On the Difference between the Maximum Multiplicity and Path Cover Number for Tree-Like Graphs

Francesco Barioli[a], Shaun Fallat[b,1,2] and Leslie Hogben[c]

[a] *School of Mathematics and Statistics, Carleton University, Ottawa, ON, K1S 5B6, Canada* (fbarioli@math.carleton.ca).

[b] *Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada* (sfallat@math.uregina.ca).

[c] *Department of Mathematics, Iowa State University, Ames, IA, 50011, U.S.A.* (lhogben@iastate.edu)

We dedicate this work to Pauline van den Driessche for her life long contributions to linear algebra and her support of the linear algebra community.

Abstract

For a given undirected graph G, the maximum multiplicity of G is defined to be the largest multiplicity of an eigenvalue over all real symmetric matrices A whose (i, j)th entry is nonzero whenever $i \neq j$ and $\{i, j\}$ is an edge in G. The path cover number of G is the minimum number of vertex-disjoint paths occurring as induced subgraphs of G that cover all the vertices of G. We derive a formula for the path cover number of a vertex-sum of graphs, and use it to prove that the vertex-sum of so-called non-deficient graphs is non-deficient. For unicyclic graphs we provide a complete description of the path cover number and the maximum multiplicity (and hence the minimum rank), and we investigate the difference between path cover number and maximum multiplicity for a class of graphs referred to as block-cycle graphs.

Key words: Graphs, minimum rank, maximum multiplicity, path cover number, vertex-sums, unicyclic graphs, block-cycle graphs.

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[2] Corresponding Author.

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1 Introduction and Preliminaries

For a given graph $G = (V, E)$ it has become clear that calculating the so-called minimum rank of G (see definition to follow) is usually very difficult. However, if G is a tree this computation is easily accomplished by noting connections between the minimum rank and other parameters called the maximum multiplicity of G and the path cover number of G (see definitions to follow). In this paper we study further the relationships between all of these parameters for more general graphs that bear some resemblance to trees.

All matrices discussed in this paper are real and symmetric. The graph $G(A)$ of an $n \times n$ matrix A has $\{1, ..., n\}$ as vertices, and as edges the unordered pairs $\{i, j\}$ such that $a_{ij} \neq 0$ with $i \neq j$. Graphs G of the form $G = G(A)$ do not have loops or multiple edges, and the diagonal of A is ignored in the determination of $G(A)$.

For the matrix A, $\sigma(A)$ denotes the spectrum of A and for $\lambda \in \sigma(A)$, $\mathrm{mult}_A(\lambda)$ denotes the multiplicity of λ. We let $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : G(A) = G\}$ denote the *minimum rank of* G, and we let $M(G) = \max\{\mathrm{mult}_A(\lambda) : \lambda \in \sigma(A) \text{ and } G(A) = G\}$ denote the *maximum multiplicity of* G. Further, $P(G)$ is the *path cover number*, namely, the minimum number of vertex disjoint paths, occurring as induced subgraphs of G, that cover all the vertices of G; $\Delta(G)$ is the maximum of $p - q$ such that the deletion of q vertices from G leaves p paths.

If we denote the order of G by $|G|$, then it is easy to see that $|G| = M(G) + \mathrm{mr}(G)$. This relation has been exploited to obtain results about the maximum possible multiplicity from results on the minimum rank, and also played a role, for example, in the fact that for trees the three parameters $M(T)$, $P(T)$ and $\Delta(T)$ are equal [JLD99]. We note further that, for arbitrary graphs, $\Delta(G) \leqslant M(G)$ can be deduced from the work in [JLD99], while $\Delta(G) \leqslant P(G)$ has been proved in [BFH]. However, as noted in [BFH], both $P(G) < M(G)$ and $M(G) < P(G)$ are possible (see Figures 1 and 2 on the following page).

If v is a vertex of a graph G, we denote by $G - v$ the subgraph of G

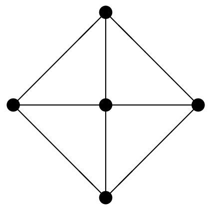

Figure 1: The 5-wheel W_5: $P(W_5) = 2$, $M(W_5) = 3$.

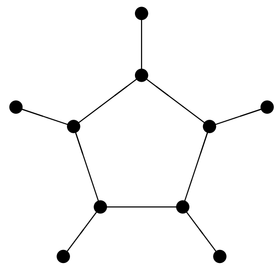

Figure 2: The 5-sun H_5: $P(H_5) = 3$, $M(H_5) = 2$.

obtained by deleting v and all edges incident with v. Any induced subgraph of G is obtained by deleting some subset of vertices. For a matrix A with $G(A) = G$, the matrix $A(v)$ will denote the principal submatrix of A obtained by deleting row and column v. In particular $G(A(v)) = G - v$. Induced subgraphs play a significant role in what follows and of particular interest is keeping track of change in the minimum rank and path cover number over certain types of induced subgraphs. Consequently, we define *rank-spread* of G at v as $r_v(G) = \operatorname{mr}(G) - \operatorname{mr}(G - v)$. We then have $0 \leqslant r_v(G) \leqslant 2$ (see, for example, [N]). Similarly we define the *path-spread* of G at v as $p_v(G) = P(G) - P(G - v)$.

Two natural graph operations are vertex-sums and edge-sums of graphs: let $G_1, \dots, G_h$ be disjoint graphs. For each i, we select a vertex $v_i \in V(G_i)$ and join all G_i's by identifying all v_i's as a unique vertex v. The resulting graph is called the *vertex-sum* at v of the graphs $G_1, \dots, G_h$, and is denoted by $\underset{v}{+} G_i$. The graphs $G_1, \dots, G_h$ are called the *constituents* of the vertex-sum. A vertex v is a *cut-vertex* of G if $G - v$ has more components than G. If at least two of $G_1, \dots, G_h$ have order greater than one, v is a cut-vertex of the vertex-sum. For graphs that can be written as a vertex-sum (at a fixed vertex) of graphs we have the following characterization of rank-spread, which will be needed later.

Theorem 1.1 [BFH, Thm. 2.3] *Let* $G = \underset{v}{+} G_i$. *Then*

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}, \tag{1}$$

that is, $\operatorname{mr}(G) = \sum_1^h \operatorname{mr}(G_i - v) + \min\left\{\sum_1^h r_v(G_i),\ 2\right\}$.

The edge-sum of two graphs is defined as follows. Let G_1 and G_2 be disjoint graphs, and let v_1 and v_2 be vertices of G_1 and G_2 respectively. If we connect G_1 and G_2 by adding the edge $e = \{v_1, v_2\}$, the resulting graph G is called *edge-sum* of G_1 and G_2 along the edge e.

Theorem 1.2 [BFH, Thm. 2.6] *Let G be the edge-sum of G_1 and G_2 along the edge $e = \{v_1, v_2\}$. Then*

$$\mathrm{mr}(G) = \begin{cases} \mathrm{mr}(G_1) + \mathrm{mr}(G_2) & \textit{if } r_{v_i}(G_i) = 2 \textit{ for at least one } i; \\ \mathrm{mr}(G_1) + \mathrm{mr}(G_2) + 1 & \textit{otherwise.} \end{cases}$$

Since in general $P(G) \neq M(G)$, we define $\eta(G) = P(G) - M(G)$, and call a graph *non-deficient* if $\eta(H) \geqslant 0$ for each induced subgraph H of G. We state the following result on edge-sums of non-deficient graphs.

Theorem 1.3 [BFH, Thm. 3.6] *The edge-sum of non-deficient graphs is non-deficient.*

Clearly, an edge-sum of graphs can be viewed as a sequence of two vertex-sums, so that Theorem 1.2 can be easily obtained from Theorem 1.1. With this in mind, in Section 2 we extend Theorem 1.3 by proving that a vertex-sum of non-deficient graphs is non-deficient as well (Theorem 2.3).

In Section 3 we will exploit this result to determine P, M and η for *partial suns*, namely, graphs consisting of a cycle with several leaves appended.

Some "trimming procedures" are presented in Section 4 which allow us to reduce a general graph to a simpler and smaller one, while keeping track of any changes to P, M and hence η. In particular, in Section 5, these trimming procedures provide a complete characterization of P, M and η for *unicyclic graphs*, namely, graphs containing a unique cycle.

In Section 6 we then consider *block-cycle graphs*, that is, graphs that can be obtained by a sequence of vertex-sums of cycles and/or edges. In Corollary 6.4 we will prove that, for a block-cycle graph G, a sharp upper bound on $\eta(G)$ is provided by the number of odd-cycles contained in G.

We conclude, in Section 7, by investigating the possible relationship between rank-spread and path-spread of a graph G. In some sense, such a relationship does not exist, in general. Indeed, we present examples showing that any possible value for $r_v(G)$ is consistent with any possible value for $p_v(G)$.

2 Non-deficient graphs

A vertex v of a graph G is called *doubly terminal* if there is a minimum path cover in which v is a path by itself. The vertex v is called *simply terminal* if v is not doubly terminal and is the endpoint of a path in some minimum path cover of G.

Lemma 2.1 *Let v be a vertex of G. Then*

i. $-1 \leqslant p_v(G) \leqslant 1$*;*

ii. $p_v(G) = 1$ *if and only if v is doubly terminal;*

iii. $p_v(G) = 0$ *if v is simply terminal.*

Proof

i. For any minimum path cover of $G - v$, this path cover together with v is a path cover of G, so $P(G) \leqslant P(G - v) + 1$, that is, $p_v(G) \leqslant 1$. On the other hand, if we delete v from a minimum path cover of G, we obtain a path cover of $G - v$ with at most $P(G) + 1$ paths, so $P(G - v) \leqslant P(G) + 1$, that is, $p_v(G) \geqslant -1$.

ii. If $p_v(G) = 1$, the path cover defined in the first part of (i.) is a minimal path cover of G, therefore v is doubly terminal. Conversely, if v is doubly terminal, from a minimum path cover of G with v singleton, we obtain a path cover of $G - v$ with $P(G) - 1$ paths, which is necessarily minimal, so that $p_v(G) = 1$.

iii. If, from a minimum path cover of G with v as endpoint of a path, we delete v, we obtain a path cover of $G - v$ with exactly $P(G)$ paths. Therefore $p_v(G) \geqslant 0$, and finally $p_v(G) = 0$ by (ii.). □

Note that it can be $p_v(G) = 0$ even if v is not simply terminal, as can be seen by considering vertex 5 of G in Figure 3 on the next page. The paths $(6, 1, 5, 4, 9)$ and $(7, 2, 3, 8)$ provide the unique minimum path cover of G, and since vertex 5 is not an endpoint, it is not simply terminal. However the paths $(6, 1, 2, 7)$ and $(8, 3, 4, 9)$ provide the unique minimum path cover for $G - (5)$, so that $p_5(G) = 0$.

We now prove a result analogous to Theorem 1.1 for the path-spread of vertex-sums of graphs.

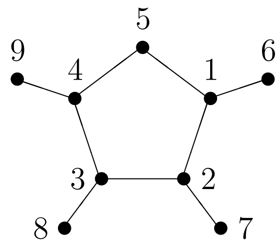

Figure 3: The graph $G = H_5 - (10)$

Proposition 2.2 *Let G be vertex-sum at v of graphs $G_1, \ldots, G_h$. Then*

$$p_v(G) = \begin{cases} \min\limits_i p_v(G_i) & \text{if } v \text{ is simply terminal in at most one of the } G_i\text{'s;} \\ -1 & \text{otherwise.} \end{cases}$$

Proof Let $p_v(G_j) = \min_i p_v(G_i)$ for some j. Let Φ_j and Φ'_i, $i = 1, \ldots, h$, $i \neq j$ be minimal path covers for G_j and $G_i - v$ respectively. Then $\Phi = \Phi_j \cup \bigcup_{i \neq j} \Phi'_i$ is a path cover for G, so that

$$\begin{aligned} p_v(G) &= P(G) - \textstyle\sum_1^h P(G_i - v) \\ &\leqslant |\Phi_j| + \textstyle\sum_{i \neq j} |\Phi'_i| - \sum_1^h P(G_i - v) \\ &= P(G_j) + \textstyle\sum_{i \neq j} P(G_i - v) - \sum_1^h P(G_i - v) \\ &= P(G_j) - P(G_j - v) \\ &= p_v(G_j) \\ &= \min_i p_v(G_i) \end{aligned}$$

Assume now $p_v(G) < \min p_v(G_i)$. We will prove that $p_v(G) = -1$ and v is simply terminal in at least two of the G_i's. Consider a minimum path cover Φ of G. If the path in Φ containing v is either a singleton or contains only vertices (other than v) of a unique constituent, say G_1, then we easily obtain $P(G) \geqslant P(G_1) + \sum_2^h P(G_i - v)$, and finally $p_v(G) \geqslant p_v(G_1)$, which is a contradiction. So the path in Φ containing v must contain vertices of two different constituents, say G_1 and G_2. Note that such a path cannot contain vertices of any further constituent. Let Φ_1, Φ_2 and Φ'_i, $i \geqslant 3$ be the path covers of G_1, G_2 and $G_i - v$'s, consisting of the pieces of Φ in G_1, G_2 and $G_i - v$'s respectively. We then have $|\Phi_1| + |\Phi_2| + \sum_3^h |\Phi'_i| = |\Phi| + 1$, that is, $P(G) \geqslant P(G_1) + P(G_2) + \sum_3^h P(G_i - v) - 1$, and finally

$$p_v(G) \geqslant p_v(G_1) + p_v(G_2) - 1. \tag{2}$$

Note that, if $p_v(G_1) = 1$, by (2) we would get $p_v(G) \geqslant p_v(G_2)$, which is impossible. Moreover, $p_v(G_1) = -1$ cannot occur, and so $p_v(G_1) = 0$, and similarly, $p_v(G_2) = 0$. Since $p_v(G) < p_v(G_i)$ for all i, necessarily $p_v(G) = -1$. Thus (2) is actually an equality, which means that Φ_1 and Φ_2 are minimal path covers for G_1 and G_2 respectively, and so v is simply terminal both in G_1 and G_2. □

Note that, even if v is simply terminal in more than one of the G_i's, it can be $p_v(G) = \min p_v(G_i) = -1$. This occurs, for instance, if v is simply terminal in G_1 and G_2, while $p_v(G_3) = -1$.

At this point we state and prove the main result of this section on vertex-sums of non-deficient graphs (cf. Theorem 1.3).

Theorem 2.3 *Let G be vertex-sum at v of graphs $G_1, \ldots, G_h$. Then G is non-deficient if and only if all of the G_i's are non-deficient.*

Proof Let H be an induced subgraph of G. We will prove that $\mathrm{mr}(H) + P(H) \geqslant |H|$. Without loss of generality we can assume H is connected. Let H_i be the subgraph induced by $V(H) \cap V(G_i)$. Since H is connected, either H is a subgraph of one G_i, or H is vertex-sum at v of H_i's. In the latter case, since G_i's are non-deficient, we have

$$\mathrm{mr}(H_i - v) + P(H_i - v) \geqslant |H_i - v| \qquad i = 1, \ldots, h; \tag{3}$$

$$\mathrm{mr}(H_i) + P(H_i) \geqslant |H_i| \qquad i = 1, \ldots, h. \tag{4}$$

From (4) we then have

$$\mathrm{mr}(H_i - v) + P(H_i - v) \geqslant |H_i - v| + 1 - r_v(H_i) - p_v(H_i). \tag{5}$$

Let $J = \{j \mid r_v(H_j) + p_v(H_j) \leqslant 0\}$ and $K = \{k \mid r_v(H_k) > 0\}$. By applying Theorem 1.1, and minimizing $\mathrm{mr}(H_i - v) + P(H_i - v)$ as in (3) for $j \notin J$, and as in (5) for $j \in J$, we obtain

$$\begin{aligned} &\mathrm{mr}(H) + P(H) = \\ &= \textstyle\sum_1^h (\mathrm{mr}(H_i - v) + P(H_i - v)) + r_v(H) + p_v(H) \\ &\geqslant |H| - 1 + \textstyle\sum_J (1 - r_v(H_j) - p_v(H_j)) + \min\{\sum_K r_v(H_k), 2\} + p_v(H) \end{aligned}$$

We obtain the desired conclusion if we prove that

$$\textstyle\sum_J (1 - r_v(H_j) - p_v(H_j)) + \min\{\sum_K r_v(H_k), 2\} + p_v(H) \geqslant 1. \tag{6}$$

If $|J| + |K| \geqslant 2$, then $\sum_J (1 - r_v(H_j) - p_v(H_j)) + \min\{\sum_K r_v(H_k), 2\} \geqslant 2$, so that, since $p_v(H) \geqslant -1$, (6) holds. We are then left to consider the case $|J| + |K| \leqslant 1$. If, for some i, $p_v(H_i) < 1$, then $i \in J \cup K$. Hence $p_v(H_i) = 1$ for each i except at most one. In particular, by Lemma 2.1, v is simply terminal in at most one of the H_i's. If, without loss of generality, we assume $p_v(H_1) = \min p_v(H_i)$, by Proposition 2.2, we have $p_v(H) = p_v(H_1)$. Finally, if $r_v(H_1) + p_v(H_1) \geqslant 1$, (6) holds, since

$$\begin{cases} \sum_J (1 - r_v(H_j) - p_v(H_j)) & \geqslant & 0 \\ \min\{\sum_K r_v(H_k), 2\} & \geqslant & r_v(H_1) \\ p_v(H) & = & p_v(H_1). \end{cases}$$

Similarly, if $r_v(H_1) + p_v(H_1) \leqslant 0$, then $1 \in J$, so that

$$\begin{aligned} &\sum_J (1 - r_v(H_j) - p_v(H_j)) + \min\{\sum_K r_v(H_k), 2\} + p_v(H) \geqslant \\ &\geqslant (1 - r_v(H_1) - p_v(H_1)) + r_v(H_1) + p_v(H_1) = 1. \end{aligned}$$

The converse is trivial. □

3 Partial n-suns

In an effort to prove which unicyclic graphs G have $\eta(G) > 0$ we first establish some preparatory results, which will be needed in this characterization. We split this analysis into two parts. The first deals with partial n-suns, and the second (see the next section) with certain kinds of graph trimming operations.

Let C_n be an n-cycle and let $U \subseteq V(C_n)$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun*. We will call a *segment* of H any maximal subset of consecutive vertices in U. The segments of H will be counted clockwise and denoted by $U_1, \ldots, U_t$. For instance, the graph H in Figure 4 on the following page has two segments, namely $U_1 = \{2\}$ and $U_2 = \{4, 5, 6\}$.

A partial n-sun is said *sub-ordinary* if it is either the n-cycle, or the n-cycle with one additional leaf, or the n-cycle plus two leaves appended to adjacent vertices of C_n. In other words, H is sub-ordinary if it has at most one segment and $|U| \leqslant 2$. On the other hand, if $U = V(C_n)$, then H is called the n-sun. If H is neither sub-ordinary nor the n-sun, H will be called an *ordinary* partial n-sun.

For n-suns we have the following result.

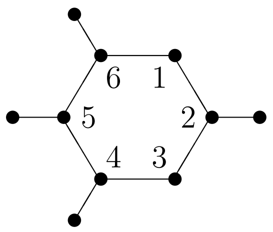

Figure 4: Partial 6-sun

Proposition 3.1 [BFH, Prop. 3.2] *Let H_n be the n-sun on $2n$ vertices. Then*

i. $P(H_n) = \left\lceil \frac{n}{2} \right\rceil$, $n \geqslant 3$;

ii. $M(H_3) = 2$;

iii. $M(H_n) = \left\lfloor \frac{n}{2} \right\rfloor$, $n \geqslant 4$.

In particular, $P(H_n) \neq M(H_n)$ exactly when n is an odd number greater than 3. The following propositions extend the previous one to partial n-suns.

Proposition 3.2 *Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Then*

$$P(H) = \begin{cases} 2 & \text{if } H \text{ is sub-ordinary} \\ \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil & \text{otherwise.} \end{cases} \tag{7}$$

Proof If H is sub-ordinary (7) can be proved by direct inspection, while, if H is the n-sun, the result follows from Proposition 3.1. Therefore, let us assume that H is ordinary, so that, either $t = 1$ and $3 \leqslant |U_1| < n$, or $t \geqslant 2$. For each $i = 1, \ldots, t$, u_i and u_i' will denote respectively the first and the last vertex of U_i, while v_i and v_i' will denote the leaves appended to u_i and u_i', respectively (see Figure 5 on the next page). Moreover, W_i will denote the (nonempty) set of vertices lying between U_i and U_{i+1} (read U_1 for U_{t+1}), and w_i, w_i' will be the first and the last vertex in W_i respectively. Finally, let z_i be the vertex before u_i'. Note that some coincidence among these vertices can occur when $|U_i| \leqslant 2$ or $|W_i| = 1$.

In a minimal path cover Φ, all of the vertices in W_i belong to the same path, say P_i. Moreover, the four vertices z_i, u_i', v_i' and w_i must lie on two distinct paths. If necessary, we alter slightly such two paths so that we can

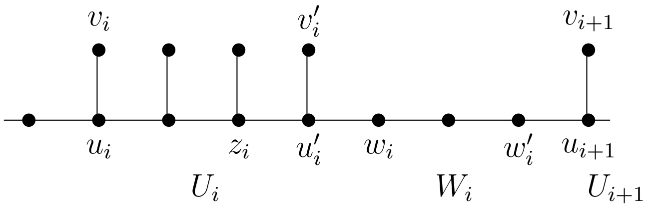

Figure 5:

assume u_i', v_i' and w_i in P_i. Similarly, if $|U_{i+1}| > 1$, we can assume u_{i+1} and v_{i+1} in P_i. Note that, since H is not sub-ordinary, P_i is an induced subgraph, as u_{i+1} is neither coincident nor adjacent to u_i'. Therefore the union of $P_1, \ldots, P_t$ covers exactly all vertices u_i, u_i', v_i, v_i' for each i, and all the W_i's. If, for some i, $|U_i| > 2$, we will need further $\left\lceil \frac{|U_i|-2}{2} \right\rceil$ paths to cover U_i (see Figure 6).

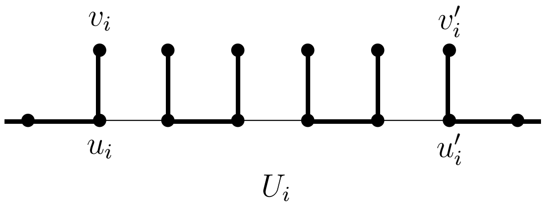

Figure 6:

Summarizing, the minimum path cover is $t + \sum_1^t \left\lceil \frac{|U_i|-2}{2} \right\rceil = \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil$, which proves (7). □

Proposition 3.3 *Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Then*

$$\Delta(H) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } H \text{ is the } n\text{-sun} \\ \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil & \text{otherwise} \end{cases} \tag{8}$$

Proof If H is sub-ordinary or is the 3-sun, (8) can be proved by direct inspection. For the remaining cases, consider the set Q obtained as follows: For each U_i, select every other vertex in U_i, beginning with the first. Clearly $|Q| = \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil$. If H is not the n-sun, the removal of Q leaves exactly $2\sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil$ paths. Therefore, by virtue of Proposition 3.2 and [BFH, Thm. 3.1],

$$\Delta(H) \geqslant 2\sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil - \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil = \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil = P(H) \geqslant \Delta(H)$$

and (8) holds. Finally, if H is the n-sun with $n > 3$, then $|Q| = \lceil \frac{n}{2} \rceil$, while the removal of Q leaves exactly n paths. By virtue of Proposition 3.1 we have

$$\Delta(H) \geqslant \left\lfloor \frac{n}{2} \right\rfloor = M(H) \geqslant \Delta(H)$$

and (8) holds. □

Proposition 3.4 *Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Then*

$$M(H) = \begin{cases} 2 & \text{if either } H \text{ is sub-ordinary or is the 3-sun} \\ \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil & \text{if } H \text{ is ordinary} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } H \text{ is the } n\text{-sun, } n > 3. \end{cases} \tag{9}$$

Proof By Theorem 2.3, H is non-deficient. Therefore $\Delta(H) \leqslant M(H) \leqslant P(H)$. In particular, if H is an ordinary partial sun, by Propositions 3.2 and 3.3 we have $\Delta(H) = M(H) = P(H) = \sum_1^t \left\lceil \frac{|U_i|}{2} \right\rceil$. If H is the n-sun, the result follows from Proposition 3.1. If H is sub-ordinary, by using Theorem 1.1 we can easily see that $\text{mr}(H) = |H| - 2$, that is, $M(H) = 2$. □

We summarize the results obtained in Propositions 3.2, 3.3 and 3.4 in Table 1 on the next page.

Recall that a graph is called non-deficient if $P(H) \geqslant M(H)$ for all induced subgraphs H of G. One of the primary interests here is to study non-deficient graphs and to characterize a large set of graphs for which $P(G) > M(G)$. Recall that $\eta(G) = P(G) - M(G)$. For partial n-suns, the above analysis implies the next result.

Corollary 3.5 *Let H be a partial n-sun. Then*

$$\eta(H) = \begin{cases} 1 & \text{if } n > 3, \text{ odd, and } H \text{ is the } n\text{-sun} \\ 0 & \text{otherwise.} \end{cases}$$

	$\Delta(H)$	$M(H)$	$P(H)$
n-cycle	0	2	2
n-cycle plus 1 leaf	1	2	2
n-cycle plus 2 consecutive leaves	1	2	2
ordinary partial sun	$\sum_1^t \left\lceil \frac{\lvert U_i\rvert}{2} \right\rceil$	$\sum_1^t \left\lceil \frac{\lvert U_i\rvert}{2} \right\rceil$	$\sum_1^t \left\lceil \frac{\lvert U_i\rvert}{2} \right\rceil$
3-sun	1	2	2
n-sun, $n > 3$	$\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	$\lceil \frac{n}{2} \rceil$

Table 1:

4 Trimming branches

In an effort to compute $\eta(G)$ for unicyclic graphs, we introduce a type of surgery on such graphs which we will refer to as 'trimming branches.'

A vertex v is said to be *appropriate* if there exist at least two pendent paths from v (in other words there are at least two components in the graph $G - v$ that are paths which were joined to v, in G, at an endpoint). In [BFH] it was shown that any appropriate vertex of a graph has rank-spread 2. A vertex v is called a *peripheral leaf* if $\delta(v) = 1$ and $\delta(u) \leqslant 2$ where $\delta(v)$ denotes the degree of v and u is the only neighbour of v. In this section we consider subgraphs of a given graph G, obtained by the following "trimming" procedures.

1. *Deletion of an appropriate vertex.* If v is an appropriate vertex, $G' = G - v$ is said to be *obtained from G by deletion of an appropriate vertex.* Proposition 4.1 deals with appropriate vertices.
2. *Deletion of an isolated path.* If one component of G is a path P, the graph $G' = G - V(P)$ is said to be *obtained from G by deletion of an isolated path.* In general, this process is required after the deletion of an appropriate vertex, which leaves two or more isolated paths.

Proposition 4.2 deals with isolated paths.

3. *Deletion of a peripheral leaf.* If v is a peripheral leaf, $G' = G - v$ is said to be *obtained from G by deletion of a peripheral leaf.* This process is studied in Proposition 4.3.

Before we come to a sequence of results on the various trimming operations above, we note the following convention that all of the parameters, Δ, P, and M are defined to be 0 for the empty graph.

Proposition 4.1 *If G' is obtained from G by deletion of an appropriate vertex v, then*

i. $\Delta(G') = \Delta(G) + 1$;

ii. $P(G') = P(G) + 1$;

iii. $M(G') = M(G) + 1$.

Proof

i. Since v is appropriate, we can always determine a subset Q of q vertices whose removal leaves p paths, with $\Delta(G) = p - q$ and such that $v \in Q$. After that, the conclusion follows easily.

ii. Since v is simply terminal in each of the two graphs induced by v and the vertices of the pendent paths, and G is the vertex-sum on v of these two graphs and everything else, by Proposition 2.2 $p_v(G) = -1$, and the result is immediate.

iii. Since, as seen, an appropriate vertex has rank-spread 2, the result is immediate. □

The following result is straightforward and is included for future reference.

Proposition 4.2 *If G' is obtained from G by deletion of an isolated path, then*

i. $\Delta(G') = \Delta(G) - 1$;

ii. $P(G') = P(G) - 1$;

iii. $M(G') = M(G) - 1$.

Proposition 4.3 *If G' is obtained from G by deletion of a peripheral leaf, then*

i. $\Delta(G') = \Delta(G)$*;*

ii. $P(G') = P(G)$*;*

iii. $M(G') = M(G)$*.*

Proof

i. Let v be a peripheral leaf and let u be its unique neighbour. We can obtain a subset Q of $V(G)$ of cardinality q whose removal leaves p paths with $\Delta(G) = p - q$ by taking only vertices of degree larger than 2. Therefore Q contains neither v nor its neighbour u. By this fact $\Delta(G') = \Delta(G)$ follows easily.

ii. It suffices to note that, in a minimal path cover, v and its neighbour u lie on the same path.

iii. If we apply Theorem 1.2 with $G_1 = G'$ and $G_2 = \{v\}$, since $r_u(G_1) \leqslant 1$ and $r_v(\{v\}) = 0$, we have $\text{mr}(G) = \text{mr}(G') + 1$, which yields $M(G') = M(G)$. □

A *trimmed form* of a graph G is an induced subgraph, obtained by a sequence of the above mentioned trimming operations, that does not contain peripheral leaves, isolated paths and/or appropriate vertices. The next proposition proves that the trimmed form is unique. Hence we say G is in *trimmed form* if it coincides with its trimmed form.

Proposition 4.4 *The trimmed form of a graph is unique.*

Proof The proof is by induction on the number of vertices, n. If $n = 1$ or 2, then result is obvious. Thus we assume that the trimmed form of any graph on fewer than n vertices is unique. Let G be any graph on n vertices. If it does not contain an appropriate vertex, or an isolated path, or a peripheral leaf, then the trimmed form $\breve{G}$ of G is G and is thus unique. Also, if only one of the trimming operations can be initially performed uniquely on G, then by induction, $\breve{G}$ is unique. So assume that at least two trimming operations can be initially performed on G.

The idea for the remainder of the proof is to show that if G_1 is obtained from G by performing one trimming operation and G_2 is obtained from G

by performing the other trimming operation, then there is a way to trim G_1 and a way to trim G_2 to obtain a common subgraph H, and hence it follows that $\breve{G}_1 = \breve{G}_2$.

Consider the case that G contains an appropriate vertex v and a peripheral leaf u. Suppose G_1 is obtained from G by deleting v and that G_2 is obtained from G by deleting u. Suppose that the paths emanating from v in G are $P_1, P_2, \ldots, P_i$, where $i \geqslant 2$. Then there are three cases to consider.

Case 1) Suppose $u \notin P_j$, for any $j = 1, 2, \ldots, i$. Then it follows that v is still an appropriate vertex in G_2 and that u is still a peripheral leaf in G_1. Hence $G_1 - u = G_2 - v$.

Case 2) $u \in P_j$, but is not adjacent to v. This case is handled by a similar argument to the one used in Case 1.

Case 3) $u \in P_j$, but is adjacent to v. Then the degree of v must be exactly two. Thus G is just a path on n vertices. It is easy to check that $\breve{G}$ is the empty graph.

The remaining cases of which types of initial trimming operations are all handled in a similar manner and their proofs are omitted here. □

If $\breve{G}$ can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then, by taking into account Propositions 4.1, 4.2 and 4.3, we define $\tau(G) = n_2 - n_1$ which we will call the *trimming index* of G.

Proposition 4.5 *The trimming index does not depend on the sequence of deletions performed to obtain* $\breve{G}$*, and*

i. $\Delta(G) = \Delta(\breve{G}) + \tau(G)$

ii. $P(G) = P(\breve{G}) + \tau(G)$

iii. $M(G) = M(\breve{G}) + \tau(G)$

iv. $\eta(G) = \eta(\breve{G})$

Proof The proof follows from Propositions 4.1, 4.2 and 4.3. □

5 Unicyclic graphs

We now turn our attention to unicyclic graphs (namely those graphs that contain a unique cycle), and characterize η for a unicyclic graph. It is useful to keep in mind that all trees are non-deficient since $P = M$ for any tree.

Lemma 5.1 *Let G be a connected unicyclic graph, and let C denote the unique induced cycle. If there exists $v \notin C$ of degree greater than 2, then G has an appropriate vertex.*

Proof Let $S = \{v \notin C \mid \delta(v) > 2\} \neq \varnothing$. For any $v \in S$, define $l(v)$ as the length of the path connecting v and C. Any vertex in S that maximizes $l(v)$ is necessarily appropriate. $\square$

Corollary 5.2 *The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun.*

Proof Let C denote the cycle of G. If $C \nsubseteq \breve{G}$, then $\breve{G} = \varnothing$. If $C \subseteq \breve{G}$, then, since $\breve{G}$ is necessarily connected, by Lemma 5.1, all the vertices of $\breve{G} - V(C)$ have degree at most 2. Therefore $\breve{G}$ consists of C plus, possibly, several paths emanating from (some of) the vertices of C. However, since $\breve{G}$ has no appropriate vertices, there will be at most one path emanating from each vertex of C, and since $\breve{G}$ has no peripheral leaves, all these paths must have length 1. Therefore $\breve{G}$ is a partial sun. $\square$

From Corollary 5.2, Corollary 3.5 and Proposition 4.5 we then have the next result for unicyclic graphs.

Corollary 5.3 *Let G be a unicyclic graph. Then*

$$\eta(G) = \begin{cases} 1 & \text{if } \breve{G} \text{ is an } n\text{-sun, } n > 3, \text{ odd;} \\ 0 & \text{otherwise.} \end{cases}$$

6 Block-cycle graphs

In the previous two sections the function $\eta(G)$ was clearly a useful quantity. In this section we extend beyond the class of unicyclic graphs to block-cycle graphs and determine sharp upper and lower bounds on $\eta(G)$.

A vertex v of a graph G is a cut vertex of G if $G - v$ has more components than G. A graph is nonseparable if it is connected and has no cut-vertices. A block of a graph is a maximal nonseparable induced subgraph. A *block-cycle graph* is a graph in which every block is either an edge or a cycle. In the study of $\eta(G)$, where G is a block-cycle graph, a central role will be played by $\text{co}(G)$, the number of blocks of G that are odd cycles of length greater than 3.

A *quasi n-sun* is the graph obtained by deleting a leaf from an n-sun. Note that in a quasi n-sun there is a unique vertex of degree 2.

Lemma 6.1 *Let v be a vertex of a partial n-sun H, and suppose $H \neq H_n$ (i.e. not the complete n-sun). Then*

i. $p_v(H) + r_v(H) = 1$*;*

ii. *v is not simply terminal in H with $p_v(H) = 0$ if and only if n is odd and greater than 3, H is a quasi n-sun and v is the only vertex of degree 2.*

Proof

i. Since H is not a complete (odd) n-sun, $\eta(H) = \eta(H - v) = 0$, and the claim follows easily.

ii. Suppose $p_v(H) = 0$ and v not simply terminal. Therefore v lies on the cycle of H. By (i.) we have $r_v(H) = 1$. Let $\bar{H}$ be the graph obtained by appending a leaf w on the vertex v, and let $\tilde{H}$ be the subgraph (edge) of $\bar{H}$ induced by v and w. Thus $\bar{H} = H \underset{v}{+} \tilde{H}$ and $r_v(\tilde{H}) = 1$. By Theorem 1.1 we obtain $r_v(\bar{H}) = 2$. Furthermore $p_v(\tilde{H}) = 0$, and since v is not simply terminal in H, by Proposition 2.2 we conclude $p_v(\bar{H}) = 0$. So $p_v(\bar{H}) + r_v(\bar{H}) = 2$, that is $\eta(\bar{H}) = \eta(\bar{H} - v) + 1 = 1$, since $\bar{H} - v$ is acyclic. Therefore, since $\bar{H}$ is unicyclic, by Corollary 5.3 the trimmed form of $\bar{H}$ must be an odd n-sun, and the claim follows easily. Conversely, let $n > 3$ odd, H be a quasi n-sun, and v be the only vertex of degree 2 in H. Moreover, let H_n be the n-sun obtained by appending an edge $\tilde{H} = \{w, v\}$ to H. From Table 1 we easily find $P(H) = \frac{n-1}{2}$, $P(H_n) = \frac{n+1}{2}$, while, by direct inspction, we see that $P(H - v) = \frac{n-1}{2}$, $P(H_n - v) = \frac{n-1}{2} + 1$. Therefore $p_v(H) = p_v(H_n) = 0$. Finally, if v were simply terminal in H, by applying Proposition 2.2 to $H_n = H \underset{v}{+} \tilde{H}$, we would find $p_v(H_n) = -1$, which is a contradiction. □

Let G be a graph containing a partial n-sun H as induced subgraph. Let C denote the unique cycle contained in H. H is said to be a *terminal* partial n-sun in G if there exists $v \in C$ such that

i. v has degree 2 in H;

ii. $G = H \underset{v}{+} G_1$ for some G_1 subgraph of G.

Note that condition (i.) implies that H is not a complete n-sun.

Proposition 6.2 *A nonempty block-cycle graph G in trimmed form always contains a terminal partial n-sun.*

Proof Without loss of generality we can assume G to be connected. We can then define a sequence of induced connected subgraphs $G_0 \subset G_1 \subset \cdots \subset G_r$, with $G_0 = \varnothing$, $G_r = G$ and $G_{i+1} = G_i \underset{v_i}{+} F_i$, where F_i is either a cycle or an edge. Let F_j be the last cycle added in this building process, and let H be F_j and all leaves appended to F_j except, if any, the leaf appended to v_j. Then H is a terminal partial n-sun. $\square$

Let $G = H \underset{v}{+} G_1$ with H a terminal partial n-sun of G. The graph $\tilde{G}$ obtained as vertex-sum at v of G_1 and $r_v(H)$ leaves is said to be *obtained from G by trimming a terminal partial sun* (see Figure 7).

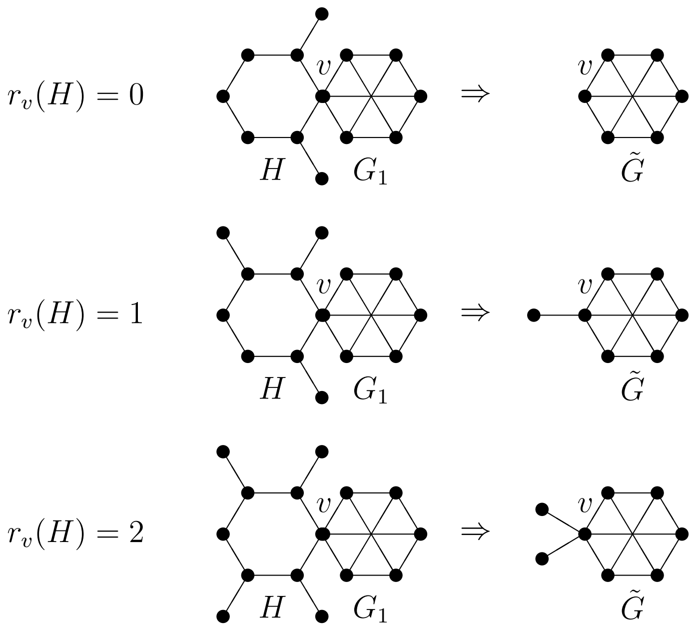

Figure 7: $\tilde{G}$ is obtained by trimming H

Theorem 6.3 *Let $G = H \underset{v}{+} G_1$ where H is a terminal partial n-sun, and let $\tilde{G}$ be the graph obtained by trimming H. Define $\rho = P(H) - 1 = M(H) - 1$. Then* *i.* $M(G) = M(\tilde{G}) + \rho$;

$$ii.\ P(G) = \begin{cases} P(\tilde{G}) + \rho + 1 & \text{if } H \text{ is an odd quasi } n\text{-sun } (n > 3) \text{ and} \\ & v \text{ is simply terminal in } G_1; \\ P(\tilde{G}) + \rho & \text{otherwise.} \end{cases}$$

$$iii.\ \eta(G) = \begin{cases} \eta(\tilde{G}) + 1 & \text{if } H \text{ is an odd quasi } n\text{-sun } (n > 3) \text{ and} \\ & v \text{ is simply terminal in } G_1; \\ \eta(\tilde{G}) & \text{otherwise.} \end{cases}$$

Proof

i. Since H is not the complete n-sun, by Lemma 6.1 we have $p_v(H) + r_v(H) = 1$. Let $\tilde{H}$ be the subgraph of $\tilde{G}$ consisting of v and the $r_v(H)$ additional leaves. Note that $\tilde{G} = \tilde{H} \underset{v}{+} G_1$. In all cases we have

$$r_v(\tilde{H}) = r_v(H) \tag{10}$$

and therefore

$$r_v(\tilde{G}) = r_v(G). \tag{11}$$

Furthermore $\mathrm{mr}(G) = \mathrm{mr}(H - v) + \mathrm{mr}(G_1 - v) + r_v(G)$ and $\mathrm{mr}(\tilde{G}) = 0 + \mathrm{mr}(G_1 - v) + r_v(\tilde{G})$, that is, by (11),

$$\mathrm{mr}(G) - \mathrm{mr}(\tilde{G}) = \mathrm{mr}(H - v). \tag{12}$$

Finally, note that

$$|G| - |\tilde{G}| = |H| - 1 - r_v(H). \tag{13}$$

Thus

$$\begin{aligned} M(G) - M(\tilde{G}) &= |G| - \mathrm{mr}(G) - |\tilde{G}| + \mathrm{mr}(\tilde{G}) \\ &= |H| - 1 - r_v(H) - \mathrm{mr}(H - v) \quad \text{by (12) and (13)} \\ &= |H| - 1 - \mathrm{mr}(H) \\ &= M(H) - 1 \\ &= \rho. \end{aligned}$$

ii. We have $P(G) = P(H - v) + P(G_1 - v) + p_v(G)$ and $P(\tilde{G}) = r_v(H) + P(G_1 - v) + p_v(\tilde{G})$. By applying Lemma 6.1.i we obtain

$$\begin{aligned} P(G) - P(\tilde{G}) &= P(H - v) + p_v(H) - 1 + p_v(G) - p_v(\tilde{G}) \\ &= P(H) - 1 + p_v(G) - p_v(\tilde{G}) \\ &= \rho + p_v(G) - p_v(\tilde{G}). \end{aligned} \tag{14}$$

Furthermore, since $\tilde{H}$ is acyclic, by (10) and Lemma 6.1.i, we obtain

$$p_v(\tilde{H}) = p_v(H). \tag{15}$$

Let us assume $p_v(G) \neq p_v(\tilde{G})$. We will prove that

a. v is simply terminal in G_1;
b. H is an odd quasi n-sun with $n > 3$.

Indeed, by applying Proposition 2.2 to G and to $\tilde{G}$, $p_v(G) \neq p_v(\tilde{G})$ implies (a.) and either

c. v is simply terminal in $\tilde{H}$ but is not simply terminal in H, or
d. v is simply terminal in H but is not simply terminal in $\tilde{H}$.

Further, note that (d.) cannot occur. Indeed, if v is simply terminal in H, then, by Lemma 2.1 and Lemma 6.1.i, we obtain $r_v(H) = 1$. Thus $\tilde{H}$ is a single edge and v is necessarily simply terminal in $\tilde{H}$. On the other hand, (c.) implies $p_v(\tilde{H}) = 0$ and, by (15), $p_v(H) = 0$. Finally, by Lemma 6.1.ii we obtain (b.). Conversely, if (a.) and (b.) hold, we have $p_v(G_1) = 0$ and, by Lemma 6.1.ii, $p_v(H) = 0$, while v is not simply terminal in H. By using Proposition 2.2, it is now easy to conclude $p_v(G) = 0$ and $p_v(\tilde{G}) = -1$, that is $p_v(G) - p_v(\tilde{G}) = 1$.

iii. Follows immediately from i. and ii. □

Corollary 6.4 *Let G be a block-cycle graph. Then $\eta(G) \leqslant \mathrm{co}(G)$.*

Proof If G' is obtained from G by deleting appropriate vertices, peripheral leaves, isolated paths, and/or by trimming a terminal partial n-sun that is not an odd quasi n-sun with $n > 3$ we have $\eta(G) = \eta(G')$ and $\mathrm{co}(G) \geqslant \mathrm{co}(G')$, thus $\eta(G) - \mathrm{co}(G) \leqslant \eta(G') - \mathrm{co}(G')$. If G' is obtained from G by trimming a terminal odd quasi n-sun with $n > 3$, we have $\eta(G) = \eta(G') + 1$ and

$\mathrm{co}(G) = \mathrm{co}(G') + 1$, that is, $\eta(G) - \mathrm{co}(G) = \eta(G') - \mathrm{co}(G')$. Since G can be reduced to the empty graph by a sequence of the above mentioned trimming procedures, we obtain $\eta(G) - \mathrm{co}(G) \leqslant \eta(\varnothing) - \mathrm{co}(\varnothing) = 0$. $\square$

There exist examples of block-cycle graphs such that $\eta(G) = \mathrm{co}(G) = k$ for any k. Consider for instance the graph consisting of a sequence of k 5-suns as shown in Figure 8.

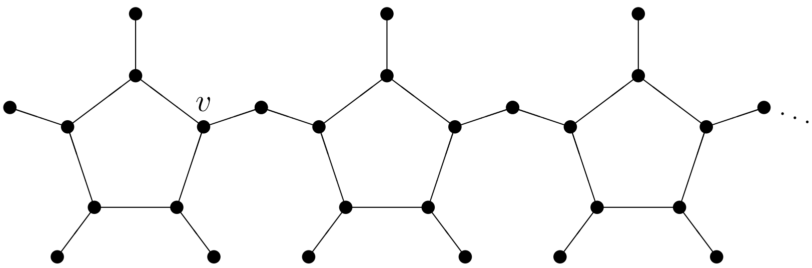

Figure 8: Sequence of 5-suns

Define H as the quasi n-sun on the left of v and G_1 the subgraph on the right of v. By trimming H we will decrease by one both $\eta(G)$ and $\mathrm{co}(G)$. Indeed, since v is simply terminal in G_1, by Theorem 6.3 we have $\eta(\tilde{G}) = \eta(G) - 1$ where $\tilde{G}$ is the graph in Figure 9.

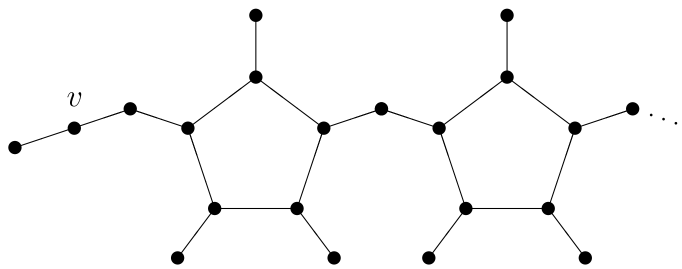

Figure 9: Same sequence after trimming a partial sun

After trimming two peripheral leaves, we are left with a sequence of $k-1$ 5-suns, and we can repeat the same procedure.

7 Rank-spread and path-spread

As stated in the introduction, for any graph G and any vertex v of G we have $r_v(G) \in \{0,1,2\}$, and it is easy to prove that each of these values can be attained. Similarly, by Lemma 2.1, $p_v(G) \in \{-1,0,1\}$, and again each of these values can occur. It is therefore natural to ask whether or not the pair $(r_v(G), p_v(G))$ can assume all of the possible values in $\{0,1,2\}\times\{-1,0,1\}$. In this section we give an answer in the positive, by presenting examples for all the possible cases. We first consider the four graphs presented in Figure 10.

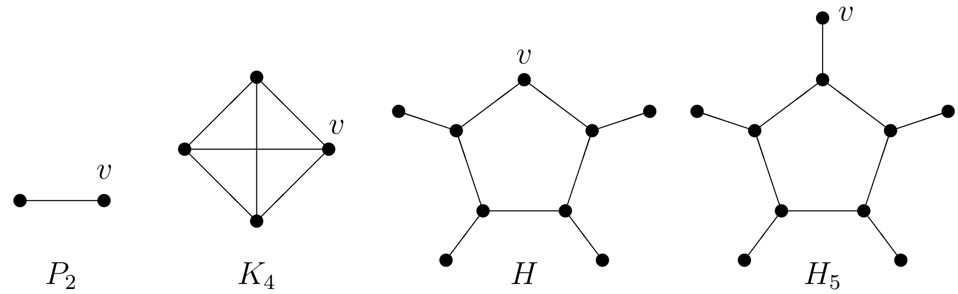

Figure 10:

By direct inspection and/or by applying results on partial suns in Section 3, it is easy to verify that

$$\begin{array}{lll} r_v(P_2) = 1; & p_v(P_2) = 0; & v \text{ simply terminal;} \\ r_v(K_4) = 0; & p_v(K_4) = 0; & v \text{ simply terminal;} \\ r_v(H) = 1; & p_v(H) = 0; & v \text{ not simply terminal;} \\ r_v(H_5) = 1; & p_v(H_5) = 1; & v \text{ doubly terminal.} \end{array} \tag{16}$$

By considering suitable vertex-sums of these four graphs, and using Theorem 1.1 and Proposition 2.2, together with (16) we obtain the following Table 2 on the following page.

References

[BFH] F Barioli, S.M. Fallat, and L. Hogben, Computation of Minimal Rank and Path Cover Number for Certain Graphs, preprint.

[F] M. Fiedler, A characterization of tridiagonal matrices, *Lin. Alg. Appl.* 2 (1969), 191–197.

G	$r_v(G)$	$p_v(G)$
$K_4 \underset{v}{+} K_4$	0	-1
K_4	0	0
$\{v\}$	0	1
$P_2 \underset{v}{+} K_4$	1	-1
P_2	1	0
H_5	1	1
$P_2 \underset{v}{+} P_2$	2	-1
$P_2 \underset{v}{+} H$	2	0
$H_5 \underset{v}{+} H_5$	2	1

Table 2: Possible combinations for rank-spread and path-spread

[JLD99] C.R. Johnson and A. Leal Duarte, The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, *Lin. Mult. Alg.* 46 (1999), 139–144.

[N] P.M. Nylen, Minimum-rank matrices with prescribed graph, *Lin. Alg. Appl.* 248 (1996), 303–316.