

# RATIONAL REALIZATION OF MAXIMUM EIGENVALUE MULTIPLICITY OF SYMMETRIC TREE SIGN PATTERNS

ATOSHI CHOWDHURY\*, LESLIE HOGBEN†, JUDE MELANCON‡, AND RANA MIKKELSON§

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**Abstract.** A sign pattern is a matrix whose entries are elements of  $\{+, -, 0\}$ ; it describes the set of real matrices whose entries have the signs in the pattern. A graph (that allows loops but not multiple edges) describes the set of symmetric matrices having a zero-nonzero pattern of entries determined by the absence or presence of edges in the graph. DeAlba et al. [3] gave algorithms for the computation of maximum multiplicity and minimum rank of matrices associated with a tree sign pattern or tree, and an algorithm to obtain an integer matrix realizing minimum rank. We extend these results by giving algorithms to obtain a symmetric rational matrix realizing the maximum multiplicity of a rational eigenvalue among symmetric matrices associated with a symmetric tree sign pattern or tree.

**Key words.** Sign pattern matrix, symmetric tree sign pattern, maximum multiplicity, rational realization, tree, graph.

**AMS subject classifications.** 05C50, 15A18, 05C05, 05C85

**1. Introduction.** Recently there has been interest in minimum rank and the related question of the maximum multiplicity of an eigenvalue of the family of symmetric matrices associated with a symmetric sign pattern, and also in realizing the extreme values with rational matrices [1]. DeAlba, Hardy, Hentzel, Hogben, and Wangsness [3] gave algorithms for the computation of minimum rank and maximum multiplicity of an eigenvalue for tree sign patterns and for trees (that allow loops and that restrict the zero-nonzero pattern of the diagonal by the absence or presence of loops). An algorithm was also given for the construction of an integer matrix realizing minimum rank (equivalent to realizing maximum multiplicity for eigenvalue 0). In this note we provide algorithms for the construction of a symmetric rational matrix realizing the maximum multiplicity of a nonzero rational eigenvalue of a symmetric tree sign pattern or tree.

For background information on sign patterns, see Brualdi and Shader [2]. For a survey of recent work, see Hall and Li [4]. We use the definitions and terminology from [3], noting here briefly a few usages that are relatively specialized.

Let  $\mathcal{N} = \{1, 2, \dots, n\}$ . All matrices discussed here are real, and with a matrix  $A$  we associate an index set  $\iota(A) \subseteq \mathcal{N}$  by which the entries are indexed, i.e.,  $A = [a_{ij}]$  with  $i, j \in \iota(A)$ . If  $A$  is a matrix and  $R \subseteq \iota(A)$ ,  $A[R]$  denotes the submatrix of  $A$  lying in rows and columns indexed by  $R$ , together with the index set  $R$ , and  $A(R) = A[\overline{R}]$  with  $\iota(A(R)) = \overline{R}$ , where  $\overline{R} = \iota(A) - R$ . The analogous notation  $\iota(Z)$ ,  $Z[R]$ ,  $Z(R)$  is used with sign patterns. If  $S$  is a set of matrices all having the same index set,  $\iota(S)$ , then we apply this same terminology to  $S$ : For  $R \subseteq \iota(S)$ ,  $S[R] = \{A[R] : A \in S\}$ ,  $S(R) = \{A(R) : A \in S\}$ .

A graph,  $G = (V(G), E(G))$ , allows loops but not multiple edges; a simple graph allows neither. If  $G$  is a graph, the *simple graph associated with  $G$* ,  $\widehat{G}$ , is obtained from  $G$  by suppressing all loops; we also use  $\widehat{G}$  to denote a simple graph. The *distance* between two vertices in a graph  $G$  is the number of edges in a shortest path between them. The *diameter* of  $G$  is the maximum distance between any two vertices of  $G$ . A *tree* is a graph  $T$  such that  $\widehat{T}$  is connected and acyclic.

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\*Department of Mathematics, Princeton University, Princeton, NJ 08544, USA (atoshic@princeton.edu), research supported by NSF grant DMS 0353880.

†Corresponding author, Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu), research supported by NSF grant DMS 0353880.

‡Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA (jude@math.lsu.edu), research supported by NSF grant DMS 0353880.

§Department of Mathematics, Iowa State University, Ames, IA 50011, USA (ranam@iastate.edu).

In this paper, we use the word *star* to refer to any simple connected graph with at least two vertices and at most one vertex of degree greater than 1; that is, a star has the form  $\widehat{K}_{1,n}$  where  $n \geq 1$ . For  $n \geq 2$ , the star  $\widehat{K}_{1,n}$  has a unique vertex of degree greater than 1, which we term its *center*. (Either of the two vertices of  $\widehat{K}_{1,1}$  may be considered its center.) A *double star* is a graph with exactly two vertices, called the *centers*, of degree greater than 1; it can be thought of as the result of joining the centers of two stars by an edge. For  $r, s \geq 1$ ,  $\widehat{DS}_{r,s}$  denotes the double star whose centers have degree  $r+1$  and  $s+1$ ; that is, one of its centers is adjacent to  $r$  leaves and the other to  $s$ .

If  $Z$  is a symmetric sign pattern, the *simple graph of  $Z$* , denoted  $\widehat{G}(Z)$ , is the simple graph with vertices  $\iota(Z)$  such that  $ij$  is an edge of  $\widehat{G}(Z)$  if and only if  $i \neq j$  and  $Z_{ij} \neq 0$ . A symmetric sign pattern  $Z$  is a *symmetric tree sign pattern* if  $\widehat{G}(Z)$  is a tree.

Most of our discussion involves symmetric sign patterns and symmetric matrices. In particular, this is true whenever we discuss the multiplicity of an eigenvalue (as opposed to the existence of an eigenvalue), so it is not necessary to distinguish algebraic and geometric eigenvalue multiplicity. However, in Section 3, we construct rational matrices having a prescribed eigenvalue, and in that section sign patterns and matrices are sometimes not required to be symmetric.

Let  $Z$  be a sign pattern and  $G$  a graph.

- $\mathcal{S}(G) = \{A : A = A^T, \iota(A) = V(G), \text{ and } a_{ij} \neq 0 \text{ if and only if } ij \text{ is an edge of } G\}$
- If  $Z$  is symmetric,  $\mathcal{S}(Z) = \{A : A = A^T, \iota(A) = \iota(Z), \text{ and } \text{sgn}(a_{ij}) = z_{ij}\}$
- $\mathcal{Q}(Z) = \{A : \iota(A) = \iota(Z) \text{ and } \text{sgn}(a_{ij}) = z_{ij}\}$

Note that  $\mathcal{S}(Z)$  and  $\mathcal{Q}(Z)$ , in contrast to  $\widehat{G}(Z)$ , depend on the diagonal entries of  $Z$ .

The multiplicity of a real number  $\lambda$  as an eigenvalue of the symmetric matrix  $A$  will be denoted by  $m_A(\lambda)$ . (Of course, if  $\lambda$  is not an eigenvalue of  $A$ , then  $m_A(\lambda)=0$ .) We define the *maximum multiplicity* of a symmetric sign pattern or graph:

- $M_\lambda(Z) = \max\{m_A(\lambda) : A \in \mathcal{S}(Z)\}$ .
- $M_\lambda(G) = \max\{m_A(\lambda) : A \in \mathcal{S}(G)\}$ .

Multiplication by a positive number does not change the signs of the entries of a matrix. Thus if  $A \in \mathcal{S}(Z)$  has eigenvalue  $\lambda$  with multiplicity  $m_A(\lambda)$  and  $r > 0$ , then  $rA \in \mathcal{S}(Z)$  and  $rA$  has eigenvalue  $r\lambda$  with  $m_{rA}(r\lambda) = m_A(\lambda)$ . This is also true for eigenvalues of matrices associated with graphs with the restriction that  $r > 0$  replaced by  $r \neq 0$ .

A set  $S$  of symmetric matrices *allows* eigenvalue  $\lambda$  if there is a matrix  $A \in S$  such that  $\lambda$  is an eigenvalue of  $A$ . The method used in [3] to find a matrix that attains the maximum multiplicity for eigenvalue  $\lambda$  cuts the graph (the tree or the graph of the symmetric tree sign pattern) into pieces, such that the set of symmetric matrices associated with each piece allows eigenvalue  $\lambda$ . The following lemma plays an important role; it is an immediate consequence of [3, Lemma 1.8].

LEMMA 1.1. *Let  $G$  be a graph and let  $Z$  be a sign pattern.*

1.  $\mathcal{S}(G)$  allows a nonzero eigenvalue if and only if  $G$  has an edge (note that a loop is an edge).
2.  $\mathcal{S}(Z)$  allows a positive (respectively, negative) eigenvalue if and only if  $Z$  has non-zero off-diagonal entry or  $Z$  has a positive (respectively, negative) diagonal entry.

In Section 3 we show that when the eigenvalue  $\lambda$  is rational, any tree or symmetric tree sign pattern that allows  $\lambda$  has a rational matrix having eigenvalue  $\lambda$ . We give an explicit construction for such a matrix; this allows the maximum multiplicity for eigenvalue  $\lambda$  to be realized by a rational matrix.

When attempting to construct a rational matrix realizing maximum multiplicity, it is natural to restrict our attention to rational eigenvalues. For example, if  $\widehat{K}_2$  is the simple complete graph on two vertices, then any symmetric matrix  $A \in \mathcal{S}(\widehat{K}_2)$  is of the form  $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ , and has eigenvalues  $\pm a$ ; thus an irrational eigenvalue cannot be realized by a rational matrix for  $\widehat{K}_2$ .

**2. Finding a symmetric rational matrix realizing maximum multiplicity.** In this section we show how to obtain a rational matrix realizing the maximum multiplicity of a rational eigenvalue for a symmetric tree sign pattern or a tree. This algorithm can be applied to a forest or forest sign pattern by executing it on each component separately.

ALGORITHM 2.1. *Let  $Z$  be a symmetric tree sign pattern (respectively, let  $T$  be a tree). Let  $\widehat{T} = \widehat{\mathcal{G}}(Z)$  (respectively, let  $\widehat{T}$  be the simple graph associated with  $T$ ),  $S = \mathcal{S}(Z)$  (respectively,  $S = \mathcal{S}(T)$ ), and let  $\lambda$  be a nonzero rational number. Construct a rational matrix  $A \in S$  having  $m_A(\lambda) = M_\lambda(Z)$  (respectively,  $m_A(\lambda) = M_\lambda(T)$ ) as follows:*

1. *In the case of a sign pattern  $Z$  and  $\lambda < 0$ , replace  $Z$  by  $-Z$ .*
2. *Apply Algorithm 2.2 below to find the subset  $Q$  of vertices to be deleted. Let the vertices of the components of  $\widehat{T} - Q$  be denoted by  $R_i, i = 1, \dots, h$ .*
3. *For each  $i = 1, \dots, h$ : If  $|R_i| > 1$ , apply one of Algorithms 3.1, 3.4, 3.9, 3.11 or Lemma 3.17 below to construct a rational matrix  $A_i \in S[R_i]$  having eigenvalue 1.*
4. *Construct a matrix  $A$  such that  $A[R_i] = A_i$  and  $A \in S$ , using 0, 1, or  $-1$  for any as yet unspecified entry.*
5. *Multiply  $A$  by  $\lambda$ .*

Note that by performing step 1, we ensure the correct sign pattern in the final matrix.

In [3] it was shown that an algorithm similar to this, without the restriction that the matrices in step 3 be rational, will produce a symmetric matrix having the given tree sign pattern or tree that realizes maximum multiplicity of eigenvalue  $\lambda$ . However, no algorithms were given for the construction of the rational matrices required in step 3. In Section 3 we provide these algorithms and thus establish that for a nonzero rational eigenvalue, maximum multiplicity can be realized by a rational matrix.

In the interest of completeness, we include a specialized version of Algorithm 2.4 from [3] as the next algorithm. First we need some notation. For a simple graph  $\widehat{G}$  and  $H \subseteq V(\widehat{G})$ , a vertex  $v$  is an  $H$ -vertex if  $v \in H$ , and a component of  $\widehat{G}$  is  $H$ -free if it does not contain any  $H$ -vertex. For a tree  $T$  and a set  $Q \subseteq V(T)$ , define  $c_1(Q)$  to be the number of components of  $T - Q$  that allow eigenvalue 1. By Lemma 1.1, this is the number of components such that the component has at least two vertices or is an isolated vertex with a loop. For a symmetric tree sign pattern  $Z$  and a set  $Q \subseteq \iota(Z)$ , define  $c_1(Q)$  to be the number of components of  $\widehat{\mathcal{G}}(Z(Q))$  such that the associated principal subpattern allows eigenvalue 1. By Lemma 1.1, this is the number of components such that the component has at least two vertices or is an isolated vertex  $v$  and  $z_{vv} = +$ . In [3] it was shown that  $M_1(Z) = c_1(Q) - |Q|$  (or,  $M_1(T) = c_1(Q) - |Q|$ ), for the set  $Q$  of vertices produced by the next algorithm.

ALGORITHM 2.2. [3] *Let  $Z$  be a symmetric tree sign pattern and  $T$  a tree. Let  $\widehat{T} = \widehat{\mathcal{G}}(Z)$  (or let  $\widehat{T}$  be the simple graph associated with  $T$ ),  $S = \mathcal{S}(Z)$  (or  $\mathcal{S}(T)$ ). Construct a set  $Q$  of indices (or vertices) as follows:*

*Initialize:  $Q = \emptyset, i = 1$ , and  $H$  is the set of all vertices of degree 3 or more in  $\widehat{T}$ .*

*While  $H \neq \emptyset$ :*

1. *Set  $\widehat{T}_i =$  the unique component of  $\widehat{T} - Q$  that contains an  $H$ -vertex.*
2. *Set  $Q_i = \emptyset$ .*
3. *Set  $W_i = \{w \in H : \text{all but possibly one component of } \widehat{T}_i - w \text{ is } H\text{-free}\}$ .*
4. *For each vertex  $w \in W_i$ :  
If  $\widehat{T}_i - w$  has at least two  $H$ -free components  $\langle R_j \rangle$  such that  $S[R_j]$  allows eigenvalue 1, then  $Q_i = Q_i \cup \{w\}$ .*
5.  *$Q = Q \cup Q_i$ .*
6. *Remove all the vertices of  $W_i$  from  $H$ .*

7. For each  $v \in H$ :  
If  $\deg_{\widehat{T}-Q} v \leq 2$ , then remove  $v$  from  $H$ .
8.  $i = i + 1$ .

**3. Construction of a matrix with a prescribed eigenvalue.** In this section, we give algorithms for construction of rational matrices in  $\mathcal{S}(Z)$  or  $\mathcal{S}(T)$  having eigenvalue 1 for any symmetric tree sign pattern  $Z$  or tree  $T$  that allows eigenvalue 1. In fact, these algorithms construct a rational matrix having eigenvalue 1 for any graph with a loop or symmetric sign pattern with a nonzero diagonal entry that allows eigenvalue 1. As in Algorithm 2.1, multiplication of the resulting matrix by a positive number  $\lambda$  yields a matrix in  $\mathcal{S}(Z)$  having eigenvalue  $\lambda$ . To obtain a negative eigenvalue,  $Z$  should be replaced by  $-Z$  before finding the matrix having eigenvalue 1.

Lemma 1.1 requires that we be able to produce a matrix having eigenvalue 1 for any symmetric tree sign pattern having an off-diagonal entry. Note that the assertion that there exists a rational matrix having eigenvalue 1 for a tree follows from the result for symmetric tree sign patterns; however, we give separate algorithms for trees, since they are much simpler.

The construction of a matrix having a prescribed eigenvalue depends not only on whether we have a tree or tree sign pattern, but also on the signs of the diagonal entries. The cases, discussed in subsections below, are: trees with at least one loop, simple trees and symmetric tree sign patterns with zero diagonal, symmetric tree sign patterns with at least one positive diagonal entry, and symmetric tree sign patterns with at least one negative diagonal entry and no positive diagonal entries.

**3.1. Graphs with loops.** Given a graph  $G$  with at least one loop, we can use the next algorithm to construct a matrix  $A \in \mathcal{S}(G)$  with rational entries and eigenvalue 1. We may renumber the vertices if necessary so that  $G$  has a loop at vertex 1. Note that the algorithm discussed in this subsection does not require the graph to be a tree.

ALGORITHM 3.1. *Let  $G$  be a graph having vertex set  $\mathcal{N}$  and a loop at vertex 1.*

1. Define the  $n \times n$  matrix  $M = [m_{ij}]$  by
  - $m_{ij} = 2$  if  $i \neq j$  and  $ij$  is an edge of  $G$ ;
  - $m_{ij} = 0$  if  $i \neq j$  and  $ij$  is not an edge of  $G$ ;
  - $m_{ii} = 1$  if  $i \neq 1$  and  $ii$  is a loop of  $G$ ;
  - $m_{ii} = -1$  if  $i \neq 1$  and  $ii$  is not a loop of  $G$ ;
  - $m_{11} = x$ .
2. Compute  $\det M = c_1x + c_0$ .
3. Set  $x = -c_0/c_1$ .
4. Set  $A = M + I$ .

LEMMA 3.2. *Algorithm 3.1 produces a rational matrix  $A \in \mathcal{S}(G)$  that has eigenvalue 1 for any graph  $G$  that has a loop.*

*Proof.*

$$\begin{aligned} \det(M) &= \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n m_{i,\pi(i)} \\ &= x \underbrace{\left( \pm 1 + \sum_{\substack{\pi \neq \mathbf{1} \\ \pi(1)=1}} \operatorname{sgn}(\pi) \prod_{i=2}^n m_{i,\pi(i)} \right)}_{c_1} + \underbrace{\sum_{\pi(1) \neq 1} \operatorname{sgn}(\pi) \prod_{i=1}^n m_{i,\pi(i)}}_{c_0}. \end{aligned}$$

Every product  $\prod m_{i,\pi(i)}$  that appears in one of the two sums above contains an off-diagonal entry of  $M$ , because  $\pi$  is not the identity mapping  $\mathbf{1}$ . Every off-diagonal entry of  $M$  is divisible

by 2, so  $c_1 \equiv 1 \pmod{2}$  and  $c_0 \equiv 0 \pmod{2}$ . Thus  $x$  can be defined,  $M$  is singular, and 1 is an eigenvalue of  $A = M + I$ . Since  $x \neq -1$ , it is clear that  $A \in \mathcal{S}(G)$ .  $\square$

**3.2. Simple trees and symmetric tree sign patterns with zero diagonal.** To construct a matrix  $A \in \mathcal{S}(\widehat{T})$  where  $\widehat{T}$  has no loops, we first observe the following lemma.

LEMMA 3.3. *Any simple tree  $\widehat{T}$  satisfies one of the following:*

1.  $\widehat{T}$  is an isolated vertex,
2.  $\widehat{T}$  is a star,
3.  $\widehat{T}$  is a double star,
4.  $\widehat{T}$  has a vertex whose deletion leaves at least two components having an edge.

*Proof.* Assume that we have a simple tree  $\widehat{T}$  which does not satisfy case 4. Then  $\widehat{T}$  does not have a path of length 4 (because if it did, we could delete the middle vertex leaving at least two components having an edge). So  $\widehat{T}$  has diameter at most 3. If  $\widehat{T}$  has diameter 0, it is an isolated vertex. If  $\widehat{T}$  has diameter 1 or 2, then it is a star. If  $\widehat{T}$  has diameter 3, it is a double star.  $\square$

We apply the following algorithm to a simple tree (or simple forest) to cut the graph up into stars, double stars, and isolated vertices. Assuming step 3 can be accomplished, Algorithm 3.4 will produce a rational matrix in  $\mathcal{S}(T)$  having eigenvalue 1, by the Interlacing Theorem [5, p. 185].

ALGORITHM 3.4. *Let  $\widehat{T}$  be a simple forest that has at least one edge. To construct a nonnegative rational matrix  $A \in \mathcal{S}(\widehat{T})$  having eigenvalue 1:*

1. Initialize  $Q = \emptyset$ .
2. While possible, choose a component  $\langle R \rangle$  of  $\widehat{T} - Q$  that is not a star, a double star, or an isolated vertex:
  - (a) Choose a vertex  $v \in R$  such that  $\langle R - v \rangle$  has at least two components having an edge.
  - (b)  $Q = Q \cup \{v\}$ .
3. For each component  $\langle R_i \rangle$  of  $\widehat{T} - Q$  that is a star or double star, use Lemma 3.7 or 3.8 to construct a matrix  $A_i$  having eigenvalue 1.
4. Construct a matrix  $A$  such that  $A[R_i] = A_i$  and  $A \in \mathcal{S}(\widehat{T})$ , using 0 or 1, as required by  $\widehat{T}$ , for any as yet unspecified entry.

In order to show explicitly how to construct a matrix for star and double star components, we provide an algorithm for finding a *Pythagorean  $(n+1)$ -tuple*, i.e., an  $(n+1)$ -tuple of positive integers satisfying  $\sum_{i=1}^n x_i^2 = x_{n+1}^2$ , which is required to construct the matrices; in practice, Pythagorean  $(n+1)$ -tuples can also be constructed quite easily (with smaller integers) by hand. Note that this algorithm overwrites  $x_j$  at stage  $j$ .

ALGORITHM 3.5. *If  $n = 1$ ,  $x_1 = x_2 = 1$ ; otherwise:  
Set  $x_1 = 3$ ,  $x_2 = 4$ ,  $x_3 = 5$  and set  $j = 3$ .*

*While  $j \leq n$ :*

1.  $k_j = \frac{x_j - 1}{2}$ .
2.  $x_j = 2k_j(k_j + 1)$ .
3.  $x_{j+1} = k_j^2 + (k_j + 1)^2$ .
4.  $j = j + 1$ .

LEMMA 3.6. *Algorithm 3.5 produces a Pythagorean  $(n+1)$ -tuple.*

*Proof.* Note first that when starting step  $j$ ,  $x_j$  is odd, since it is the sum of two consecutive squares, so  $k_j$  is an integer. For  $n > 3$ , assume that  $\sum_{i=1}^{n-1} x_i^2 = x_n^2$ , where  $(x_1, \dots, x_n)$  is the  $n$ -tuple produced by repeating the steps of the algorithm  $n - 3$  times (that is, until Step 4 yields  $j = n$ ). In the last iteration, let  $y = 2k_n + 1$ , i.e., the value of  $x_n$  when beginning this iteration. Then  $y^2 + x_n^2 = x_{n+1}^2$ . But  $y^2 = \sum_{i=1}^{n-1} x_i^2$ , so  $\sum_{i=1}^n x_i^2 = x_{n+1}^2$ , as desired.  $\square$

The next lemma gives a method for explicitly constructing a rational matrix in  $\mathcal{S}(\widehat{K}_{1,n})$  having eigenvalue 1.

LEMMA 3.7. *Let  $\widehat{K}_{1,n}$  have center 1. For  $n \geq 2$ , a matrix  $A \in \mathcal{S}(\widehat{K}_{1,n})$  with rational entries and eigenvalue 1 is given by*

$$\begin{aligned} a_{1k} = a_{k1} &= \frac{x_{k-1}}{x_{n+1}} \text{ for } k \neq 1, \\ a_{ij} &= 0 \text{ otherwise,} \end{aligned}$$

where  $(x_1, x_2, \dots, x_{n+1})$  is a Pythagorean  $(n+1)$ -tuple.

*Proof.*

$$\det(A - I) = (-1)^{n+1} - (-1)^{n-1} \left( \sum_{i=1}^n \frac{x_i^2}{x_{n+1}^2} \right) = (-1)^{n+1} \left( 1 - \frac{1}{x_{n+1}^2} \sum_{i=1}^n x_i^2 \right) = 0$$

so 1 is an eigenvalue of  $A$ .  $\square$

Next we give a method for explicitly constructing a rational matrix in  $\mathcal{S}(\widehat{DS}_{r,s})$  that has eigenvalue 1.

LEMMA 3.8. *Let  $\widehat{DS}_{r,s}$  have centers 1 and  $r+s+2$ . A matrix  $A \in \mathcal{S}(\widehat{DS}_{r,s})$  with rational entries and eigenvalue 1 is given by*

$$\begin{aligned} a_{1,k+1} = a_{k+1,1} &= \frac{x_k}{2x_{r+1}} \text{ for } k \in \{1, \dots, r\}, \\ a_{1,r+s+2} = a_{r+s+2,1} &= \frac{3}{4}, \\ a_{r+s+2,r+s+2-k} = a_{r+s+2-k,r+s+2} &= \frac{y_k}{2y_{s+1}} \text{ for } k \in \{1, \dots, s\}, \\ a_{ij} &= 0 \text{ otherwise,} \end{aligned}$$

where  $(x_1, x_2, \dots, x_{r+1})$  is a Pythagorean  $(r+1)$ -tuple and  $(y_1, y_2, \dots, y_{s+1})$  is a Pythagorean  $(s+1)$ -tuple.

*Proof.*

$$\begin{aligned} \det(A - I) &= \det((A - I)[\{1, \dots, r+1\}]) \det((A - I)[\{r+2, \dots, r+s+2\}]) - (-1)^{r+s} \left( \frac{3}{4} \right)^2 \\ &= (-1)^{r+s} \left( \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{1}{4} \right) - \left( \frac{3}{4} \right)^2 \right) = 0 \end{aligned}$$

so 1 is an eigenvalue of  $A$ .  $\square$

Any symmetric tree sign pattern is similar by a diagonal  $\pm$  similarity to a symmetric tree sign pattern having all off-diagonal entries equal to  $+$  [3]. Hence signed versions of the matrices in Lemmas 3.7 and 3.8 also have eigenvalue 1, which implies that Algorithm 3.4 (with step 4 now using 0, 1 or  $-1$ ) can be applied to symmetric tree sign patterns with zero diagonal.

**3.3. Sign patterns with at least one positive diagonal entry.** Let  $Z$  be an  $n \times n$  sign pattern (with  $n \geq 2$ ) that has at least one positive diagonal entry. In this section we construct a matrix  $A \in \mathcal{Q}(Z)$  (and if  $Z$  is symmetric,  $A \in \mathcal{S}(Z)$ ) with rational entries and eigenvalue 1. Note that we may assume  $z_{11} = +$  because  $Z$  is permutation similar to a sign pattern with  $+$  in the 1,1-position.

ALGORITHM 3.9. Let  $Z = [z_{ij}]$  be an  $n \times n$  sign pattern indexed by  $\mathcal{N}$ , with  $z_{11} = +$ .

1. Define  $M \in \mathbb{R}^{n \times n}$  by
 
$$\begin{aligned} m_{ij} &= z_{ij} \frac{2^{-n}}{n!} \text{ if } i \neq j; \\ m_{ii} &= -2 \text{ if } z_{ii} = -; \\ m_{ii} &= -1 \text{ if } z_{ii} = 0; \\ m_{ii} &= 1 \text{ if } z_{ii} = + \text{ and } i \neq 1; \\ m_{11} &= x. \end{aligned}$$
2. Compute  $\det M = c_1 x + c_0$ .
3. Set  $x = -c_0/c_1$ .
4. Set  $A = M + I$ .

A related example is given after Algorithm 3.11.

LEMMA 3.10. Algorithm 3.9 produces a rational matrix  $A \in \mathcal{Q}(Z)$  that has eigenvalue 1. If  $Z$  is symmetric, then  $A \in \mathcal{S}(Z)$ .

*Proof.*

$$\det(M) = x \underbrace{\left( \prod_{i=2}^n m_{ii} + \sum_{\substack{\pi \neq \mathbf{1} \\ \pi(1)=1}} \operatorname{sgn}(\pi) \prod_{i=2}^n m_{i,\pi(i)} \right)}_{c_1} + \underbrace{\sum_{\pi(1) \neq 1} \operatorname{sgn}(\pi) \prod_{i=1}^n m_{i,\pi(i)}}_{c_0}.$$

We must show that  $c_1 \neq 0$  and  $a_{11} = x + 1 > 0$ . We do so by finding appropriate bounds on  $c_1$  and  $c_0$ . Since every nonzero off-diagonal entry of  $M$  has magnitude  $\frac{2^{-n}}{n!}$ , while no entry has magnitude greater than 2, we have the following inequality for any  $\pi \neq \mathbf{1}$ , where  $u = 2$  if  $\pi(1) = 1$  and  $u = 1$  otherwise:

$$\left| \prod_{i=u}^n m_{i,\pi(i)} \right| \leq \left( \frac{2^{-n}}{n!} \right)^2 (2^{n-2}) < \frac{1}{4(n!)}.$$

Since there are  $(n-1)! - 1$  permutations  $\pi \neq \mathbf{1}$  satisfying  $\pi(1) = 1$  and  $n! - (n-1)!$  permutations

$\pi$  satisfying  $\pi(1) \neq 1$ ,  $\left| \sum_{\substack{\pi \neq \mathbf{1} \\ \pi(1)=1}} \operatorname{sgn}(\pi) \prod_{i=2}^n m_{i,\pi(i)} \right|$  and  $\left| \sum_{\pi(1) \neq 1} \operatorname{sgn}(\pi) \prod_{i=1}^n m_{i,\pi(i)} \right|$  are both less than  $\frac{1}{4}$ .

Combining these bounds with the fact that  $|\prod_{i=2}^n m_{ii}| \geq 1$ , we find that  $|c_1| \geq \frac{3}{4}$  and  $|c_0| \leq \frac{1}{4}$ ; in particular,  $|c_0| < |c_1|$ , and  $c_1 \neq 0$ . Thus  $x = \frac{-c_0}{c_1} > -1$ , so  $A \in \mathcal{Q}(Z)$  and has 1 as an eigenvalue. If  $Z$  is symmetric, then  $A \in \mathcal{S}(Z)$ .  $\square$

**3.4. Sign patterns with at least one negative and no positive diagonal entries.** If the symmetric sign pattern  $Z$  has at least one negative diagonal entry, no positive diagonal entries, and has a connected graph, we can construct a matrix  $A \in \mathcal{S}(Z)$  with rational entries and eigenvalue 1 using Algorithm 3.11 for  $n \geq 3$ ; the  $n = 2$  case is handled in Lemma 3.17. As in the previous

subsection, we may assume that  $z_{11} = -$ . In addition, since  $\widehat{\mathcal{G}}(Z)$  is assumed to be connected, we may assume that  $z_{12}$  is nonzero.

A computation in the next algorithm uses a submatrix that is not principal. If  $M$  is a square matrix with index set  $\iota(M)$ , and  $R, C \subseteq \iota(M)$ , then  $M(R, C)$  denotes the submatrix obtained from  $M$  by deleting the rows in  $R$  and the columns in  $C$ ; as this submatrix is used in this computation only, we do not formally attach an index set.

ALGORITHM 3.11. Let  $Z = [z_{ij}]$  be a symmetric  $n \times n$  sign pattern ( $n \geq 3$ ) indexed by  $\mathcal{N}$ , with  $z_{11} = -$ , for all  $i = 2, \dots, n$ ,  $z_{ii} \neq +$ , and  $z_{12} \neq 0$ .

1. Define  $M = [m_{ij}] \in \mathbb{R}^{n \times n}$  by
  - $m_{12} = m_{21} = z_{12}2(n-1)$ ;
  - $m_{ij} = z_{ij} \frac{2-n}{n!}$  if  $i \neq j$ ,  $(i, j) \neq (1, 2)$ ,  $(i, j) \neq (2, 1)$ ;
  - $m_{ii} = -2$  if  $z_{ii} = -$  and  $i \neq 1$ ;
  - $m_{ii} = -1$  if  $z_{ii} = 0$ ;
  - $m_{11} = x$ .
2. For  $1 < k, j \leq n$ , set
  - $b_1 = \det(M(\{1\}))$ ,  $b_j = \det(M(\{1, j\}))$  and  $b_{kj} = \det(M(\{1, k\}, \{1, j\}))$ .
3. Set

$$x = \frac{1}{b_1} \left( m_{12}^2 b_2 + \sum_{j=3}^n m_{1j}^2 b_j + \sum_{j=2}^n \sum_{k \neq 1, j} (-1)^{j+k} m_{k1} m_{1j} b_{kj} \right).$$

4. Set  $A = M + I$ .

Before proving that Algorithm 3.11 produces a matrix  $A \in \mathcal{S}(Z)$  that has eigenvalue 1, we illustrate it with an example.

EXAMPLE 3.12. Let  $Z = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 & 0 & 0 \\ + & 0 & + & - & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & - & + & - \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & - \end{bmatrix}$ . The graph (including loops) of

$Z$  is shown in Figure 3.1.

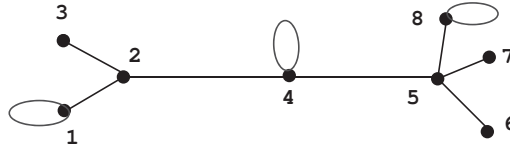


FIG. 3.1. The graph of  $Z$  in Example 3.12

The  $8 \times 8$  matrix  $M$  produced in step 1 has the following entries  $m_{ij}$ :

- $m_{11} = x$ ,  $m_{44} = m_{88} = -2$ , and all other diagonal entries are equal to  $-1$ .
- $m_{12} = m_{21} = 14$ , and all other nonzero off-diagonal entries are equal to  $\pm \frac{1}{10321920}$ .

For steps 2 and 3, only  $b_1 = -\frac{45404818745330534108155084817}{11351204686333112966184960000}$  and  $b_2 = \frac{35514010828799}{8878502707200}$  need be computed, and  $x = m_{12}^2 \frac{b_2}{b_1} = -\frac{8899344474084909978628600627200}{45404818745330534108155084817}$ . If we set  $x$  to this value and set  $A = M + I$ , then  $A \in \mathcal{S}(Z)$  and  $A$  has eigenvalue 1. (These computations were done using *Mathematica*.)

As can be seen from this example, the formulas in the algorithms give rise to exceedingly large integers in exact arithmetic. No attempt was made to optimize the values chosen; the off-diagonal entries are chosen unnecessarily small (i.e., unnecessarily large integer denominator) to facilitate the proofs below. In practice, a little experimentation with a computer algebra system quickly yields an exact solution involving much smaller integers. This comment applies to Algorithm 3.9 also. Furthermore, it is unnecessary to compute  $b_1, b_2, b_j, b_{kj}$  individually; the key idea is to find  $x$  (of the right sign) making  $M$  singular and then set  $A = M + I$ . Using the values given in Algorithm 3.11 for the diagonal entries (except for  $x$ ) and  $m_{12} = m_{21} = 14$  in this example, and setting the remaining off-diagonal entries equal to  $\pm 1$  fails (the value found for  $x$  has the wrong sign), but choosing the off-diagonal entries to be  $\pm \frac{1}{2}$  is successful:

$$\text{Let } M = \begin{bmatrix} x & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -2 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -2 \end{bmatrix}. \text{ Then } \det M = -\frac{9x}{16} - 196, \text{ so if we set}$$

$$x = -\frac{3136}{9} \text{ and } A = M + I = \begin{bmatrix} -\frac{3127}{9} & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -1 \end{bmatrix}, \text{ then } A \in \mathcal{S}(z) \text{ and } A$$

has eigenvalue 1.

For the proof that Algorithm 3.11 produces a matrix  $A \in \mathcal{S}(Z)$  that has eigenvalue 1, we need three lemmas that bound the values of  $b_1, b_j$  and  $b_{kj}$ .

LEMMA 3.13.  $\frac{11}{12} \leq \frac{11}{12} \prod_{i=2}^n |m_{ii}| < |b_1| < \frac{13}{12} \prod_{i=2}^n |m_{ii}|$ , and  $\text{sgn}(b_1) = (-1)^{n-1}$ .

*Proof.*

$$b_1 = \sum_{\pi} \text{sgn}(\pi) \prod_{i=2}^n m_{i,\pi(i)} = \prod_{i=2}^n m_{ii} + \sum_{\pi \neq \mathbf{1}} \text{sgn}(\pi) \prod_{i=2}^n m_{i,\pi(i)},$$

where  $\pi$  ranges over permutations of  $\mathcal{N} \setminus \{1\}$ . When  $\pi \neq \mathbf{1}$ , there are at least two  $i \in \mathcal{N} \setminus \{1\}$  for which  $i \neq \pi(i)$ , and so  $m_{i,\pi(i)}$  has magnitude at most  $\frac{2^{-n}}{n!}$ . All other entries of  $M$  are bounded in magnitude by 2, so

$$\left| \sum_{\pi \neq \mathbf{1}} \prod_{i=2}^n m_{i,\pi(i)} \right| \leq \sum_{\pi \neq \mathbf{1}} \left( \left( \frac{2^{-n}}{n!} \right)^2 2^{n-2} \right) < \frac{(n-1)!}{4(n!)} \leq \frac{1}{12}.$$

For  $i \geq 2$ ,  $|m_{ii}| \geq 1$ , so  $|\prod_{i=2}^n m_{ii}| \geq 1$  and thus  $\frac{11}{12} \leq \frac{11}{12} |\prod_{i=2}^n m_{ii}| < |b_1| < \frac{13}{12} |\prod_{i=2}^n m_{ii}|$  and  $\text{sgn}(b_1) = \text{sgn}(\prod_{i=2}^n m_{ii})$ . Since  $m_{ii} < 0$  for every  $i \in \mathcal{N} \setminus \{1\}$ ,  $\text{sgn}(b_1) = (-1)^{n-1}$ , as desired.  $\square$

LEMMA 3.14. For any  $j \neq 1$ ,  $\frac{11}{12} \leq \frac{11}{12} \prod_{i \neq 1,j} |m_{ii}| < |b_j| < \frac{13}{12} \prod_{i \neq 1,j} |m_{ii}|$  and  $\text{sgn}(b_j) = (-1)^{n-2}$ .

*Proof.* If  $n = 3$ ,  $b_j = m_{kk}$  where  $k \neq 1, j$ , and the result is clear, so assume  $n \geq 4$ . As in the proof of Lemma 3.13,  $b_j = \prod_{i \notin \{1, j\}} m_{ii} + \sum_{\pi \neq \mathbb{1}} \prod_{i \in \mathcal{N} \setminus \{1, j\}} m_{i, \pi(i)}$ , and

$$\left| \sum_{\pi \neq \mathbb{1}} \prod_{i \in \mathcal{N} \setminus \{1, j\}} m_{i, \pi(i)} \right| \leq \sum_{\pi \neq \mathbb{1}} \left( \left( \frac{2^{-n}}{n!} \right)^2 2^{n-4} \right) \leq \frac{(n-2)!}{16(n!)} < \frac{1}{12}.$$

For  $i \in \mathcal{N} \setminus \{1, j\}$ ,  $|m_{ii}| \geq 1$ , so  $\frac{11}{12} \leq \frac{11}{12} \prod_{i \neq 1, j} |m_{ii}| < |b_j| < \frac{13}{12} \prod_{i \neq 1, j} |m_{ii}|$  and  $\text{sgn}(b_j) = \text{sgn} \left( \prod_{i \in \mathcal{N} \setminus \{1, j\}} m_{ii} \right) = (-1)^{n-2}$ , as desired.  $\square$

LEMMA 3.15. *For any  $j, k \geq 2$  such that  $j \neq k$ ,  $|b_{kj}| \leq \frac{1}{48}$ .*

*Proof.* Let  $\pi$  range over the set of bijections from  $\mathcal{N} \setminus \{1, k\}$  to  $\mathcal{N} \setminus \{1, j\}$ , and write

$$|b_{kj}| \leq \sum_{\pi} \left| \prod_{i \in \mathcal{N} \setminus \{1, k\}} m_{i, \pi(i)} \right| \leq \sum_{\pi} \frac{2^{n-3} 2^{-n}}{n!} = \frac{(n-2)!}{8(n!)} \leq \frac{1}{48}$$

(since  $n \geq 3$ ).  $\square$

Now we can prove Algorithm 3.11 succeeds.

LEMMA 3.16. *Algorithm 3.11 produces a rational matrix  $A \in \mathfrak{S}(Z)$  that has eigenvalue 1.*

*Proof.* The determinant of  $M$  can be computed by the Laplace expansion:

$$\begin{aligned} \det M &= x \det M(\{1\}) - \sum_{j=2}^n (-1)^j m_{1j} \det M(\{1\}, \{j\}) \\ &= x \det M(\{1\}) - \sum_{j=2}^n (-1)^j m_{1j} \sum_{k=2}^n (-1)^k m_{k1} \det M(\{1, k\}, \{1, j\}) \\ &= x b_1 - \sum_{j=2}^n m_{1j}^2 b_j - \sum_{j=2}^n \sum_{k \neq 1, j}^n (-1)^{j+k} m_{1j} m_{k1} b_{kj}. \end{aligned} \tag{3.1}$$

By Lemma 3.13,  $b_1 \neq 0$ , so  $x$  can be defined in step 3:

$$x = \left( m_{12}^2 \frac{b_2}{b_1} + \sum_{j=3}^n m_{1j}^2 \frac{b_j}{b_1} + \sum_{j=2}^n \sum_{k \neq 1, j}^n (-1)^{j+k} m_{1j} m_{k1} \frac{b_{kj}}{b_1} \right).$$

From (3.1) we see that  $M$  is singular for this value of  $x$ , so 1 is an eigenvalue of  $A$ . It is clear that  $A$  is rational, but we still need to show that  $a_{11} = x + 1 < 0$  in order to show  $A \in \mathfrak{S}(Z)$ . We do so by first showing that  $|x| > 1$  and then that  $x < 0$ .

From Lemmas 3.13 and 3.14 and the fact that  $1 \leq |m_{jj}|$ , we know that for  $j \geq 3$ ,

$$\left| m_{1j}^2 \frac{b_j}{b_1} \right| < \left( \frac{2^{-n}}{n!} \right)^2 \frac{13/12}{11/12} < 1.$$

From Lemmas 3.13 and 3.15, for  $j, k \geq 2$ ,  $j \neq k$ ,

$$\left| m_{1j} m_{k1} \frac{b_{kj}}{b_1} \right| < 2(n-1) \left( \frac{2^{-n}}{n!} \right) \frac{1/48}{11/12} < 1.$$

Therefore

$$\left| \sum_{j=3}^n m_{1j}^2 \frac{b_j}{b_1} + \sum_{j=2}^n \sum_{k \in \mathcal{N} \setminus \{1, j\}} (-1)^{j+k} m_{1j} m_{k1} \frac{b_{kj}}{b_1} \right| < \sum_{j=3}^n 1 + \sum_{j=2}^n \sum_{k \neq 1, j} 1 = n(n-2). \quad (3.2)$$

By Lemma 3.13 and 3.14 with  $j = 2$  and the fact that  $|m_{jj}| \leq 2$ ,  $\left| \frac{b_2}{b_1} \right| > \frac{1}{2} \frac{11/12}{13/12} > \frac{1}{4}$ , so

$$\left| m_{12}^2 \frac{b_2}{b_1} \right| = 4(n-1)^2 \left| \frac{b_2}{b_1} \right| > (n-1)^2. \quad (3.3)$$

By equations (3.2) and (3.3),

$$\begin{aligned} |x| &\geq \left| m_{12}^2 \frac{b_2}{b_1} \right| - \left| \sum_{j=3}^n m_{1j}^2 \frac{b_j}{b_1} + \sum_{j=2}^n \sum_{k \neq 1, j} (-1)^{j+k} m_{1j} m_{k1} \frac{b_{kj}}{b_1} \right| \\ &> (n-1)^2 - n(n-2) = 1, \end{aligned}$$

and  $\text{sgn}(x) = \text{sgn}\left(m_{12}^2 \frac{b_2}{b_1}\right) = \text{sgn}\left(\frac{b_2}{b_1}\right) = -$ , since by Lemmas 3.13 and 3.14,  $b_1$  and  $b_2$  have opposite signs. Thus  $x < -1$  and  $x + 1 < 0$ , so  $A \in \mathcal{S}(Z)$ .  $\square$

Since the above was for  $n \geq 3$  we now consider  $n = 2$ . There are two cases to consider:

$$Z_1 = \begin{bmatrix} - & \pm \\ \pm & 0 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} - & \pm \\ \pm & - \end{bmatrix}.$$

LEMMA 3.17. *The matrices*

$$A_1 = \begin{bmatrix} -8 & \pm 3 \\ \pm 3 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & \pm 2 \\ \pm 2 & -1 \end{bmatrix}$$

satisfy  $A_i \in \mathcal{S}(Z_i)$  and 1 is an eigenvalue of  $A_i$  for  $i = 1, 2$ .

**4. Conclusion.** The case  $\lambda = 0$  was done in [3], so the following theorems have now been established.

**THEOREM 4.1.** *For any symmetric tree sign pattern  $Z$  and any rational number  $\lambda$ , there is a rational matrix  $A \in \mathcal{S}(Z)$  such that  $m_A(\lambda) = M_\lambda(Z)$ . Such a matrix can be constructed by the algorithms in Sections 2 and 3 and in [3].*

**THEOREM 4.2.** *For any tree  $T$  and any rational number  $\lambda$ , there is a rational matrix  $A \in \mathcal{S}(T)$  such that  $m_A(\lambda) = M_\lambda(T)$ . Such a matrix can be constructed by the algorithms in Sections 2 and 3 and in [3].*

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