{N}$. We claim that $\ell'_2 = \mathtt{S}$: Otherwise, $\ell'_2 = \mathtt{A}$, and therefore each off-diagonal entry of B' must be nonzero, This would imply that $B' = J_5 - I_5$, which is impossible since $J_5 - I_5$ is nonsingular. Thus, $\ell'_2 = \mathtt{S}$.

Therefore B' must have a singular 2×2 principal submatrix, which must be $O_{2\times 2}$. We may assume, without loss of generality, that $B'[\{1,2\}] = O_{2\times 2}$. Since every 4×4 principal submatrix of B' is nonsingular, each row (and column) of B' must contain at least two nonzero entries (otherwise B' would have a 4×4 principal submatrix with a row consisting of only zeros); without loss of generality, we may assume that $b'_{13} = b'_{14} = 1$. Similarly, the second row (and column) must contain at least two nonzero entries, implying that at least one of b'_{23} and b'_{24} must be nonzero; we may assume that $b'_{23} = 1$. Since $B'[\{1,2\}] = O_{2\times 2}$, and because every 4×4 principal submatrix of B' is nonsingular, every 2×2 submatrix of $B'[\{1,2\},\{3,4,5\}]$ must be nonsingular, implying that $b'_{24} = 0$, and consequently that $b'_{25} = 1$ and $b'_{15} = 0$. It follows that $0 = \det(B') = 2(b'_{45} - b'_{34} - b'_{35})$; thus, $b'_{45} - b'_{34} - b'_{35} = 0$. Now we have ${b'_{34}}^2 = (b'_{45} - b'_{35})^2 = \det B'[\{1,3,4,5\}] \neq 0$ and ${b'_{35}}^2 = (b'_{45} - b'_{34})^2 = \det B'[\{2,3,4,5\}] \neq 0$, implying that $b'_{34} = b'_{35} = 1$, and therefore that $b'_{45} = 2$, a contradiction, since B' is a (0,1) adjacency matrix. □

Using Propositions 4.2 and 4.4, we can deduce that an epr-sequence of order $n = 2, 3, 4$, or 5 is attainable by a (0,1) adjacency matrix if and only if the realization provided in [4, Tables 2–5] is a (0,1) adjacency matrix (that is, the listed matrix is an adjacency matrix, a matrix of the form $(J_s - I_s) \oplus 0_{n-s}$, or a zero matrix).

Remark 4.5. It follows from Proposition 4.4 that the epr-sequences `NSSAN` and `NAAAN` cannot be realized by a (0,1) adjacency matrix; however, they are attainable by real symmetric matrices (see [4, Example 5.6]) and, as the next example shows, by Hermitian adjacency matrices.

Example 4.6. The Hermitian adjacency matrices $M_{\mathtt{NAAAN}}$ and $M_{\mathtt{NSSAN}}$ below have epr-sequences `NAAAN` and `NSSAN`, respectively.

$$M_{\mathtt{NAAAN}} = \begin{bmatrix} 0 & 1 & 1 & i & i \\ 1 & 0 & 1 & i & -i \\ 1 & 1 & 0 & -i & -i \\ -i & -i & i & 0 & 1 \\ -i & i & i & 1 & 0 \end{bmatrix}, \ M_{\mathtt{NSSAN}} = \begin{bmatrix} 0 & 0 & i & 0 & -i \\ 0 & 0 & 0 & 1 & 1 \\ -i & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ i & 1 & 1 & 1 & 0 \end{bmatrix}.$$

5. Probabilistic techniques for constructing attainable sequences. In this section we use probabilistic methods to establish that any epr-sequence that begins with an `N`, followed by zero or more `A`s, `S`s, and `N`s in that order can be realized by a Hermitian matrix.

Theorem 5.1. *Every epr-sequence of the form* $\mathtt{N}\overline{\mathtt{A}}\,\overline{\mathtt{S}}\,\overline{\mathtt{N}}$ *that does not end in* `S` *is attained by a Hermitian matrix.*

Proof. First we introduce notation for some parameters that specify which epr-sequence of the form $\mathtt{N}\overline{\mathtt{A}}\,\overline{\mathtt{S}}\,\overline{\mathtt{N}}$ is desired. If there is at least one `A`, let a denote the position of the last `A`; otherwise set $a = 1$. If there is at least one `S`, let s denote the position of the last `S`; otherwise set $s = a$. As usual, let n denote the order of the matrix. In general we have $1 \le a \le s \le n$. The zero matrix of size n satisfies the case $s = 1$, so henceforth we assume $s \ge 2$. The fact that the sequence does not end in `S` is equivalent to requiring that $s > a$ implies $n > s$.

Let D be the $s \times s$ diagonal matrix whose first $s-1$ diagonal entries are 1, followed by a single -1. We give a construction of an $s \times n$ matrix T in such a way that, with probability 1, the $n \times n$ matrix $B = T^* D T$ has the desired epr-sequence (in particular, *some* matrix T works). We divide T into two blocks: Its first $s-1$ rows will be called U, and its last row will have 1 in every entry. The block U is further divided into blocks consisting of an identity matrix of width $s-1$, a vector $\mathbf{z}$ (of width 1), and a matrix R of width $n-s$, as follows:

$$T = \begin{bmatrix} U \\ \mathbb{1}_n^T \end{bmatrix}, \quad U = \begin{bmatrix} I_{s-1} & \mathbf{z} & R \end{bmatrix}.$$

The columns of R are chosen randomly and independently from the set of unit vectors in $\mathbb{C}^{s-1}$. (To be concrete, their real and imaginary parts are chosen from the standard measure on the unit sphere of dimension $2s-3$ embedded in $\mathbb{R}^{2s-2}$.) In the extreme case $s=2$, for example, the columns of R are scalars chosen from the unit circle in the complex plane, which generically means that no two of them are equal and none equal 1.

The vector $\mathbf{z}$ is constructed in a way that depends on the values of a and s, according to these three cases:

1. For $s=a$, $\mathbf{z}$ is a random unit vector in $\mathbb{C}^{s-1}$, like the columns of R.
2. For $s>a$ and $a=1$, $\mathbf{z}$ is a repetition of the first column of I_{s-1}.
3. For $s>a$ and $a>1$, $\mathbf{z}=[z_j]$ is given by

$$\begin{aligned} z_1 &= \frac{1}{a} + i\sqrt{\frac{a-1}{2a}}, \\ z_2 &= \frac{1}{a} - i\sqrt{\frac{a-1}{2a}}, \\ z_3 = \ldots \; = \; z_a \quad &= \; \frac{1}{a}, \\ z_{a+1} = \ldots \; = \; z_{s-1} \; &= \; 0. \end{aligned}$$

In all cases $\mathbf{z}$ is a unit vector in $\mathbb{C}^{s-1}$. In both Case 2 and Case 3, or in other words whenever the desired epr-sequence contains at least one $\mathtt{S}$, $\mathbf{z}$ is designed in such a way that the sum of its entries is 1 and such that its entries are nonzero in exactly the first a rows.

Let $\operatorname{epr}(B)=\ell_1\cdots\ell_n$. To complete the proof we show the four necessary conclusions:

(A) $\ell_k=\mathtt{N}$ for $k=1$,
(B) $\ell_k=\mathtt{A}$ for $1<k\le a$,
(C) $\ell_k=\mathtt{S}$ for $a<k\le s$, and
(D) $\ell_k=\mathtt{N}$ for $s<k\le n$,

which must hold with probability 1 in all cases. Given $\alpha\subseteq[n]$, we let T_α and U_α denote respectively the matrices $T[[s],\alpha]$ and $U[[s-1],\alpha]$ that select the subset α of columns. Since $B[\alpha]=T_\alpha^* D T_\alpha$, the rank of $B[\alpha]$ is at most the rank of D, namely s, which gives us Conclusion (D).

We define the matrix $C=U^*U$. Since every column of U is a unit vector, C is a complex correlation matrix. We observe that for every $\alpha\subseteq[n]$, letting $k=|\alpha|$,

$$B[\alpha]=T_\alpha^* D T_\alpha = U_\alpha^* U_\alpha - \mathbb{1}_k\mathbb{1}_k^T = C[\alpha]-J_k. \tag{5.1}$$

For $\alpha=\{j\}$, this becomes

$$B[\{j\}] = U^*_{\{j\}} U_{\{j\}} - 1 = C[\{j\}] - J_1 = 0,$$

which gives us Conclusion (A).

In Case 1, Conclusion (C) is trivial. For Cases 2 and 3, define a subset $\beta = [a] \cup \{s\}$ of size $a+1$. Since the nonzero entries of $\mathbf{z}$ lie in its first a rows and have a sum of 1, the last row of T_β is the sum of its first a rows, and all other rows are zero. It follows that the columns of T_β form a dependent set. Thus for any α containing β as a subset, and in particular for $\alpha = [k-1] \cup \{s\}$ in the range $a < k \le s$, the columns of T_α are also dependent and $B[\alpha] = T^*_\alpha D T_\alpha$ is singular. This shows that $\ell_k \ne \mathtt{A}$ for any $a < k \le s$, which gives half of Conclusion (C).

Given $\alpha \subseteq [n]$ with $|\alpha| = k$, we have shown that $B[\alpha]$ is singular when $k > s$, when $k = 1$, or when β is defined (that is, $s > a$) and $\beta \subseteq \alpha$. In fact we will show that these are, with probability 1, the only conditions giving rise to singular $B[\alpha]$.

To that end we make the assumption $\beta \not\subseteq \alpha$, and establish the following three claims:

(i) If $k \ge 2$, then $C[\alpha] \ne J_k$ with probability 1.
(ii) If $k \le s-1$, then the columns of U_α are independent with probability 1.
(iii) If $k = s$, then the columns of T_α are independent with probability 1.

Claim (i) is equivalent, for $k \ge 2$, to the assertion that, with probability 1, at least two of the unit vector columns of U_α are not equal. The only two columns of U that could be equal, by construction and with positive probability, occur in Case 2 and are precisely the two columns of U_β, so Claim (i) always holds.

Claim (ii) is verified by induction on k. For $k = 1$ the fact that the columns of U are unit vectors suffices. Suppose then that $\alpha = \{\alpha_1, \dots, \alpha_k\}$ for some $1 < k \le s-1$ and that columns $\{\alpha_1, \dots, \alpha_{k-1}\}$ of U are independent. If $\alpha_k < s$, U_α is an independent subset of the columns of I_{s-1}. If $\alpha_k = s$ in Case 2 or 3, then $\beta \not\subseteq \alpha$ means that one of the nonzero entries $1, \dots, a$ of $\mathbf{z}$ is the only nonzero entry in its row of U_α, allowing row reduction to a subset of the columns of I_{s-1}. The remaining possibilities are the randomly chosen vectors: either $\alpha_k > s$ or $\alpha_k = s = a$. The first $k-1$ columns of U_α span a subspace of dimension $k-1 < s-1$ in which a randomly chosen unit vector will not lie, giving with probability 1 an independent set of columns for U_α.

For Claim (iii), $k = s$ and $\beta \not\subseteq \alpha$ imply that the last column of T_α must come from one of the randomly chosen columns of U. This corresponds in T to a set in $\mathbb{C}^s$ of real dimension $2s-3$, always with last entry equal to 1. Allowing arbitrary complex scaling, which does not affect independence, expands this to a set of real dimension $2s-1$ in $\mathbb{C}^s$, which in particular is higher than the real dimension $2s-2$ of the complex span of the first $s-1$ columns of T_α. It follows that with probability

1 the columns of T_α are independent.

Suppose now that α is a set for which $B[\alpha]$ has not already been shown to be singular. In Case 1 this corresponds to $1 < k \le s$, and in Cases 2 and 3 this corresponds to $1 < k \le s$ and $\beta \not\subseteq \alpha$. For $1 < k < s$, Claims (ii) and (i) establish with probability 1 that $C[\alpha]$ is positive definite, with minimum eigenvalue $\epsilon > 0$, and not equal to J_k. By (5.1), $B[\alpha] = C[\alpha] - J_k$ is thus a nonzero Hermitian matrix with zeros on the diagonal, implying that it must have at least one strictly negative eigenvalue. But $B[\alpha]$ is a rank-one perturbation of $C[\alpha]$, with at most one eigenvalue lower than ϵ. It follows that $B[\alpha]$ is nonsingular. For $k = s$, Claim (iii) establishes that T_α is an invertible $s \times s$ matrix, and so $B[\alpha] = T_\alpha^* D T_\alpha$ is also nonsingular.

Since β, when defined, satisfies $|\beta| = a + 1$, $k \le a$ implies either that β is not defined or that $\beta \not\subseteq \alpha$, which are now sufficient to deduce Conclusion (B). In Case 1, Conclusion (C) is trivial, and in Cases 2 and 3 we have $n > s$ and in particular that the index $s + 1$ exists. The nonsingularity of $B[\alpha]$ for the sets $\alpha = [k-1] \cup \{s+1\}$ in the range $a < k \le s$ is now sufficient to give $\ell_k \ne \mathtt{N}$ for $a < k \le n$, giving the other half of Conclusion (C) and completing the proof. □

6. Epr-sequences over $\mathbb{H}_n \setminus \mathbb{R}_n$ for $n \le 5$. Sequences attainable over $\mathbb{R}_n$ are listed in [4, Tables 2 - 5]. Note that attain($\mathbb{H}_1$) = attain($\mathbb{R}_1$) and attain($\mathbb{H}_2$) = attain($\mathbb{R}_2$). The only order 3 epr-sequences that are not attainable over the real numbers are `NAN`, `NNA`, `NSA`, and `SNA`. Since `NNA`, `NSA`, and `SNA` are prohibited by the `NN` Theorem, the `NSA` Theorem, and Proposition 1.6, `NAN` is the only epr-sequence of order 3 that could be attained by a Hermitian matrix but not by a real symmetric matrix. Furthermore, `NAN` is in fact attained by a tournament (see Theorem 4.3).

We list all attainable sequences over $\mathbb{H}_n$ that are not attainable over $\mathbb{R}_n$ for $n = 4$ and 5 in Tables 6.1 and 6.2 below. By the Inverse Theorem, the attainability of $\ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$ implies the attainability of $\ell_{n-1}\ell_{n-2}\cdots\ell_1\mathtt{A}$, and vice versa; thus, for the sake of brevity, we say that $\ell_{n-1}\ell_{n-2}\cdots\ell_1\mathtt{A}$ is the "inverse of $\ell_1\ell_2\cdots\ell_{n-1}\mathtt{A}$." Again, for brevity, when the attainability of a sequence is established with a realization that is a tournament, we simply say "tournament," instead of providing a matrix. Hermitian adjacency matrices (that are not tournaments) are also identified in the table. If no realization is provided for a sequence, then a result is cited.

To complete the classification, we need some more matrix examples.

Example 6.1. Matrices for Tables 6.1 and 6.2.

$$M_{\mathtt{AANSN}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & i & i \\ 1 & -1 & 2 & -i+1 & 0 \\ 1 & -i & i+1 & 2 & -i+1 \\ 1 & -i & 0 & i+1 & 2 \end{bmatrix},\ M_{\mathtt{NASAA}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & i \\ 1 & 1 & 1 & -i & 0 \end{bmatrix},$$

$$M_{\texttt{NASAN}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & i & 1 & 1 \\ 1 & -i & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & i \\ 1 & 1 & 1 & -i & 0 \end{bmatrix},\ M_{\texttt{NASSA}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & i & 1 & 1-i \\ 1 & -i & 0 & 1-i & -1 \\ 1 & 1 & 1+i & 0 & 2+i \\ 1 & 1+i & -1 & 2-i & 0 \end{bmatrix},$$

$$M_{\texttt{NSNAN}} = \begin{bmatrix} 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -i & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, M_{\texttt{SANSN}} = \begin{bmatrix} -1 & -i+1 & i & -2i-1 & i+1 \\ i+1 & 0 & -i-1 & i+1 & -2i+2 \\ -i & i-1 & 1 & i & i+1 \\ 2i-1 & -i+1 & -i & -1 & i+1 \\ -i+1 & 2i+2 & -i+1 & -i+1 & 0 \end{bmatrix}.$$

For $n = 4$, there are 54 epr-sequences that end in `N` or `A`. Thirty-nine of these sequences are attained by matrices in $\mathbb{H}_n$, because 5 are listed in Table 6.1 and 34 are attainable over the reals [4, Table 4]. Of the remaining 15 sequences, 7 are not attainable by the `NN` Theorem and 5 more are forbidden by the `NSA` Theorem. The remaining 3 sequences are forbidden by Proposition 1.6 or Proposition 2.1. For each unattainable sequence, the specific reason that it is forbidden is listed in [3], and similarly for order 5.

Table 6.1

All epr-sequences of order 4 that can be attained by Hermitian matrices but not by real symmetric matrices.

epr-sequence	Hermitian matrix	Result
`NANA`	tournament	Theorem 4.3
`NANN`		Remark 3.5
`NASA`	$M_{\texttt{NASAA}}(\{1\})$ (Hermitian adjacency matrix)	Example 6.1
`NASN`	$M_{\texttt{NASNN}}(\{5\})$	Theorem 5.1
`SANA`		inverse of `NASA`

For $n = 5$, there are 162 epr-sequences ending in `A` or `N`. Of these 162 sequences, we discard the 33 sequences containing the prohibited subsequences `NNA` and `NNS` (`NN` Theorem), leaving 129 potentially attainable sequences. Of the 129 sequences remaining, 16 contain `NSA`, which we may also discard (`NSA` Theorem); that leaves 113 sequences. Among these 113 sequences, 5 are of the form $\cdots$`ASN`$\cdots$`A`, which are forbidden (`NSA` Theorem); that leaves 108 sequences. Discarding the 6 sequences having one of the prohibited *initial* subsequences `ANAN, ANAS` and `SANA` leaves 102 sequences (see Propositions 2.1 and 2.3). The epr-sequences `AANAN, SSNAN, NANAA` and `NSSNA` are each forbidden (see Corollary 2.2 and Propositions 3.7 and 2.4), and, thus, are discarded, leaving 98 sequences. Among the remaining 98 sequences, 8 have the unattainable form `SN`$\cdots$`A`$\cdots$ (see Proposition 1.6), which are also discarded; that leaves 90 sequences, which we claim are all attainable. Of these 90 sequences, 75 are the sequences attainable by real symmetric matrices (see [4, Table 5]). The remaining 15 sequences appear in Table 6.2, which are those attainable by Hermitian matrices but not by real symmetric matrices.

Table 6.2
All epr-sequences of order 5 that can be attained by Hermitian matrices but not by real symmetric matrices.

epr-sequence	Hermitian matrix	Result
`AANSN`	$M_{\texttt{AANSN}}$	Example 6.1
`ANAAN`		Theorem 3.3
`ASANA`		inverse of `NASAA`
`NAANA`		Theorem 3.3
`NANAN`	tournament	Theorem 4.3
`NANNN`		Remark 3.5
`NANSN`		Remark 3.5
`NASAA`	$M_{\texttt{NASAA}}$ (Hermitian adjacency matrix)	Example 6.1
`NASAN`	$M_{\texttt{NASAN}}$ (Hermitian adjacency matrix)	Example 6.1
`NASNN`		Theorem 5.1
`NASSA`	$M_{\texttt{NASSA}}$	Example 6.1
`NASSN`		Theorem 5.1
`NSNAN`	$M_{\texttt{NSNAN}}$ (Hermitian adjacency matrix)	Example 6.1
`SANSN`	$M_{\texttt{SANSN}}$	Example 6.1
`SSANA`		inverse of `NASSA`

A natural question now arises: Are all the sequences starting with `N` in the tables above attainable by a Hermitian adjacency matrix? Observe that each sequence (starting with `N`) whose attainability was not established with a Hermitian adjacency matrix, starts with `NA` and does not have `A` in the 4th position. For a Hermitian adjacency matrix, this pattern is not allowed by Proposition 4.2, implying that any sequence starting with `N` listed in Table 6.1 or 6.2 is attainable by a Hermitian adjacency matrix if and only if the realization provided in these tables is a Hermitian adjacency matrix.

We conclude by noting that, for $n = 2, 3, 4, 5$, the set of epr-sequences attainable by an $n \times n$ Hermitian adjacency matrix but not by a (0,1) adjacency matrix consists of `NAAAN`, `NSSAN` (see Remark 4.5), `NAN`, and each sequence in Tables 6.1 and 6.2 whose corresponding realization is a Hermitian adjacency matrix.

7. Relationships for attainable epr-sequences. Here we summarize the relationships regarding attainability of epr sequences of the various classes of matrices that we consider. In addition to the notation $\mathbb{R}_n, \mathbb{C}_n$ and $\mathbb{H}_n$ already defined, we denote the $n \times n$ $(0,1)$ graph adjacency matrices by $\mathbb{G}_n$, and the $n \times n$ Hermitian adjacency matrices of mixed graphs by $\mathbb{D}_n$.

Clearly $\text{attain}(\mathbb{R}_n) \subseteq \text{attain}(\mathbb{C}_n)$, $\text{attain}(\mathbb{R}_n) \subseteq \text{attain}(\mathbb{H}_n)$, $\text{attain}(\mathbb{G}_n) \subseteq \text{attain}(\mathbb{D}_n)$, $\text{attain}(\mathbb{G}_n) \subseteq \text{attain}(\mathbb{R}_n)$, and $\text{attain}(\mathbb{D}_n) \subseteq \text{attain}(\mathbb{H}_n)$. All five classes $\text{attain}(\mathbb{R}_n), \text{attain}(\mathbb{C}_n), \text{attain}(\mathbb{H}_n), \text{attain}(\mathbb{G}_n)$, and $\text{attain}(\mathbb{D}_n)$ are distinct (examples are cited below).

The epr-sequence `NAN` shows $\text{attain}(\mathbb{H}_n) \not\subseteq \text{attain}(\mathbb{C}_n)$ [4, Proposition 2.8 and Example 2.9]. For $\text{attain}(\mathbb{C}_n) \not\subseteq \text{attain}(\mathbb{R}_n)$ see [2, Example 6.8] (when containment fails for pr-sequences it necessarily also fails for epr-sequences). An obvious open question is the epr-version of a question raised in [1, p. 235].

Question 7.1. Is $\text{attain}(\mathbb{C}_n) \subset \text{attain}(\mathbb{H}_n)$?

For $(0,1)$ graph adjacency matrices, Hermitian mixed graph adjacency matrices, real symmetric matrices, and (complex) Hermitian matrices, the relationships among attainable epr-sequences are known, and in the next table we summarize these relationships. If there is an example of an epr-sequence attainable in one class and not in another, an example is given; otherwise, a dash – denotes an impossible combination. There are many possible examples, but we have selected small and/or meaningful ones (e.g., for a sequence not attainable by the adjacency matrix of a graph or mixed graph, we have selected an example beginning with `N`).

Table 7.1

Attainability of epr-sequences by various classes of matrices

	$\in \text{attain}(\mathbb{G}_n)$	$\in \text{attain}(\mathbb{D}_n)$	$\in \text{attain}(\mathbb{R}_n)$	$\in \text{attain}(\mathbb{H}_n)$
$\notin \text{attain}(\mathbb{G}_n)$	–	`NAN`	`NAAN`	`NAAN`
$\notin \text{attain}(\mathbb{D}_n)$	–	–	`NAAN`	`NAAN`
$\notin \text{attain}(\mathbb{R}_n)$	–	`NAN`	–	`NAN`
$\notin \text{attain}(\mathbb{H}_n)$	–	–	–	–

REFERENCES

[1] W. Barrett, S. Butler, M. Catral, S. M. Fallat, H.T. Hall, L. Hogben, P. van den Driessche, M. Young. The principal rank characteristic sequence over various fields. *Linear Algebra and its Applications*, 459: 222-236, 2014.

[2] R. A. Brualdi, L. Deaett, D. D. Olesky, P. van den Driessche. The principal rank characteristic sequence of a real symmetric matrix. *Linear Algebra and its Applications*, 436: 2137-2155, 2012.

[3] S. Butler, M. Catral, H. T. Hall, L. Hogben, X. Martínez-Rivera, B. Shader, P. van den Driessche. Supporting documentation for Hermitian epr-sequences, `http://orion.math.iastate.edu/lhogben/research/eprlistH.pdf`.

[4] S. Butler, M. Catral, S. M. Fallat, H. T. Hall, L. Hogben, P. van den Driessche, M. Young. The

enhanced principal rank characteristic sequence. *Linear Algebra and its Applications* (2015). dx.doi.org/10.1016/j.laa.2015.03.023

[5] S. M. Fallat, D. D. Olesky, P. van den Driessche. The enhanced principal rank characteristic sequence for skew-symmetric matrices. *Linear Algebra and its Applications* (2015). dx.doi.org/10.1016/j.laa.2015.08.014

[6] K. Guo, B. Mohar. Hermitian adjacency matrix of digraphs and mixed graphs. Under review. http://arxiv.org/abs/1505.01321

[7] O. Holtz and H. Schneider. Open problems on GKK $\tau-$matrices. *Linear Algebra and its Applications*, 345: 263-267, 2002.

[8] J. Liu, X. Li. Hermitian-adjacency matrices and Hermitian energies of mixed graphs. *Linear Algebra and its Applications*, 466: 182-207, 2015.

[9] X. Martínez-Rivera. Classification of families of pr- and epr-sequences. Under review.

[10] C. A. McCarthy, A.T. Benjamin. Determinants of the tournaments. *Mathematics Magazine*, 69 (2): 133-135, 1996.