Cospectral bipartite graphs for the normalized Laplacian

by

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DEDICATION

I would like to dedicate this thesis to my wife Lacey and to my daughter Eva. Their support made this research possible.

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CHAPTER 1. General Introduction

Spectral graph theory attempts to answer a simple question: If we have a large data structure, what information can we capture with relatively few numbers? Specifically, can we go from storing the graph (n^2 data points) to the spectrum (n data points) of a matrix associated with the graph, or even just a few eigenvalues of large and small magnitude? An immediate question arises: Can the eigenvalues of the adjacency matrix[1] of a graph be used to completely reconstruct the graph (the "Inverse Eigenvalue Problem")? It is easy to show that this is not possible. The graphs in Figure 1.1 have the same spectrum $(-2, 0, 0, 0, 2)$. One of these graphs is disconnected, so it might seem that this is an anomaly associated with the connectedness of a graph. However, there are two connected graphs on six vertices which are cospectral (Figure 1.2). Now, one might be tempted to think that the spectrum is a poor gauge of a graph. But there are many graphs which can be recovered via their spectrum. In fact, there are papers by van Dam and Haemers which address this subject [9, 10].

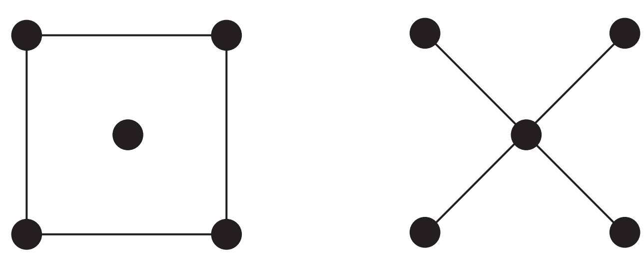

Figure 1.1 Cospectral graphs

Spectral graph theory has a rich background. See [2] and [8] for an overview of spectral graph theory and [3] and [5] for an overview as concerned with the normalized Laplacian. We say a graph G is *determined by its spectrum* (DS) for a particular type of matrix associated with the

[1] Formal definitions of a graph and the matrices associated with it are given in Section 1.1

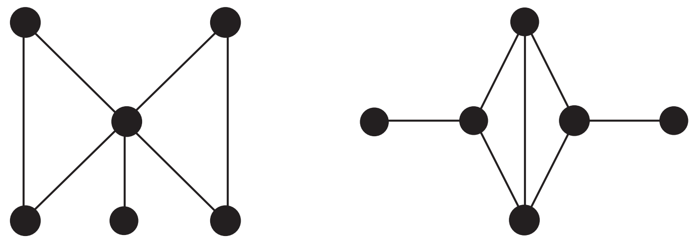

Figure 1.2 Cospectral connected graphs

graph if there is no nonisomorphic graph which has the same spectrum for that type of matrix. Conversely, we say two graphs G and H are *cospectral* with respect to a given type of matrix associated with the graph if these matrices share the same spectrum including multiplicity. It is known that many graphs have a cospectral mate [13, 16]. Clearly, nonisomorphic graphs with the same spectrum demonstrate the weaknesses of a certain type of matrix. In order to understand these weaknesses it is helpful to look at the structure of cospectral graphs. For the normalized Laplacian, little is known about cospectral pairs; see [4] for a few constructions of cospectral graphs for the normalized Laplacian.

1.1 Definitions and Notation

A *graph* $G = (V(G), E(G))$ is a pair of sets of vertices $V(G)$ and edges $E(G)$. If G is understood, we often write $V = V(G)$ and $E = E(G)$. An *edge* is a two element subset of V. If the pair $\{u, v\} \in E$, we say that u and v are *adjacent* in G or they are *neighbors* in G. The neighborhood of u in G, denoted $N_G(u)$, is the set of neighbors of u in G. The *degree* of a vertex v in G, denoted $\deg_G v$, is the cardinality of its neighborhood in G. An *isolated vertex* v in G is a vertex with $\deg_G v = 0$. A *subgraph* of G is a graph (V', E') where $V' \subseteq V$ and $E' \subseteq E$. The *path of order* n, P_n, is the graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$. A graph is *connected* if for all vertices u and v in G there is a path in G beginning at u and ending at v. If a graph is not connected, we say it is *disconnected*. A *component* of G is a maximal connected subgraph of G. The *cycle of length* n, C_n, is the graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

A connected graph is a *tree* if it does not contain a cycle as a subgraph. A graph is *bipartite* if $V = X\dot{\cup}Y$ and $N_G(x) \subseteq Y$ for all $x \in X$ and $N_G(y) \subseteq X$ for all $y \in Y$ for some partition X and Y of V. A well known result of graph theory is that a graph is bipartite if and only if it contains no cycles of odd length. Thus, a tree is necessarily bipartite. A *bipartite component* of G is a component of G that is bipartite.

Given a matrix N, the *characteristic polynomial* of N is $p_N(x) := \det(xI - N)$. The *spectrum* of a matrix N is the multiset of roots of $p_N(x)$ including multiplicity. The *diagonal matrix* denoted by $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ is the matrix with entries

$$\mathrm{diag}(a_1, a_2, \ldots, a_n)_{ij} = \begin{cases} a_i & \text{if } i = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Although there are many different ways of interpreting a graph as a matrix, we are particularly concerned with four variants: adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices. The *adjacency matrix* of a graph G, denoted $A(G)$, is the symmetric matrix indexed by the ordered set $(v_1, \ldots, v_n)$ of vertices of G with $(A)_{uv} = 1$ if u and v are adjacent in G and 0 otherwise. The *diagonal degree matrix* of G is $D(G) := \mathrm{diag}(\deg_G v_1, \ldots, \deg_G v_n)$. The *Laplacian matrix* of G is $L(G) := D(G) - A(G)$, the *signless Laplacian matrix* of G is $Q(G) := D(G) + A(G)$, and the *normalized Laplacian matrix* of G, $\mathcal{L}(G)$ is given by

$$\mathcal{L}(G)_{uv} = \begin{cases} 1 & \text{if } u = v \text{ and } \deg v \neq 0, \\ -\frac{1}{\sqrt{\deg u \deg v}} & \text{if } u \sim v, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if G has no isolated vertices $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2}$. Note that $A(G)$ is often written A if the graph G is implied, and this applies to $D(G), L(G), Q(G)$ and $\mathcal{L}(G)$ as well. Define $M(\lambda, t, G) := \lambda I_n - A(G) + tD(G)$ where G is a graph on n vertices. The *generalized characteristic polynomial* of G is $\phi(\lambda, t, G) := \det(M(\lambda, t, G))$. Note that if G has no isolated vertices, $\phi(x, 0, G) = p_{A(G)}(x)$, $\phi(-x, 1, G) = (-1)^n p_{L(G)}(x)$, $\phi(x, -1, G) = p_{Q(G)}(x)$, and $\phi(0, -x + 1, G) = (-1)^n \det(D) p_{\mathcal{L}(G)}(x)$. Thus if two graphs have the same generalized characteristic polynomial, they are simultaneously cospectral with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices.

1.2 Spectral Graph Theory

Each of these types of matrices comes with a set of strengths and weaknesses. In the table below we summarize the four types of matrices and four different structural properties of a graph and indicate which properties can be determined by the eigenvalues. A "No" answer indicates the existence of two non-isomorphic graphs which have the same spectrum but differ in the indicated structure. This table may also be found in [3].

Matrix	Bipartite	# Components	# Bipartite Components	# Edges
Adjacency	Yes	No	No	Yes
Laplacian	No	Yes	No	Yes
Signless Laplacian	No	No	Yes	Yes
Normalized Laplacian	Yes	Yes	Yes	No

We will now discuss this table in detail.

1.2.1 Adjacency Matrix

As stated in the table, the adjacency matrix can detect whether a graph is bipartite. In fact, it just comes down to inspecting the symmetry of the spectrum.

Theorem 1.2.1 [6] *If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of $A(G)$, then G is bipartite if and only if $\lambda_i = -\lambda_{n+1-i}$ for $i = 1, \ldots, n$.*

While the adjacency can tell *if* a graph is bipartite, it is incapable of determining the number of bipartite components in a graph. The Saltire pair is a demonstration of this fact. Note that this pair also implies that the adjacency matrix cannot count the number of components. However, it can count edges.

Theorem 1.2.2 [6] *If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of a graph with n vertices and m edges, then $\sum_i \lambda_i^2 = 2m$.*

1.2.2 Laplacian Matrix

The graphs shown in Figure 1.3 are cospectral with respect to the Laplacian matrix. They demonstrate that the Laplacian matrix cannot detect whether a graph is bipartite. This also implies the Laplacian matrix is incapable of counting the number of bipartite components of a graph. However, the Laplacian matrix can count the components and edges of a graph.

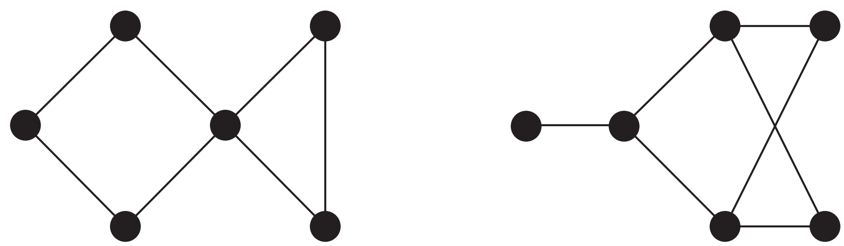

Figure 1.3 Graphs cospectral with respect to Laplacian matrix

Theorem 1.2.3 [5] *If the multiplicity of 0 as an eigenvalue of $L(G)$ is c, then G has c components.*

The following is a well known result.

Theorem 1.2.4 *If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of the Laplacian matrix of a graph n vertices and m edges, then $\sum_i \lambda_i = 2m$.*

Proof. We know $\operatorname{tr}(L) = \operatorname{tr}(D) = \sum_{v \in V(G)} \deg v = 2m$. However, $\operatorname{tr}(L) = \sum_i \lambda_i$. ∎

1.2.3 Signless Laplacian Matrix

The signless Laplacian can detect neither if a graph is bipartite nor the number of its components. The graphs in Figure 1.4 are cospectral with respect to the signless Laplacian matrix and demonstrate this fact. However, the signless Laplacian is capable of counting the number of bipartite components in a graph.

Theorem 1.2.5 [7] *If the multiplicity of 0 as an eigenvalue of $Q(G)$ is b, then G has b bipartite components.*

The following is a well known result.

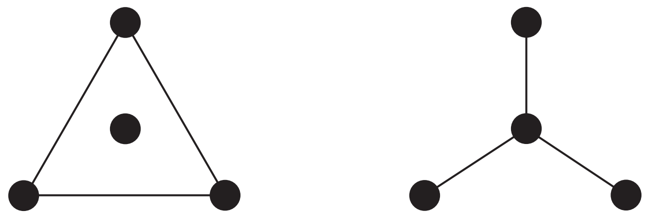

Figure 1.4 Graphs cospectral with respect to signless Laplacian matrix

Theorem 1.2.6 *If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of the signless Laplacian matrix of a graph with n vertices and m edges, then $\sum_i \lambda_i = 2m$.*

The proof is identical to the proof of Theorem 1.2.4.

1.2.4 Normalized Laplacian Matrix

On our short list of attributes, the normalized Laplacian detects three of the four characteristics. The follow theorems are due to Chung [5].

Theorem 1.2.7 [5] *Assume G is a graph and $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of $\mathcal{L}(G)$. The following statements are equivalent:*

1. *G is bipartite.*
2. *G has i components and $\lambda_j = 2$ for $1 \leq j \leq i$.*
3. *For each λ_i, $2 - \lambda_i = \lambda_j$ for some j.*

Theorem 1.2.8 [5] *If the multiplicity of 0 as an eigenvalue of $\mathcal{L}(G)$ is c, then G has c components.*

In order to show that the normalized Laplacian can count the number of bipartite components of a graph, we make the following simple observation: the normalized Laplacian spectrum of a graph is the union of the normalized Laplacian spectra of its components [5].

Theorem 1.2.9 *If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of $\mathcal{L}(G)$ where α is the multiplicity of 2, then the number of bipartite components of G is $\alpha + n - \sum_i \lambda_i$.*

Proof. We wish to count the number of bipartite components of a graph G using the spectrum of its normalized Laplacian matrix. The cardinality of the spectrum is n. This tells us G has n vertices. Further, we know that the diagonal entry in $\mathcal{L}(G)$ corresponding to each vertex in G is 1 unless the vertex is isolated (then the value is 0). Thus, the number of isolated vertices in G is $\ell = n - \operatorname{tr}(\mathcal{L}) = n - \sum_i \lambda_i$. Thus, we may remove ℓ zeros from the spectrum and our spectrum now corresponds to G with all isolated vertices removed. By Theorem 1.2.8, we know each component must have a exactly one 0 in its spectrum. Thus, by Theorem 1.2.7, each bipartite component must have exactly one 2 in its spectrum. Thus, $\alpha + \ell$ is the number of bipartite components in G. □

Unfortunately, the normalized Laplacian is not capable of counting the edges of a graph. The example given in Figure 1.5 shows a graph cospectral with one of its proper subgraphs with respect to the normalized Laplacian.

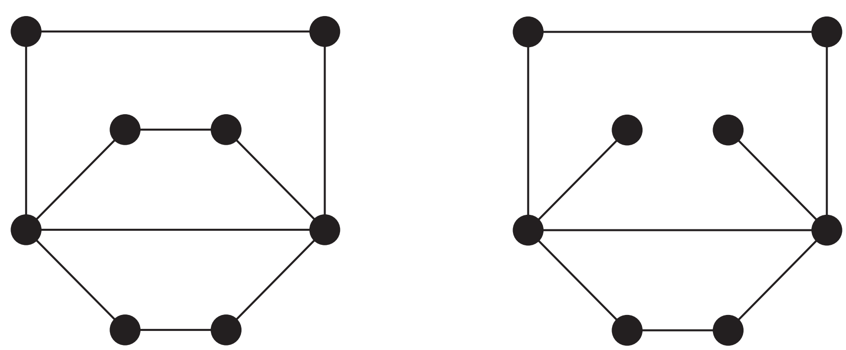

Figure 1.5 Graph cospectral with a proper subgraph with respect to normalized Laplacian matrix

1.3 Dissertation Organization

This dissertation is organized in the format of a dissertation containing journal papers. In the general introduction, pertinent background information is presented.

Chapter 2 contains the paper "Constructions of cospectral bipartite graphs for the normalized Laplacian" [14] submitted to the *Electronic Journal of Linear Algebra*. In the paper, we construct two infinite families of trees that are pairwise cospectral with respect to the normalized Laplacian. We also use the normalized Laplacian applied to weighed graphs to give new constructions of cospectral pairs of bipartite unweighted graphs.

Chapter 3 contains the paper "Almost all trees are normalized Laplacian cospectral" [15] submitted to *Linear Algebra and its Applications*. In the paper, we show that almost all trees have a cospectral mate for the normalized Laplacian matrix as well. We also show that almost every tree is cospectral with another tree with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices simultaneously by showing that almost all trees have a mate with the same generalized characteristic polynomial.

Chapter 4 is for general conclusions. Results are summarized and recommendations for future research are also presented.

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CHAPTER 2. Constructions of cospectral bipartite graphs for the normalized Laplacian

A paper submitted to the *Electronic Journal of Linear Algebra*

Steven Osborne

Abstract

We construct two infinite families of trees that are pairwise cospectral with respect to the normalized Laplacian. We also use the normalized Laplacian applied to weighed graphs to give new constructions of cospectral pairs of bipartite unweighted graphs.

2.1 Introduction

The spectrum of graphs is a well studied problem [1, 4, 5]. We say two graphs G and H are cospectral with respect to a given matrix described by the graph (e.g., adjacency, Laplacian, etc.) if these matrices share the same spectrum including multiplicity. It is known that many graphs have a cospectral mate [8, 9]. However, it is difficult to determine which graphs do not have a cospectral mate. The goal of spectral graph theory is to know how much information about the graph we can determine given its spectrum [7]. Clearly, distinct graphs with the same spectrum demonstrate the weaknesses of a certain matrix. In order to understand these weaknesses it is helpful to look at the structure of cospectral graphs. For the normalized Laplacian little is known about cospectral pairs; see [3] for constructions of cospectral graphs for the normalized Laplacian. In this paper we will give some new constructions of bipartite graphs which are cospectral for the normalized Laplacian, including the first known example of an infinite family of cospectral trees.

We will consider the spectrum of a graph as it pertains to the normalized Laplacian matrix. We will first consider simple graphs and then we will introduce weighted graphs in Section 2.4. The adjacency matrix of a graph G, denoted $A(G)$, is the matrix indexed by the ordered set $(v_1, \ldots, v_n)$ of vertices of G with $(A)_{uv} = 1$ if u and v are adjacent in G and 0 otherwise. The diagonal degree matrix of G is $D(G) := \operatorname{diag}(\deg_G v_1, \ldots, \deg_G v_n)$. The Laplacian matrix of G is $L(G) := D(G) - A(G)$ and the normalized Laplacian matrix of G is $\mathcal{L}(G) := D(G)^{-1/2} L(G) D(G)^{-1/2}$ where by convention each isolated vertex contributes a 0 to the spectrum of $\mathcal{L}(G)$. Thus,

$$\mathcal{L}(G)_{uv} = \begin{cases} 1 & \text{if } u = v \text{ and } \deg v \neq 0, \\ -\frac{1}{\sqrt{\deg u \deg v}} & \text{if } u \sim v, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

See [4] for an overview of the normalized Laplacian. Note that $A(G)$ is often written A if the graph G is implied, and this applies to $D(G), L(G)$ and $\mathcal{L}(G)$ as well. The distance between two vertices u and v in G, denoted $d_G(u, v)$, is the length of the shortest path connecting u and v in G. The diameter of a graph G is $\operatorname{diam}(G) := \max_{u,v \in G} d_G(u, v)$. The neighborhood of a vertex v in G, denoted $N_G(v)$, is the set of vertices adjacent to v in G. Two vertices u and v are *twins* in a graph G if $N_G(v) = N_G(u)$ (this implies u and v are not adjacent).

2.2 0-Eigenvectors of the adjacency matrix of trees

We know that 0 and 2 appear in the spectrum of the normalized Laplacian of a tree [4]. However, 1 often appears with relatively high multiplicity. As $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$, a 0 in the spectrum of A implies a 1 in the spectrum of $\mathcal{L}$ and a set of linearly independent eigenvectors for eigenvalue 0 of A are related to a set of linearly independent eigenvectors for eigenvalue 1 of $\mathcal{L}$. Therefore, we begin by examining the structure of 0-eigenvectors of the adjacency matrix of a tree.

Definition 2.2.1 An *every-other tree*, denoted T_{eo}, is a set of vertices of a tree of T such that

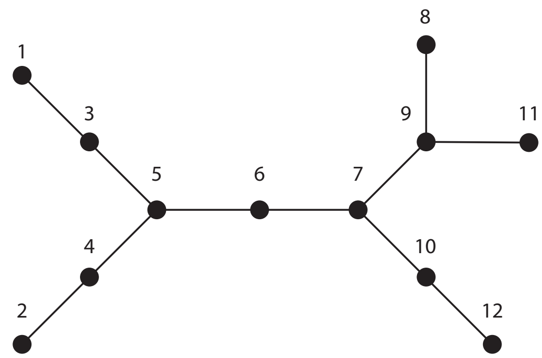

Figure 2.1 $\mathcal{T}_1 = \{8, 11\}$ and $\mathcal{T}_2 = \{1, 2, 5, 7, 8, 12\}$ are every-other trees. The leaves of $\mathcal{T}_2$ are $1, 2, 8, 12$.

1. for every $v \in T$, $|N_T(v) \cap T_{\text{eo}}| \in \{0, 2\}$, and

2. for every vertex $u \in T_{\text{eo}}$, there exists at least one vertex $w \in T_{\text{eo}}$ such that $d_T(u, w) = 2$.

Define a *leaf of* T_{eo} to be a vertex $u \in T_{\text{eo}}$ such that $d_T(u, w) > 2$ for all but one $w \in T_{\text{eo}}$. The vector $\mathbf{x}$ with ± 1 for each v in T_{eo} such that vertices in T_{eo} that are distance two apart in T have opposite sign and 0 elsewhere is the *standard representation vector for* T_{eo}. See Figure 2.1 for examples.

Note that it is a simple observation that any every-other tree must have at least two leaves. Also, the standard representation vector $\mathbf{x}$ for T_{eo} can be constructed with no conflicting signs by assigning a leaf of T_{eo} the value 1 in $\mathbf{x}$. Given the structure of T_{eo}, the rest of $\mathbf{x}$ is now completely determined. We will eventually be able to decompose 0-eigenvectors of $A(T)$ using every-other trees. For now, we will investigate the properties of an every-other tree.

Proposition 2.2.2 *Given a tree T, an every-other tree T_{eo} of T, and $v \in T_{\text{eo}}$, then*

1. v is not adjacent in T to any vertex in T_{eo}, and

2. v is a leaf of T_{eo} if and only if v is a leaf of T.

Proof.

1. Assume not. Then as v must not be adjacent to only one vertex in T_{eo} there exists two vertices u_1 and u_2 in T_{eo} such that v is adjacent to u_1 and u_2. Now, u_1 and u_2 are not adjacent and $|N_T(u_2) \cap T_{\text{eo}}| = 2$, so there exists a u_3 in T_{eo} such that u_2 and u_3 are adjacent. Continuing in this fashion we can create an infinite sequence of unique vertices, which is a contradiction.

2. Assume v is a leaf of T_{eo} and that $u \in T_{\text{eo}}$ such that $d_T(v, u) = 2$. Assume that v, w, u is the path of length 2 in T. Now, if v were to have a neighbor $w' \neq w$ in T, then w' must have a neighbor $u' \neq v, u$ in T such that $u' \in T_{\text{eo}}$, thus contradicting the fact that v is a leaf of T_{eo}. Hence, v must be a leaf of T. Now, assume that v is not a leaf of T_{eo} but that v is a leaf of T. Let w be the neighbor of v in T. Then w must have two neighbors u and u' that are in T_{eo} since v must be distance two from a least two vertices in T_{eo}. However, this implies that $|N_T(w) \cap T_{\text{eo}}| \geq 3$, which is a contradiction. □

We now make the connection between every-other trees and 0-eigenvectors of $A(T)$.

Theorem 2.2.3 *Given a tree T and an every-other tree T_{eo} of T, a standard representation vector $\mathbf{x}$ of T_{eo} is a 0-eigenvector of $A(T)$.*

Proof. Assume v is a vertex of T. If v is not adjacent to any vertex in T_{eo} then $(A\mathbf{x})(v) = 0$ by the definitions of A and $\mathbf{x}$ (note that this covers the case where $v \in T_{\text{eo}}$). If v is adjacent to u, w in T_{eo} then $(A\mathbf{x})(v) = -1 + 1 = 0$. Therefore $A\mathbf{x} = \mathbf{0}$. □

We may also decompose arbitrary 0-eigenvectors of $A(T)$ using every-other trees.

Observation 2.2.4 *Given a λ-eigenvector $\mathbf{x}$ of $A(G)$,*

$$\sum_{v \sim u} \mathbf{x}(v) = \lambda \mathbf{x}(u).$$

Proposition 2.2.5 *Given a tree T and 0-eigenvector $\mathbf{x}$ of $A(T)$,*

1. *if u and w are adjacent in T, then $\mathbf{x}(u) = 0$ or $\mathbf{x}(w) = 0$, and*

2. *T has a leaf v such that $\mathbf{x}(v) \neq 0$.*

Proof.

1. Assume not. Then by Observation 2.2.4, w has a neighbor $v_1 \neq u$ such that $\mathbf{x}(v_1) \neq 0$. Similarly, v_1 has a neighbor $v_2 \neq w$ such that $\mathbf{x}(v_2) \neq 0$. Proceeding in this fashion we derive an infinite sequence $\{v_1, v_2, \dots\}$ such that $\mathbf{x}(v_i) \neq 0$. As T is a tree, each vertex in the sequence is unique and we reach a contradiction.

2. Assume not. Then there exists a vertex v_1 such that v_1 is not a leaf and $\mathbf{x}(v_1) \neq 0$. Now none of the neighbors of v_1 are leaves else we have a contradiction to Observation 2.2.4. Let w_1 be a neighbor of v_1. Then, by Observation 2.2.4, w_1 has a neighbor v_2 other than v_1 such that $\mathbf{x}(v_2) \neq 0$. Similarly, choose w_2 a neighbor of v_2 other than w_1 (possible as $\deg v_2 \geq 2$) and w_2 must have a nonzero neighbor $v_3 \neq v_2$. Proceeding in this fashion we derive an infinite sequence $\{v_1, v_2, \dots\}$ such that $\mathbf{x}(v_i) \neq 0$ because no leaf l has $\mathbf{x}(l) \neq 0$. However, as T is a tree, each vertex in the sequence is unique and we reach a contradiction. □

The preceding proposition is also a result of [6]. Note that the preceding implies that every 0-eigenvector of $A(T)$, T a tree, has at least two nonzero leaves. This proposition gives rise to Algorithm 1. The fact that the algorithm produces an every-other tree from an arbitrary 0-eigenvector of $A(T)$ is straightforward.

Theorem 2.2.6 *For T a tree, if $\mathbf{y}$ is a 0-eigenvector of $A(T)$, then $\mathbf{y} = \alpha_1\mathbf{y}_1 + \cdots + \alpha_k\mathbf{y}_k$ where $\mathbf{y}_i$ is a standard representation vector for an every-other tree $T_{\text{eo}}^{(i)}$ of T.*

Proof. Apply Algorithm 1 to $\mathbf{y}$ to find an every-other tree $T_{\text{eo}}^{(1)}$ in the support of $\mathbf{y}$. Let $\alpha_1 = \min_{v \in T_{\text{eo}}^{(1)}}\{\mathbf{y}(v)\}$ and v_1 some vertex in $T_{\text{eo}}^{(1)}$ such that $\mathbf{y}(v_1) = \alpha_1$. Let $\mathbf{y}_1$ be a standard representation vector of $T_{\text{eo}}^{(1)}$ such that $\mathbf{y}_1(v_1) = 1$. Then $\mathbf{y} - \alpha_1\mathbf{y}_1$ is a 0-eigenvector for $A(T)$ and support of $\mathbf{y} - \alpha_1\mathbf{y}_1$ is smaller than the support of $\mathbf{y}$. Apply Algorithm 1 to $\mathbf{y} - \alpha_1\mathbf{y}_1$ to obtain $T_{\text{eo}}^{(2)}, \alpha_2, \mathbf{y}_2$ and v_2. Repeat the process until the support of $\mathbf{y}$ has been exhausted. Then $\mathbf{y} = \alpha_1\mathbf{y}_1 + \cdots + \alpha_k\mathbf{y}_k$. □

Corollary 2.2.7 *Given a tree T, if k is the dimension of the null space of $A(T)$ then there exists a set of k every-other trees $\{T_{\text{eo}}^{(1)}, \dots, T_{\text{eo}}^{(k)}\}$ such that the set of standard representation vectors for $T_{\text{eo}}^{(i)}$ is linearly independent.*

Algorithm 1 Algorithm for finding every-other trees in 0-eigenvectors of $A(T)$

Input: $\mathbf{y}$, a 0-eigenvector for $A(T)$ and ℓ, a leaf of T in the support of $\mathbf{y}$
Output: S, an every-other tree of T contained in the support of $\mathbf{y}$ and containing ℓ

```
S = {ℓ}
NextVertices = {ℓ}
stop = False
RequiredLeaves = 2
while not stop do
    Vertices = NextVertices
    NextVertices = {}
    for v ∈ Vertices do
        RequiredLeaves = RequiredLeaves + max{0, |N_T(v)| − 2}
        for w in N_T(v) \ S do                    ▷ Proposition 2.2.5 guarantees that y(w) = 0
            There is a u ∼ w such that sign(y(u)) = −sign(y(v))        ▷ Observation 2.2.4
            NextVertices = NextVertices ∪ {u}
        end for
    end for
    S = S ∪ NextVertices
    if number of leaves in S = RequiredLeaves then
        stop = True
    end if
end while
```

Proof. Assume $\mathbf{y}_1, \ldots, \mathbf{y}_k$ is a basis for the null space of $A(T)$ and $\mathbf{y}_i$ is a linear combination of $\mathbf{y}_{i1}, \ldots, \mathbf{y}_{ik_i}$ with $\mathbf{y}_{ij}$ a standard representation vector of an every-other tree. Then $\{\mathbf{y}_{ij}\}$ spans the null space of $A(T)$. Thus there exists a basis for $A(T)$ that is a subset of $\{\mathbf{y}_{ij}\}$. □

We now make the connection to the normalized Laplacian by use of harmonic eigenvectors [4]. If $\mathbf{y}$ is an eigenvector of $\mathcal{L}$ for λ, then $\mathbf{z} = D^{-1/2}\mathbf{y}$ is a *harmonic λ-eigenvector* of $\mathcal{L}$.

Observation 2.2.8 *A vector $\mathbf{z}$ is a harmonic λ-eigenvector of $\mathcal{L}$ if and only if $\mathbf{z}$ is a λ-eigenvector of $D^{-1}L$. Further, given a harmonic λ-eigenvector $\mathbf{z}$ of $\mathcal{L}$,*

$$\sum_{v \sim u} \mathbf{z}(v) = (1 - \lambda)\mathbf{z}(u)d(u).$$

Thus, a vector is a 0-eigenvector of A if and only if it is a harmonic 1-eigenvector of $\mathcal{L}$.

Proposition 2.2.9 *Given a tree T, T has a set of k every-other trees whose standard representation vectors form a linearly independent set if and only if the multiplicity of 1 as an eigenvalue of $\mathcal{L}(T)$ is at least k.*

Proof. Assume T has a set of k every-other trees whose standard representation vectors form a linearly independent set. This set induces k linearly independent 0-eigenvectors for $A(T)$ by Theorem 2.2.3. This set is also a set of k linearly independent harmonic 1-eigenvectors for $\mathcal{L}(T)$ which induces a set of k linearly independent 1-eigenvectors for $\mathcal{L}(T)$. Now assume the multiplicity of 1 as an eigenvalue of $\mathcal{L}(T)$ is at least k. Thus, the multiplicity of 0 as an eigenvalue of $A(T)$ is at least k. Thus by Corollary 2.2.7, T has a set of at least k every-other trees whose standard representation vectors form a linearly independent set. □

2.3 An example of an infinite family of cospectral pairs of trees

Now that we have established properties for lower bounding the multiplicity of 1 in the the spectrum of $\mathcal{L}$, we can use this to establish a pair of trees as cospectral with respect to the normalized Laplacian. We begin with some useful propositions.

Proposition 2.3.1 *If a P_7 (as enumerated in Figure 2.2) is an induced subgraph of a given graph G so that vertex 7 is the only vertex in P_7 with any neighbors in $G - P_7$ (i.e. $N_G(i) \subset P_7$*

for $i = 1, \ldots, 6$), then $p(t) = t^2 - 2t + 1/4$ is a factor of $p_{\mathcal{L}}(t)$, the characteristic polynomial of $\mathcal{L}(G)$. Further, $\mathcal{L}(G)$ has eigenvectors for the roots of $p(t)$ with support contained in P_7.

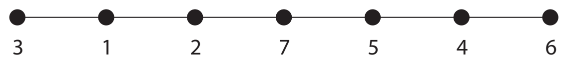

Figure 2.2 P_7

Proof. Let G be a graph with P_7 an induced subgraph as described in Proposition 2.3.1. Enumerate the vertices of G such that the P_7 is labeled as in Figure 2.2 with the remaining vertices labeled arbitrarily. Let $\lambda > \mu$ be the roots of $p(t)$. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ with entries indexed by the vertices of G. Let

$$x_i = \begin{cases} -\sqrt{3} & i = 1, \\ 1 & i = 2, 3, \\ \sqrt{3} & i = 4, \\ -1 & i = 5, 6, \\ 0 & \text{else}, \end{cases} \qquad y_i = \begin{cases} \sqrt{3} & i = 1, \\ 1 & i = 2, 3, \\ -\sqrt{3} & i = 4, \\ -1 & i = 5, 6, \\ 0 & \text{else}, \end{cases}$$

and $L_D = L(G)D(G)^{-1}$. Thus, $(L_D)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1/\deg j & \text{if } i \sim j, \text{ and} \\ 0 & \text{else.} \end{cases}$

Then a routine computation gives $L_D\mathbf{x} = \lambda\mathbf{x}$ and $L_D\mathbf{y} = \mu\mathbf{y}$. We conclude the proof by noting that $\mathcal{L} \sim L_D$ as $D^{1/2}\mathcal{L}D^{-1/2} = L_D$. □

Proposition 2.3.2 *If the graph W (see Figure 2.3) is an induced subgraph of a given graph G so that vertex 9 is the only vertex in W with any neighbors in $G - W$ (i.e. $N_G(i) \subset W$ for $i = 1, \ldots, 8$), then $p(t) = t^2 - 2t + 1/4$ is a factor of the characteristic polynomial of $\mathcal{L}(G)$, $p_{\mathcal{L}(G)}(t)$.*

Proof. Let G be a graph with W an induced subgraph as described in Proposition 2.3.2. Enumerate the vertices of G such that the W is labeled as in Figure 2.3 with the remaining

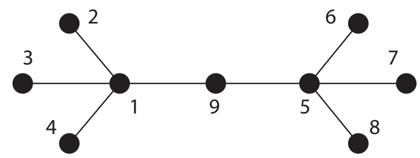

Figure 2.3 W

vertices labeled arbitrarily. Let $\lambda > \mu$ be the roots of $p(t)$. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ with entries indexed by the vertices of G. Let

$$x_i = \begin{cases} -2\sqrt{3} & i = 1, \\ 1 & i = 2,3,4, \\ 2\sqrt{3} & i = 5, \\ -1 & i = 6,7,8, \\ 0 & \text{else}, \end{cases} \qquad y_i = \begin{cases} 2\sqrt{3} & i = 1, \\ 1 & i = 2,3,4, \\ -2\sqrt{3} & i = 5, \\ -1 & i = 6,7,8, \\ 0 & \text{else}, \end{cases}$$

and $L_D = L(G)D(G)^{-1}$. Then a routine computation gives $L_D\mathbf{x} = \lambda\mathbf{x}$ and $L_D\mathbf{y} = \mu\mathbf{y}$. The result holds as $\mathcal{L}(G) \sim L_D$. □

Definition 2.3.3 Define the sequence of graphs G_k as follows: begin with the graph G_1 (Figure 2.4) on $3 + 4 \cdot 1$ vertices.

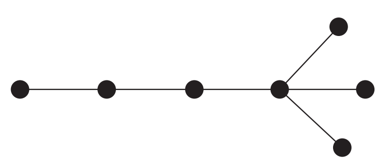

Figure 2.4 G_1

To construct G_k, a graph on $3 + 4k$ vertices, attach $k - 1$ additional P_3's and leaves to the vertex with degree > 2 (call this vertex v_{hd}). Note that $\deg v_{hd} = 2(k+1)$. Enumerate the vertices in the following fashion: let $1, 2, \ldots, k+2$ be the leaves that are adjacent to v_{hd} and $k + 3j$ for $j = 1, 2, \ldots, k$ be the non-leaf neighbors of v_{hd}. Then let $k + 3j + 1$ be the degree 2 neighbor of $k + 3j$ and $k + 3j + 2$ the degree 1 neighbor of $k + 3j + 1$. Finally, the vertex

$3+4k$ is v_{hd}. See Figures 2.5 and 2.6 for G_2 (which is shown with enumerated vertices) and G_3 respectively.

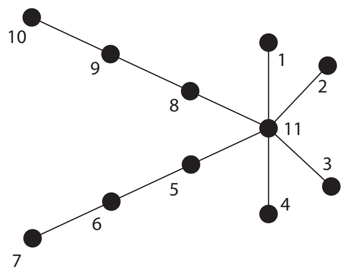

Figure 2.5 G_2

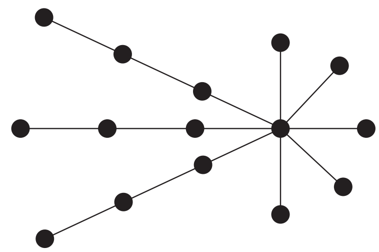

Figure 2.6 G_3

Proposition 2.3.4 *$p_k(t) = t^2 - 2t + \frac{2k+1}{4k+4}$ is a factor of $p_{\mathcal{L}(G_k)}(t)$.*

Proof. The following is a construction of the eigenvectors of $L_D := L(G_k)D(G_k)^{-1}$ for $\lambda > \mu$, the roots of $p_k(t)$. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ with entries indexed by the vertices of G_k. Let

$$x_i = \begin{cases} -k & i = 1,2,\ldots,k+2, \\ 1 & i = k+3j, \\ -a_k & i = k+3j+1, \\ k+1 & i = k+3j+2, \\ ka_k & i = 3+4k, \end{cases} \qquad y_i = \begin{cases} -k & i = 1,2,\ldots,k+2, \\ 1 & i = k+3j, \\ a_k & i = k+3j+1, \\ k+1 & i = k+3j+2, \\ -ka_k & i = 3+4k, \end{cases}$$

for $j = 1,2,\ldots,k$ and $a_k = \sqrt{(2k+3)(k+1)}$. Clearly the roots of $p_k(t)$ are

$$1 \pm \sqrt{\frac{2k+3}{4(k+1)}} = 1 \pm \frac{a_k}{\deg v_{hd}}$$

For $i = 1,2,\ldots,k+2$,

$$(L_D\mathbf{x})_i = -k - \frac{ka_k}{\deg v_{hd}} = -k\lambda = (\lambda\mathbf{x})_i$$

as i is only adjacent to v_{hd}. For $i = k+3j$ for $j = 1,2,\ldots,k$,

$$(L_D\mathbf{x})_i = 1 + \frac{1}{2}a_k - \frac{ka_k}{\deg v_{hd}} = 1 + (\lambda-1)(k+1) - k(\lambda-1) = \lambda = (\lambda\mathbf{x})_i$$

as i is adjacent to $i+1$ (the middle of a P_3) and v_{hd}. For $i = k+3j+1$ for $j = 1, 2, \ldots, k$,

$$(L_D\mathbf{x})_i = -\frac{1}{2} - a_k - (k+1) = -a_k\lambda = (\lambda\mathbf{x})_i$$

since i is adjacent to $i-1$, a non-leaf neighbor of v_{hd} and $i+1$, a leaf which is not adjacent to v_{hd}.

For $i = k+3j+2$ with $j = 1, 2, \ldots, k$,

$$(L_D\mathbf{x})_i = \frac{a_k}{2} + (k+1) = \frac{\deg v_{hd}(\lambda - 1) + \deg v_{hd}}{2} = \frac{\lambda \deg v_{hd}}{2} = (k+1)\lambda = (\lambda\mathbf{x})_i$$

as i is only adjacent to $i-1$, the middle of a P_3. Lastly, for $i = 3+4k$,

$$(L_D\mathbf{x})_i = k(k+2) - \frac{k}{2} + ka_k = k\left(\frac{1}{2} + a_k + (k+1)\right) = ka_k\lambda = (\lambda\mathbf{x})_i$$

since i is adjacent to $k+2$ leaves, which have value $-k$ in $\mathbf{x}$, and k degree 2 vertices, which have value 1 in $\mathbf{x}$. Thus, we have $L_D\mathbf{x} = \lambda\mathbf{x}$. Analogous computations show that $L_D\mathbf{y} = \mu\mathbf{y}$.
□

Definition 2.3.5 Define the sequence of graphs H_k as follows: begin with the graph $H_1 \equiv G_1$ on $3 + 4 \cdot 1$ vertices with the vertices labeled as shown in Figure 2.7.

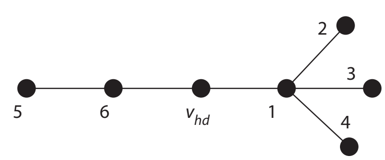

Figure 2.7 H_1

To construct H_k, a graph on $3+4k$ vertices, add $k-1$ additional copies of $H_1[\{1,2,3,4\}]$ and add an edge between v_{hd} and vertex 1 in each copy of $H_1[\{1,2,3,4\}]$. Note that $\deg v_{hd} = k+1$. Enumerate the vertices in the following fashion: let $1+4j$ for $j = 0, 1, \ldots, k-1$ be the degree 4 neighbors of v_{hd} and let $2+4j$, $3+4j$, and $4+4j$ be the neighbors of $1+4j$. Finally, let $4k+1$ be the remaining leaf in G_k, $4k+2$ the degree 2 vertex and $4k+3$ be v_{hd}. See Figures 2.8 and 2.9 for H_2 (which is shown with enumerated vertices) and H_3 respectively.

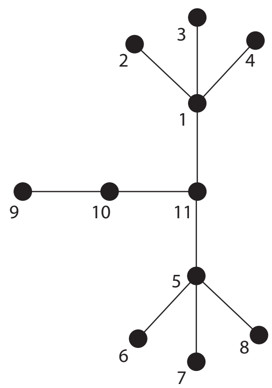

Figure 2.8 H_2

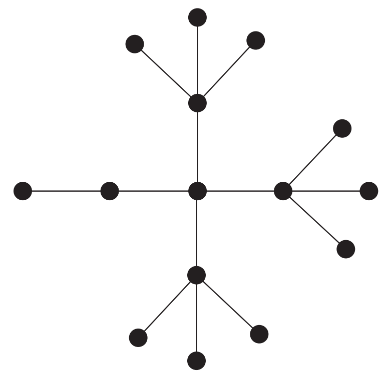

Figure 2.9 H_3

Proposition 2.3.6 *$p_k(t) = t^2 - 2t + \frac{2k+1}{4k+4}$ is a factor of $p_{\mathcal{L}(H_k)}(t)$.*

Proof. The following is a construction of the eigenvectors of $L_D := L(H_k)D(H_k)^{-1}$ for $\lambda > \mu$, the roots of $p_k(t)$. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ with entries indexed by the vertices of H_k. Let

$$x_i = \begin{cases} b_k & i = 1+4j, \\ -1 & i = 2+4j, 3+4j, 4+4j, \\ 2k & i = 4k+1, \\ -kb_k & i = 4k+2, \\ k & i = 4k+3, \end{cases} \qquad y_i = \begin{cases} b_k & i = 1+4j, \\ 1 & i = 2+4j, 3+4j, 4+4j, \\ -2k & i = 4k+1, \\ -kb_k & i = 4k+2, \\ -k & i = 4k+3, \end{cases}$$

for $j = 0, 1, \ldots, k-1$ and $b_k = \sqrt{\frac{4(2k+3)}{k+1}}$. Thus, $\lambda = 1 + \frac{1}{4}b_k,\ \mu = 1 - \frac{1}{4}b_k$. For $i = 1+4j$ with $j = 0, 1, \ldots, k-1$,

$$(L_D\mathbf{x})_i = b_k + 3 - \frac{k}{k+1} = b_k + \frac{2k+3}{k+1} = b_k + \frac{1}{4}{b_k}^2 = (\lambda\mathbf{x})_i$$

as i is adjacent to $i+1,\ i+2,\ i+3,$ and $4k+3$. For $i = 2+4j,\ 3+4j,$ or $4+4j$ with $j = 0, 1, \ldots, k-1$,

$$(L_D\mathbf{x})_i = -\frac{1}{4}b_k - 1 = -\lambda = (\lambda\mathbf{x})_i$$

as i is only adjacent to $1+4j$. For $i = 4k+1$,

$$(L_D\mathbf{x})_i = 2k + \frac{k}{2}b_k = 2k\left(1 + \frac{1}{4}b_k\right) = 2k\lambda = (\lambda\mathbf{x})_i$$

as i is only adjacent to $i+1$. For $i = 4k+2$,

$$(L_D\mathbf{x})_i = -2k - kb_k - \frac{k}{k+1} = -k\left(b_k + \frac{2k+3}{k+1}\right) = -k\left(b_k + \frac{1}{4}{b_k}^2\right) = -kb_k\lambda = (\lambda\mathbf{x})_i$$

as i is adjacent to $i-1$ and $i+1$. For $i = 4k+3$,

$$(L_D\mathbf{x})_i = -\frac{k}{4}b_k + \frac{k}{2}b_k + k = k\left(1 + \frac{1}{4}b_k\right) = k\lambda = (\lambda\mathbf{x})_i$$

as i is adjacent to $1+4j$ with $j = 0, 1, \ldots, k-1$ and to $i-1$. Thus, we have $L_D\mathbf{x} = \lambda\mathbf{x}$. Similar computations show that $L_D\mathbf{y} = \mu\mathbf{y}$. We conclude by noting that $\mathcal{L}(H_k) \sim L_D$. ∎

Theorem 2.3.7 *For all positive integers k, G_k and H_k have the same spectrum (including multiplicity) with respect to the normalized Laplacian.*

Proof. Consider the graphs G_k and H_k for a given k. Both G_k and H_k are bipartite graphs, so t and $t-2$ are factors of both $p_{\mathcal{L}(G_k)}(t)$ and $p_{\mathcal{L}(H_k)}(t)$. The graph G_k has a set of $2k+1$ every-other trees whose standard representation vectors form a linearly independent set: $\{1,2\}$ and $\{1, j+2\}, \{1, k+3j, k+3j+2\}$ for $j = 1, 2, \ldots, k$. Thus $(t-1)^{2k+1}$ is a factor of $p_{\mathcal{L}(G_k)}(t)$ by Proposition 2.2.9. Similarly, H_k also has $2k+1$ every-other trees whose standard representation vectors form a linearly independent set: $\{2, 6, \ldots, 2+4(k-1)\} \cup \{4k+1, 4k+3\}$ and $\{2+4j, 3+4j\}, \{2+4j, 4+4j\}$ for $j = 0, 1, \ldots, k-1$. Hence, $(t-1)^{2k+1}$ is a factor of $p_{\mathcal{L}(H_k)}(t)$ as well.

The graph G_k has $k-1$ copies of the P_7 described in Proposition 2.3.1 beginning at vertex $k+5$ and ending at vertex $5+k+3j$ for $j = 1, \ldots, k-1$. These induce $k-1$ linearly independent eigenvectors of the eigenvalues described in Proposition 2.3.1, so $(t^2 - 2t + 1/4)^{k-1}$ is a factor of $p_{\mathcal{L}(G_k)}(t)$. Similarly, H_k has $k-1$ copies of the graph W as described in Proposition 2.3.2 containing the vertices $1, 2, 3, 4, 3+4k$ and $4j+1, 4j+2, 4j+3, 4j+4$ for $j = 1, \ldots, k-1$, so $(t^2 - 2t + 1/4)^{k-1}$ is a factor of $p_{\mathcal{L}(H_k)}(t)$ as well. Finally, Propositions 2.3.4 and 2.3.6 guarantee that $p_k(t)$ is a factor of both $p_{\mathcal{L}(G_k)}(t)$ and $p_{\mathcal{L}(H_k)}(t)$. This is the complete factorization of $p_{\mathcal{L}(G_k)}(t)$ and $p_{\mathcal{L}(H_k)}(t)$ as $\deg(p_{\mathcal{L}(G_k)}(t)) = \deg(p_{\mathcal{L}(H_k)}(t)) = 4k+3$. Hence

$$p_{\mathcal{L}(G_k)}(t) = p_{\mathcal{L}(H_k)}(t) = t(t-2)(t-1)^{2k+1}\left(t^2 - 2t + \frac{1}{4}\right)^{k-1}\left(t^2 - 2t + \frac{2k+1}{4k+4}\right). \; \blacksquare$$

2.4 Using the weighted normalized Laplacian

A weighted graph is a graph in which each edge is assigned a positive value and each non-edge is assigned 0. The degree of a vertex is the sum of its incident edge weights. We define the adjacency matrix $A(\mathcal{G}) = (a_{ij})$ for a weighted graph $\mathcal{G}$ with vertices $\{v_1, v_2, \ldots, v_n\}$, edges $e_{ij} = \{v_i, v_j\}$, and edge weights w_{ij} by $a_{ij} = w_{ij}$. The normalized Laplacian matrix associated with a weighted graph $\mathcal{G}$ is $\mathcal{L}(\mathcal{G}) := I - D^{-1/2}A(\mathcal{G})D^{-1/2}$ where $D = \text{diag}(\deg v_1, \deg v_2, \ldots, \deg v_n)$. The definition of harmonic eigenvectors of $\mathcal{L}(\mathcal{G})$ is analogous to the earlier definition with non-weighted graphs, i.e. if $\mathbf{y}$ is an eigenvector of $\mathcal{L}(\mathcal{G})$ for λ, then $\mathbf{z} = D^{-1/2}(\mathcal{G})\mathbf{y}$ is a harmonic λ-eigenvector of $\mathcal{L}(\mathcal{G})$. Note that if G is an unweighted graph and $\mathcal{G}$ is a weighted graph with the same vertices and edges as G in which each edge is given weight 1, then $A(G) = A(\mathcal{G})$ and $\mathcal{L}(G) = \mathcal{L}(\mathcal{G})$.

Proposition 2.4.1 *Assume G is a graph, $v_1, \ldots, v_k$ are pairwise twins for some $k \geq 1$ and $\mathcal{G}$ is G where each edge is given weight 1. Then*

$$p_{\mathcal{L}(G)}(t) = (t-1)^{k-1} p_{\mathcal{L}(\mathcal{G}')}(t)$$

where $\mathcal{G}'$ is $\mathcal{G}$ with $v_1, \ldots, v_{k-1}$ removed and the weights of the edges incident with v_k multiplied by k.

Proof. It is a straight forward verification to show that the vector $\mathbf{x}^{(i)}$ with

$$\mathbf{x}^{(i)}(v_j) = \begin{cases} 1 & \text{if } j = k \\ -1 & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

is a 1-eigenvector of $\mathcal{L}(\mathcal{G})$ for $i = 1, \ldots, k-1$. We may form the remaining eigenvectors of $\mathcal{L}(\mathcal{G})$, $\mathbf{y}^{(\ell)}$ for eigenvalue μ_ℓ, so that $\mathbf{y}^{(\ell)} \perp \mathbf{x}^{(j)}$ for all ℓ and j. Hence, $\mathbf{y}^{(\ell)}(v_j) = c_\ell$ for $j = 1, \ldots, k$ for some constant c_ℓ. Consider the harmonic eigenvectors which correspond to the vectors $\mathbf{w}^{(\ell)} \in \mathbb{R}^{n-k+1}$ where $\mathbf{w}^{(\ell)}(v_j) = \mathbf{y}^{(\ell)}(v_j)$ for $j = k, \ldots, n$, i.e., $D^{-1/2}\mathbf{w}^{(\ell)}$. Since $\deg_{\mathcal{G}} v = \deg_{\mathcal{G}'} v$, Observation 2.2.8 gives $\mathcal{L}(\mathcal{G}')\mathbf{w}^{(\ell)} = \mu_\ell \mathbf{w}^{(\ell)}$. The result follows. ∎

2.4.1 Characterization of diameter 3 trees

Using this knowledge of the weighted normalized Laplacian we may classify trees which are cospectral with diameter three trees with respect to the normalized Laplacian. The *weighted path* $(a_1, a_2, \ldots, a_{n-1})$ is the weighted graph with vertices $\{v_1, \ldots, v_n\}$, edges $\{e_{12}, e_{23}, \ldots, e_{n-1\,n}\}$ and edges weights $\{w_{12} = a_1, \ldots, w_{n-1\,n} = a_{n-1}\}$.

Proposition 2.4.2 *The characteristic polynomial of the weighted path* $(m, 1, n-m-2)$ *is*

$$p_1(t) = t(t-2)\left(t^2 - 2t + \frac{n-1}{(n-m-1)(m+1)}\right)$$

and the characteristic polynomial of the weighted path $(m, 1, 1, n-m-3)$ *is*

$$p_2(t) = t(t-2)(t-1)\left(t^2 - 2t + \frac{n-1}{2(n-m-2)(m+1)}\right).$$

Proof. Via the similarity $\mathcal{L}(\mathcal{G}) \sim L(\mathcal{G})D^{-1}(\mathcal{G})$, we have

$$\mathcal{L}((m,1,n-m-2)) \sim \begin{bmatrix} 1 & -\frac{m}{m+1} & 0 & 0 \\ -1 & 1 & \frac{1}{m-n+1} & 0 \\ 0 & -\frac{1}{m+1} & 1 & -1 \\ 0 & 0 & -\frac{m-n+2}{m-n+1} & 1 \end{bmatrix}$$

which has the characteristic polynomial $p_1(t)$ and

$$\mathcal{L}((m,1,1,n-m-3)) \sim \begin{bmatrix} 1 & -\frac{m}{m+1} & 0 & 0 & 0 \\ -1 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{m+1} & 1 & \frac{1}{m-n+2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & -\frac{m-n+3}{m-n+2} & 1 \end{bmatrix}$$

which has the characteristic polynomial $p_2(t)$. ∎

Proposition 2.4.3 *1. If T_1 is a tree with diameter 3 on n vertices (i.e. of the form in Figure 2.10), then*

$$p_{\mathcal{L}(T_1)}(t) = t(t-2)(t-1)^{n-4}\left(t^2 - 2t + \frac{n-1}{(n-m-1)(m+1)}\right)$$

for some $1 \leq m \leq \lfloor n/2 \rfloor - 1$.

2. If T_2 is a tree with diameter 4 on n vertices and of the form in Figure 2.11, then

$$p_{\mathcal{L}(T_2)}(t) = t(t-2)(t-1)^{n-4}\left(t^2 - 2t + \frac{n-1}{2(n-m-2)(m+1)}\right)$$

for some $1 \leq m \leq \lfloor n/2 \rfloor - 1$.

Proof

1. Since T_1 is a diameter 3 tree , then it has the structure shown in Figure 2.10 for some $1 \leq m_1 \leq \lfloor n/2 \rfloor - 1$. Thus by Proposition 2.4.1, T_1 has the characteristic polynomial $(t-1)^{n-4} p_{\mathcal{L}(\mathcal{T}_1')}(t)$ where $\mathcal{T}_1'$ is the weighted path $(m, 1, n-m-2)$. The result follows from Proposition 2.4.2.

2. T_2 has the structure shown in Figure 2.11 for some $1 \leq m \leq \lfloor n/2 \rfloor - 1$. Thus by Proposition 2.4.1, T_2 has the characteristic polynomial $(t-1)^{n-5} p_{\mathcal{L}(\mathcal{T}_2')}(t)$ where $\mathcal{T}_2'$ is the weighted path $(m, 1, 1, n-m-3)$. The result follows from Proposition 2.4.2. □

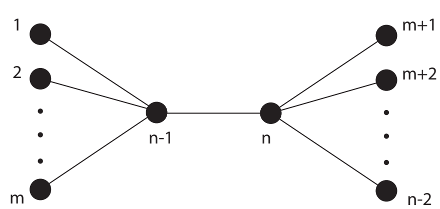

Figure 2.10 T_1

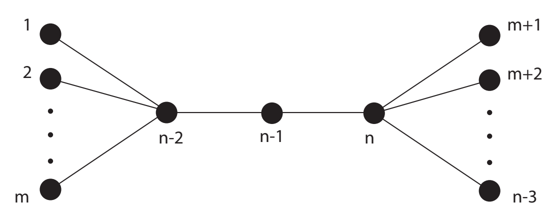

Figure 2.11 T_2

Corollary 2.4.4 *The following are consequences of Proposition 2.4.3.*

1. The diameter 3 tree on $7k+5$ vertices with degree sequence $(2+4k, 3+3k, 1, \ldots, 1)$ is cospectral with the diameter 4 tree of the form T_2 on $7k+5$ vertices with degree sequence $(3+6k, k+1, 2, 1, \ldots, 1)$.

2. *The diameter 3 tree on $7k+6$ vertices with degree sequence $(4+4k, 2+3k, 1, \ldots, 1)$ is cospectral with the diameter 4 tree of the form T_2 on $7k+6$ vertices with degree sequence $(4+6k, k+1, 2, 1, \ldots, 1)$.*

Figures 2.12 and 2.13 are cospectral graphs by Corollary 2.4.3 case 1, and Figures 2.14 and 2.15 are cospectral graphs by Corollary 2.4.3 case 2.

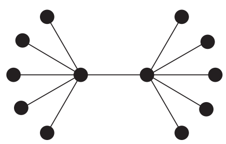

Figure 2.12 T_1

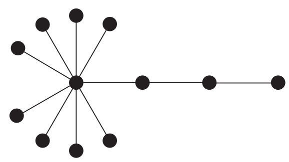

Figure 2.13 T_2

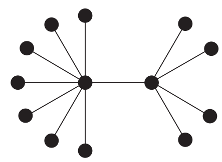

Figure 2.14 T_1

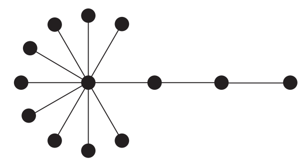

Figure 2.15 T_2

Theorem 2.4.5 *No two distinct diameter 3 trees are cospectral.*

Proof. Assume S_1 and S_2 are cospectral diameter 3 trees. Without loss of generality, by Proposition 2.4.3,

$$p_{\mathcal{L}(S_1)}(t) = t(t-2)(t-1)^{n-4}\left(t^2 - 2t - \frac{n-1}{(n-m_1-1)(m_1+1)}\right)$$

$$p_{\mathcal{L}(S_2)}(t) = t(t-2)(t-1)^{n-4}\left(t^2 - 2t - \frac{n-1}{(n-m_2-1)(m_2+1)}\right)$$

for $1 \leq m_1 \leq m_2 \leq \lfloor n/2 \rfloor - 1$. As S_1 and S_2 are cospectral, $(n-m_1-1)(m_1+1) = (n-m_2-1)(m_2+1)$. For purposes of contradiction, assume $m_1 + k = m_2$ for some $k \geq 1$. Then

$$\begin{aligned}
(n-m_2+k-1)(m_2-k+1) &= (n-m_2-1)(m_2+1) \\
(n-m_2-1)(m_2-k+1) + k(m_2-k+1) &= (n-m_2-1)(m_2+1) \\
(n-m_2-1)(m_2+1) + k(2m_2+2-n-k) &= (n-m_2-1)(m_2+1) \\
k(2m_2+2-n-k) &= 0 \\
2(m_2+1)-n-k &= 0
\end{aligned}$$

However, $m_2 + 1 \leq n/2$, thus $2(m_2+1) - n \leq 0$. But $-k$ is strictly negative and we arrive at a contradiction. Therefore, $m_1 = m_2$ and S_1 and S_2 are isomorphic. $\square$

Theorem 2.4.6 *The only trees which are cospectral with a diameter 3 tree are of the form T_2 in Figure 2.11.*

Proof. Assume that T_1 is a diameter 3 tree on n vertices and T is a tree cospectral with T_1. The multiplicity of 1 as an eigenvalue of $\mathcal{L}(T_1)$ is $n-4$ by Proposition 2.4.3. Thus T must have diameter greater than 3 by Theorem 2.4.5. Trees with diameter greater than 4 have at least 6 distinct eigenvalues, thus the multiplicity of 1 is at most $n-5$. Therefore T has diameter 4. Let v be the vertex in T such that $d_T(v,u) \leq 2$ for all $u \in T$.

Case 1: v has a leaf as a neighbor. No every-other tree in T contains v. Thus if T_{eo} is an every-other tree in T, $T_{\text{eo}} = \{v_1, v_2\}$ for v_1, v_2 leaves of the same vertex. There are are most $n-6$ every-other trees of this form whose standard representations form a linearly independent set of vectors. Thus, the multiplicity of 1 is at most $n-6$.

Case 2: v does not have a leaf as a neighbor. Assume that $v_1, v_2, \ldots, v_\ell$ are the neighbors of v and $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ are the neighbors of v_i. Let $T_{\text{eo}}^{(0)} = \{v, v_{11}, v_{21}, \ldots, v_{\ell 1}\}$ and $T_{\text{eo}}^{(ij)} = \{v_{i1}, v_{ij}\}$ for $i = 1, \ldots, \ell$ and $j = 2, \ldots, k_i$. Let $\mathbf{y}_0$ be the standard representation vector of $T_{\text{eo}}^{(0)}$ with $\mathbf{y}_0(v) = -1$ and $\mathbf{y}_{ij}$ be the standard representation vector of $T_{\text{eo}}^{(ij)}$ with $\mathbf{y}_{ij}(v_{i1}) = 1$. The set $I = \{\mathbf{y}_0\} \cup \{\mathbf{y}_{ij} : i = 1, \ldots, \ell,\ j = 2, \ldots, k_\ell\}$ is linearly independent. Assume T_{eo} is an every-other tree of T. If T_{eo} does not contain v, then $T_{\text{eo}} = \{v_{ij}, v_{ij'}\}$ for some i, j, j'. Thus, any standard representation vector of T_{eo} is the linear combination of $\mathbf{y}_{ij}$ and $\mathbf{y}_{ij'}$. If T_{eo} does contain v then $T_{\text{eo}} = \{v, v_{1j_1}, v_{2j_2}, \ldots, v_{\ell j_\ell}\}$ for some $j_1, \ldots, j_\ell$. Thus, any standard representation vector of T_{eo} is the linear combination of $\mathbf{y}_0, \mathbf{y}_{1j_1}, \ldots, \mathbf{y}_{\ell j_\ell}$. Thus I is a basis for the space of standard representation vectors of all every-other trees of v. By construction, $|I| = n - 2\ell$. Therefore, the multiplicity of 1 as an eigenvector of $\mathcal{L}(T)$ is at most $n - 2\ell$ by Corollary 2.2.7 so $\ell = 2$, i.e. $T \equiv T_2$. □

There are diameter three trees which are cospectral with non-trees. See Figure 2.16 for the smallest example. The non-tree graphs in Figure 2.16 give rise to another interesting application of the weighted normalized Laplacian.

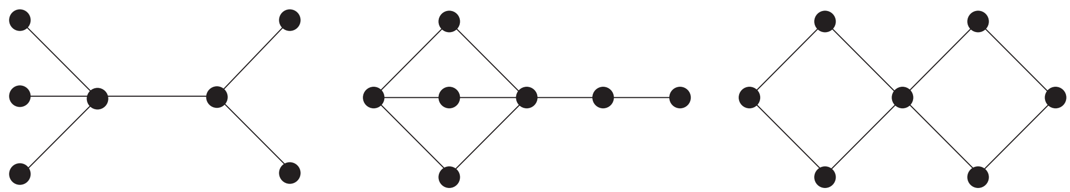

Figure 2.16 A cospectral family

2.4.2 Balanced two-weighted paths

It is well known that the eigenvalues of the adjacency matrix of a cycle on n vertices, C_n, are $\lambda_k = \omega^k + \omega^{(n-1)k}$ for $k = 0, 1, \ldots, n-1$ with $\omega = e^{2\pi i/n}$. The eigenvector for λ_k can be described as $\mathbf{v}^{(k)} = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})^T$. Now, as C_n is 2-regular, $\mathcal{L}(C_n)\mathbf{v}^{(k)} = \left(1 - \frac{1}{2}\lambda_k\right)\mathbf{v}^{(k)}$. Define $\mu_k = 1 - \frac{1}{2}\lambda_k$.

Proposition 2.4.7 *Assume that $n = 2\ell$ and $\hat{\mathbf{v}}^{(k)}$ is a vector of length $\ell + 1$ with*

$$\hat{v}_i^{(k)} = \begin{cases} \sqrt{2} & i = 1 \\ \omega^{(i-1)k} + \omega^{(n-i+1)k} & i = 2, \ldots, \ell \\ (-1)^k\sqrt{2} & i = \ell + 1 \end{cases}$$

Then $\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)} = \mu_k \hat{\mathbf{v}}^{(k)}$ where $P_{\ell+1}$ is the path on $\ell + 1$ vertices.

Proof. For $i =$

$$1: \qquad \left(\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)}\right)_1 = \sqrt{2} - \frac{1}{\sqrt{2}}\left(\omega^k + \omega^{(n-1)k}\right) = \sqrt{2}\mu_k = \mu_k \hat{v}_1^{(k)},$$

$$2: \qquad \left(\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)}\right)_2 = -1 + \omega^k + \omega^{(n-1)k} - \frac{1}{2}\omega^{2k} - \frac{1}{2}\omega^{(n-2)k} = \mu_k \hat{v}_2^{(k)},$$

$$3, \ldots, \ell-1: \qquad \left(\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)}\right)_i = -\frac{1}{2}\hat{v}_{i-1}^{(k)} + \hat{v}_i^{(k)} - \frac{1}{2}\hat{v}_{i+1}^{(k)} =$$

$$-\frac{1}{2}\left(\omega^{(i-2)k} + \omega^{(n-i+2)k}\right) + \omega^{(i-1)k} + \omega^{(n-i+1)k} - \frac{1}{2}\left(\omega^{ik} + \omega^{(n-i)k}\right) = \mu_k \hat{v}_i^{(k)},$$

$$\ell: \qquad \left(\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)}\right)_\ell = -\frac{1}{2}\left(\omega^{(\ell-2)k} + \omega^{(\ell+2)k}\right) + \omega^{(\ell-1)k} + \omega^{(\ell+1)k} - (-1)^k = \mu_k \hat{v}_\ell^{(k)},$$

$$\ell+1: \qquad \left(\mathcal{L}(P_{\ell+1})\hat{\mathbf{v}}^{(k)}\right)_{\ell+1} = -\frac{1}{\sqrt{2}}\left(\omega^{(\ell-1)k} + \omega^{(\ell+1)k}\right) + \sqrt{2}(-1)^k =$$

$$\sqrt{2}\left(-\frac{1}{2}(-1)^k\omega^{(n-1)k} - \frac{1}{2}(-1)^k\omega^k + (-1)^k\right) = (-1)^k\sqrt{2}\mu_k = \mu_k \hat{v}_{\ell+1}^{(k)}.$$

The result follows. ∎

Proposition 2.4.8 *If* $\hat{\mathbf{v}}^{(k)}$ *is defined as in Proposition 2.4.7,* $\hat{v}_i^{(k)} = (-1)^k \hat{v}_{l-i+2}^{(k)}$ *for* $i = 1, 2, \ldots, \lceil l/2 \rceil$.

Proof. This is clear for $i = 1$. Thus if $i > 1$,

$$\hat{v}_{l-i+2}^{(k)} = \omega^{(l-i+1)k} + \omega^{(l+i-1)k} = \omega^{(n-i+1)k+lk} + \omega^{(i-1)k+lk} = (-1)^k \hat{v}_i^{(k)}.$$

∎

Definition 2.4.9 For $a, b > 0$, the *balanced two-weighted path* $P_{s,a,b}$ is the weighted path on $2s+1$ vertices,

$(a, a, \ldots, a, b, b, \ldots, b)$, such that a and b both appear s times.

We will now show that weighting the path in this manner does not affect the spectrum.

Proposition 2.4.10 *Assume $\ell = 2s$ and $\dot{\mathbf{v}}^{(k)}$ is a vector of length $\ell+1$ with*

$$\dot{v}_i^{(k)} = \begin{cases} \hat{v}_i^{(k)} & i = 1, \ldots, s \\ \frac{\sqrt{a+1}}{\sqrt{2a}}\hat{v}_i^{(k)} & i = s+1 \\ \frac{1}{\sqrt{a}}\hat{v}_i^{(k)} & i = s+2, \ldots, 2s+1 \end{cases}$$

for k even and

$$\dot{v}_i^{(k)} = \begin{cases} \hat{v}_i^{(k)} & i = 1, \ldots, s \\ \frac{\sqrt{a+1}}{\sqrt{2a}}\hat{v}_i^{(k)} & i = s+1 \\ \sqrt{a}\hat{v}_i^{(k)} & i = s+2, \ldots, 2s+1 \end{cases}$$

for k odd where $\hat{v}_i^{(k)}$ is defined as in Proposition 2.4.7. Then $\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)} = \mu_k \dot{\mathbf{v}}^{(k)}$ for all $k = 0, 1, \ldots, \ell$. Hence, $P_{s,a,1}$ and P_{2s+1} are cospectral.

Proof. By the normalization, $(\mathcal{L}(P_{s,a,1}))_{ij} = (\mathcal{L}(P_{2s+1}))_{ij}$ for all pairs

$$(i,j) \notin \{(s, s+1), (s+1, s), (s+1, s+2), (s+2, s+1)\}.$$

Assume k is even. Then $\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_i = \left(\mathcal{L}(P_{s,a,1})\hat{\mathbf{v}}^{(k)}\right)_i = \mu_k \hat{v}_i^{(k)} = \mu_k \dot{v}_i^{(k)}$ for $i = 1, \ldots, s-1$. And $\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_i = \frac{1}{\sqrt{a}}\left(\mathcal{L}(P_{s,a,1})\hat{\mathbf{v}}^{(k)}\right)_i = \frac{1}{\sqrt{a}}\mu_k \hat{v}_i^{(k)} = \mu_k \dot{v}_i^{(k)}$ for $i = s+3, \ldots, 2s+1$. Therefore, we only need to consider $i = s, s+1, s+2$. For $i = s$,

$$\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_s = -\frac{1}{2}\dot{v}_{s-1}^{(k)} + \dot{v}_s^{(k)} - \frac{\sqrt{a}}{\sqrt{2(a+1)}}\dot{v}_{s+1}^{(k)} = -\frac{1}{2}\hat{v}_{s-1}^{(k)} + \hat{v}_s^{(k)} - \frac{1}{2}\hat{v}_{s+1}^{(k)} = \mu_k \hat{v}_s^{(k)} = \mu_k \dot{v}_s^{(k)}.$$

For $i = s+1$, (Note: below we use the fact that $\hat{v}_s^{(k)} = \hat{v}_{s+2}^{(k)}$ from Proposition 2.4.8)

$$\begin{aligned} \left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_{s+1} &= -\frac{\sqrt{a}}{\sqrt{2(a+1)}}\dot{v}_s^{(k)} + \dot{v}_{s+1}^{(k)} - \frac{1}{\sqrt{2(a+1)}}\dot{v}_{s+2}^{(k)} \\ &= -\frac{\sqrt{a}}{\sqrt{2(a+1)}}\hat{v}_s^{(k)} + \frac{\sqrt{a+1}}{\sqrt{2a}}\hat{v}_{s+1}^{(k)} - \frac{1}{\sqrt{2a(a+1)}}\hat{v}_{s+2}^{(k)} \\ &= \frac{\sqrt{a+1}}{\sqrt{2a}}(-\hat{v}_s^{(k)} + \hat{v}_{s+1}^{(k)}) \\ &= \frac{\sqrt{a+1}}{\sqrt{2a}}\mu_k \hat{v}_{s+1}^{(k)} = \mu_k \dot{v}_{s+1}^{(k)}. \end{aligned}$$

For $i = s+2$,

$$\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_{s+2} = -\frac{1}{\sqrt{2(a+1)}}\dot{v}_{s+1}^{(k)} + \dot{v}_{s+2}^{(k)} - \frac{1}{2}\dot{v}_{s+3}^{(k)} = \frac{1}{\sqrt{a}}\mu_k \hat{v}_{s+2}^{(k)} = \mu_k \dot{v}_{s+2}^{(k)}.$$

Now assume k is odd. As before, we only need to consider the cases where $i = s, s+1, s+2$. For $i = s$, the calculations are the same as when k was even. For $i = s+1$, using the fact that $\hat{v}_s^{(k)} = -\hat{v}_{s+2}^{(k)}$,

$$\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_{s+1} = \frac{\sqrt{a}}{\sqrt{2(a+1)}}\left(-\hat{v}_s^{(k)} + \hat{v}_{s+1}^{(k)} - \hat{v}_{s+2}^{(k)}\right) = \mu_k \dot{v}_{s+1}^{(k)}.$$

Finally, for $i = s+2$,

$$\left(\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)}\right)_{s+2} = \frac{1}{\sqrt{a}}\left(-\frac{1}{2}\hat{v}_{s+1}^{(k)} + \hat{v}_{s+2}^{(k)} - \frac{1}{2}\hat{v}_{s+3}^{(k)}\right) = \frac{1}{\sqrt{a}}\mu_k \hat{v}_{s+2}^{(k)} = \mu_k \dot{v}_{s+2}^{(k)}.$$

Therefore, $\mathcal{L}(P_{s,a,1})\dot{\mathbf{v}}^{(k)} = \mu_k \dot{\mathbf{v}}^{(k)}$. □

Given a weighted graph, scaling all the edge weights by a fixed amount does not change the spectrum of the normalized Laplacian. Thus we get the following corollary.

Corollary 2.4.11 *$P_{s,a,b}$ and P_{2s+1} are cospectral for the normalized Laplacian for any $a, b \in \mathbb{N}$.*

2.4.3 An example of cospectral bipartite graphs

We will now examine an application of using two-weighted paths to determine the spectrum of a simple graph.

Definition 2.4.12 *Consider the path P_{2s+1} with vertices $v_1, \ldots, v_{2s+1}$ and edges $e_{12}, \ldots, e_{2s\ 2s+1}$. The graph $D_{s,a,b}$ on $n := (a+b+2)s+1$ vertices is the graph P_{2s+1} in which $a-1$ twins of v_{s-2k} for $k = 0, 1, \ldots, \lceil s/2 \rceil$ have been added and $b-1$ twins of v_{s+2k} for $k = 1, 2, \ldots, \lceil s/2 \rceil$ have been added.*

Theorem 2.4.13 *$D_{s,a,b}$ has the characteristic polynomial*

$$p_{\mathcal{L}(D_{s,a,b})}(t) = (t-1)^{n-(2s+1)} p_{\mathcal{L}(P_{2s+1})}(t).$$

Proof. Apply Proposition 2.4.1 to $D_{s,a,b}$ iteratively until no twins remain. The resulting graph is $P_{s,a,b}$. Corollary 2.4.11 completes the proof. □

Note that Theorem 2.4.13 implies the spectrum of $D_{s,a,b}$ depends only on s and n. The spectrum of the graph in Figure 2.17 is related to the spectrum of the graph in Figure 2.19 by Theorem 2.4.13. The graphs in Figures 2.19 and 2.20 are cospectral by Corollary 2.4.11. And the spectrum of the graph in Figure 2.18 is related to the spectrum of the graph in Figure 2.20 by Theorem 2.4.13. Thus, the graphs in Figures 2.17 and 2.18 are cospectral.

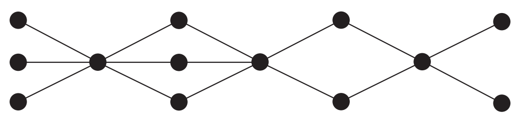

Figure 2.17 $D_{3,3,2}$

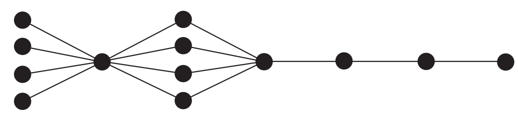

Figure 2.18 $D_{3,4,1}$

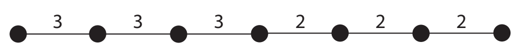

Figure 2.19 $P_{3,3,2}$

Figure 2.20 $P_{3,4,1}$

2.5 Concluding Remarks

We showed how to use every-other trees to generate 0-eigenvectors for the adjacency matrix of trees. We demonstrated an infinite family of cospectral pairs of trees using these every-other trees. Given its ease of application, it would be interesting to find an analogous method for generating these 0-eigenvectors for general bipartite graphs. We also demonstrated the usefulness of the weighted normalized Laplacian in determining the spectrum of a graph with many twins. Through the use of weighted graphs, we found large families of cospectral unweighted bipartite graphs. Certainly the weighted normalized Laplacian can be applied to more than twins. Further investigation will explain more cospectral graphs. Many of the cospectral pairs discussed in this paper were discovered using the mathematics software *Sage* [10]. The use of computational software is useful in determining cospectral mates for small graphs. There are many more cases of cospectral pairs which have yet to be explained. For example, the constructions covered in the paper are for $n \equiv 3 \mod 4$ and $n \equiv 5, 6 \mod 7$ for $n \geq 10$. This leaves many n without an example of a cospectral pair.

Conjecture 2.5.1 *Most trees have a cospectral mate for the normalized Laplacian.*

Table 2.1 Number of trees on n vertices, f_n, number of trees on n vertices cospectral with another tree with respect to the normalized Laplacian, g_n, number of trees on n vertices cospectral with another tree with respect to the adjacency matrix, h_n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
f_n	1	1	1	2	3	6	11	23	47	106	235	551	1301	3159	7741	19320	48629	123867	317955
g_n	0	0	0	0	0	0	0	0	2	4	12	28	56	122	242	464	815	1776	2442
h_n	0	0	0	0	0	0	0	2	10	8	54	119	415	826	2470	5246	14944	32347	84118

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CHAPTER 3. Almost all trees are normalized Laplacian cospectral

A paper submitted to *Linear Algebra and its Applications*

Steven Osborne

Abstract

It is known that almost all trees have a cospectral mate for the adjacency, Laplacian and signless Laplacian matrices. We show that almost all trees have a cospectral mate for the normalized Laplacian matrix as well. We also show that almost every tree is cospectral with another tree with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices simultaneously by showing that almost all trees have a mate with the same generalized characteristic polynomial.

3.1 Introduction

The spectrum of graphs is a well studied problem [1, 4, 5]. We say two graphs G and H are cospectral with respect to a given matrix described by the graph (e.g., adjacency, Laplacian, etc.) if these matrices share the same spectrum including multiplicity. It is known that many graphs have cospectral mates while other graphs are uniquely determined by their spectrum [6]. Families of graphs cospectral for the normalized Laplacian are given in [3, 10]. Further, it is known that almost all trees have a cospectral mate with respect to the adjacency [11], Laplacian and signless Laplacian matrices [9]. We show that this is also true for the normalized Laplacian matrix.

All graphs are simple, undirected, and finite. The *adjacency matrix* of a graph G, denoted $A(G)$, is the matrix indexed by the ordered set $(v_1, \ldots, v_n)$ of vertices of G with $(A)_{uv} = 1$

if u and v are adjacent in G and 0 otherwise. The *diagonal degree matrix* of G is $D(G) := \operatorname{diag}(\deg_G v_1, \ldots, \deg_G v_n)$. The *Laplacian matrix* of G is $L(G) := D(G) - A(G)$, the *signless Laplacian matrix* of G is $Q(G) := D(G) + A(G)$, and the *normalized Laplacian matrix* of G is $\mathcal{L}(G) := D(G)^{-1/2}L(G)D(G)^{-1/2}$ where by convention each isolated vertex contributes a 0 to the spectrum of $\mathcal{L}(G)$. See [4] for an overview of the normalized Laplacian. Note that $A(G)$ is often written A if the graph G is implied, and this applies to $D(G), L(G), Q(G)$ and $\mathcal{L}(G)$ as well.

Given a matrix N, the *characteristic polynomial* of N is $p_N(x) := \det(xI - N)$. Given a graph G, a vertex set S and a matrix N described by G, define $N_S(G)$ to be $N(G)$ with the rows and columns corresponding to the vertices in S deleted. A *rooted graph* (G, u) is a graph G with a fixed vertex u. The *coalescence* of two rooted graphs (G, u) and (H, v), denoted $G \cdot H$, is the graph formed by identifying u and v. We say two rooted graphs (G, u) and (H, v) are *cospectrally rooted* if $p_{A(G)}(x) = p_{A(H)}(x)$, $p_{A_u(G)}(x) = p_{A_v(H)}(x)$, *Laplacian cospectrally rooted* if $p_{L(G)}(x) = p_{L(H)}(x)$, $p_{L_u(G)}(x) = p_{L_v(H)}(x)$, *signless Laplacian cospectrally rooted* if $p_{Q(G)}(x) = p_{Q(H)}(x)$, $p_{Q_u(G)}(x) = p_{Q_v(H)}(x)$, and *normalized Laplacian cospectrally rooted* if $p_{\mathcal{L}(G)}(x) = p_{\mathcal{L}(H)}(x)$, $p_{\mathcal{L}_u(G)}(x) = p_{\mathcal{L}_v(H)}(x)$ and $\deg_u(G) = \deg_v(H)$.

3.2 Characteristic polynomial

Determining the characteristic polynomial of the adjacency matrix of a graph can be tedious. To simplify the process, Schwenk [12] introduced a number of graph decomposition techniques to derive the characteristic polynomial of a graph from its subgraphs.

Theorem 3.2.1 [12] *Let u be a vertex in G and $\mathcal{C}(u)$ be the collection of cycles in G containing u. Then*

$$p_{A(G)}(x) = xp_{A(G-u)}(x) - \sum_{w \sim u} p_{A(G-u-w)}(x) - 2 \sum_{Z \in \mathcal{C}(u)} p_{A(G-V(Z))}(x)$$

He then described the characteristic polynomial of the coalescence of two rooted graphs by applying Theorem 3.2.1 to the identified vertices.

Corollary 3.2.2 [12] *If* (G,u) *and* (H,v) *are rooted graphs, then*

$$p_{A(G\cdot H)}(x) = p_{A(G)}(x)p_{A(H-v)}(x) + p_{A(G-u)}(x)p_{A(H)}(x) - xp_{A(G-u)}(x)p_{A(H-v)}(x).$$

Note the fact that the dependence of $p_{A(G\cdot H)}$ upon G only involves $p_{A(G)}$ and $p_{A(G-u)} = p_{A_u(G)}$. Thus, we have the following corollary.

Corollary 3.2.3 [12] *If* (G,u) *and* (H,v) *are cospectrally rooted then* $G\cdot K$ *and* $H\cdot K$ *are cospectral with respect to the adjacency matrix for any rooted graph* (K,w).

In a previous paper [11], Schwenk showed that the graphs (S_1,v_1) and (S_2,v_2) shown in Figures 3.1 and 3.2 are cospectrally rooted. Using the same process as Schwenk, Guo et al. [7] extended Theorem 3.2.1 and Corollaries 3.2.2 and 3.2.3 to the Laplacian, signless Laplacian, and normalized Laplacian matrices. However, while the matrix $A_v(G)$ corresponds to $A(G-v)$, the matrix $N_v(G)$ does not correspond to $N(G-v)$ for $N = L, Q, \mathcal{L}$.

Theorem 3.2.4 [7] *Given fixed rooted graphs* (G,v) *and* (H,v) *and an arbitrary rooted graph* (K,w), *if* (G,u) *and* (H,v) *are Laplacian (signless Laplacian, normalized Laplacian) cospectrally rooted then* $G\cdot K$ *and* $H\cdot K$ *are cospectral with respect to the Laplacian (signless Laplacian, normalized Laplacian) matrix.*

It is an easy computation to show that (T_1,v_1) and (T_2,v_2) shown in Figures 3.3 and 3.4 are Laplacian and signless Laplacian cospectrally rooted. A simple computation also shows that (G_1,v_1) and (G_2,v_2) shown in Figures 3.5 and 3.6 are normalized Laplacian cospectrally rooted.

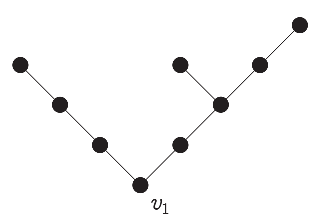

Figure 3.1 (S_1,v_1)

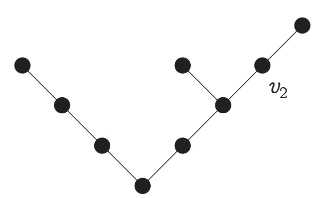

Figure 3.2 (S_2,v_2)

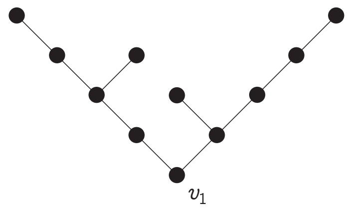

Figure 3.3 (T_1, v_1)

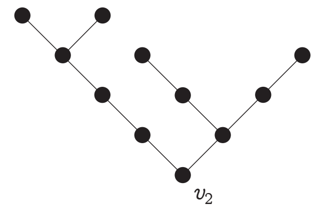

Figure 3.4 (T_2, v_2)

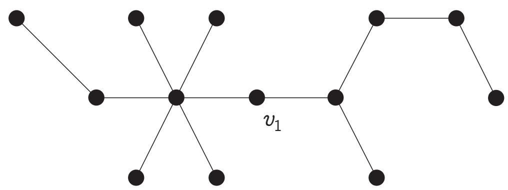

Figure 3.5 (G_1, v_1)

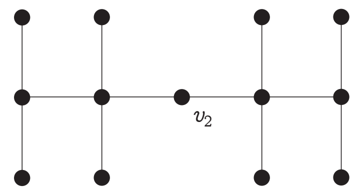

Figure 3.6 (G_2, v_2)

3.3 Almost all trees are cospectral

A *branch* of a tree T at a vertex v is a maximal subtree of T containing v as a leaf. The union of one or more branches at v is called a *limb* at v. Schwenk showed that given a rooted tree (S, v), almost all trees have (S, v) as a limb [11]. In this context, 'almost all' means that the proportion of trees without the limb (S, v) tends to 0 as the number of vertices grows. Thus, almost all trees T have (S_1, v_1) as a limb. Now, if $T' = T - S_1 + v_1$, then T is cospectral with $T' \cdot S_2$. It also follows that almost all trees T have (T_1, v_1) as limb. Thus if $T' = T - T_1 + v_1$, then T is (signless) Laplacian cospectral with $T' \cdot T_2$. This is an alternate proof of McKay's result [9]. Finally, almost all trees T have (G_1, v_1) as limb. Thus if $T' = T - G_1 + v_1$, then T is normalized Laplacian cospectral with $T' \cdot G_2$.

Theorem 3.3.1 *Almost all trees have a cospectral mate with respect to the adjacency (Laplacian, signless Laplacian, normalized Laplacian) matrix.*

The limbs used by McKay to establish the result for (signless) Laplacian matrix had 15 vertices and also reestablished the result for the adjacency matrix. The limbs shown here,

(T_1, v_1) and (T_2, v_2), are on 11 vertices but work only for the (signless) Laplacian. A computer search has shown these to be the smallest (signless) Laplacian cospectrally rooted trees. Similar computer searches have shown (S_1, v_1) and (S_2, v_2) and (G_1, v_1) and (G_1, v_2) to be the smallest adjacency and normalized Laplacian cospectrally rooted trees. The graphs (H_1, v_1) and (H_2, v_2) as shown in Figures 3.7 and 3.8 are cospectrally rooted, (signless) Laplacian cospectrally rooted and normalized Laplacian cospectrally rooted. A computer search has shown this to be the smallest such pair. This pair gives us the following result.

Theorem 3.3.2 *Almost all trees T have a mate T' such that T and T' are simultaneously cospectral with respect to all of the adjacency, (signless) Laplacian, and normalized Laplacian matrices.*

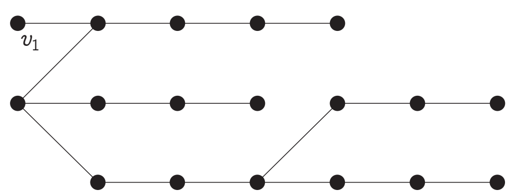

Figure 3.7 (H_1, v_1)

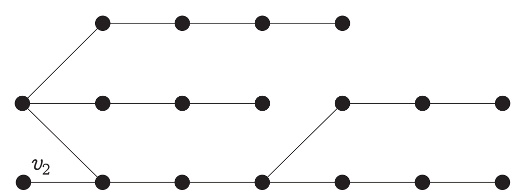

Figure 3.8 (H_2, v_2)

Theorem 3.3.2 is actually a special case of a more general result. Define $M(\lambda, t, G) := \lambda I_n - A(G) + tD(G)$ where G is a graph on n vertices. The *generalized characteristic polynomial* of G is $\phi(\lambda, t, G) := \det(M(\lambda, t, G))$. We often shorten these expressions to $M(G)$ and $\phi(G)$ or simply M and ϕ. See [8] for an overview on characterizing graphs by their generalized characteristic polynomials. Note that $\phi(x, 0, G) = p_{A(G)}(x)$, $\phi(-x, 1, G) = (-1)^n p_{L(G)}(x)$, $\phi(x, -1, G) = p_{Q(G)}(x)$, and $\phi(0, -x+1, G) = (-1)^n \det(D) p_{\mathcal{L}(G)}(x)$. Thus if two graphs have the same generalized characteristic polynomial, they are cospectral with respect to the adjacency, Laplacian, signless Laplacian, and normalized Laplacian matrices. We now extend the results of Schwenk [12] and Guo et al. [7] to the generalized characteristic polynomial.

Theorem 3.3.3 *Let u be a vertex in G and $\mathcal{C}(u)$ be the collection of cycles in G containing u.*

Then

$$\phi(G) = (\lambda + d(u)t)\det(M_u(G)) - \sum_{w\sim u}\det(M_{\{u,w\}}(G)) - 2\sum_{Z\in\mathcal{C}(u)}\det(M_Z(G))$$

where $d(u)$ is the degree of vertex u in G.

Proof. Enumerate the vertices of G by $v_1 = u, v_2, \ldots, v_n$ and assume $M(G) = (m_{ij})$. Thus

$$m_{ij} = \begin{cases} \lambda + d(v_i)t & \text{if } i = j, \\ -1 & \text{if } v_iv_j \in E(G), \\ 0 & \text{else.} \end{cases}$$

We may express $\phi(G) = \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^n m_{i\sigma(i)}$. So consider $\sigma \in S_n$ and let $s_\sigma := \operatorname{sgn}(\sigma)m_{1\sigma(1)}m_{2\sigma(2)}\cdots m_{n\sigma(n)}$. Write σ as a disjoint union of cycles $\sigma_1\sigma_2\cdots\sigma_\ell$ with $1 \in \sigma_1$. Partition S_n into P_1, P_2, P_3 where $P_i = \{\sigma \in S_n : |\sigma_1| = i\}$ for $i = 1, 2$ and $P_3 = \{\sigma \in S_n : |\sigma_1| \geq 3\}$. All $\sigma \in P_1$ fix 1 so

$$\sum_{\sigma\in P_1}\operatorname{sgn}(\sigma)\prod_{i=1}^n m_{i\sigma(i)} = (\lambda + d(v_1)t)\left(\sum_{\sigma\in P_1}\operatorname{sgn}(\sigma)\prod_{i=2}^n m_{i\sigma(i)}\right) = (\lambda + d(v_1)t)\det(M_{v_1}(G)).$$

Note that $s_\sigma \neq 0$ if and only if $v_iv_{\sigma(i)} \in E(G)$ or $\sigma(i) = i$ for $i = 1, \ldots, n$. Therefore for $\sigma \in P_2$ with $s_\sigma \neq 0$, $\sigma(1) = j$ where v_j is a neighbor of v_1. Thus

$$\begin{aligned}\sum_{\sigma\in P_2}\operatorname{sgn}(\sigma)\prod_{i=1}^n m_{i\sigma(i)} &= \sum_{\sigma\in P_2}\operatorname{sgn}(\sigma)m_{1\sigma(1)}m_{\sigma(1)1}\prod_{i\neq 1,\sigma(1)} m_{i\sigma(i)} \\ &= \sum_{\sigma\in P_2}\operatorname{sgn}(\sigma)\prod_{i\neq 1,j} m_{i\sigma(i)} = -\sum_{v_j\sim v_1}\det(M_{\{v_1,v_j\}}(G))\end{aligned}$$

as each $\sigma \in P_2$ corresponds to a $\sigma' \in S_{[n]\setminus\{1,\sigma(1)\}}$ such that $\sigma(j) = \sigma'(j)$ for $j \neq 1, \sigma(1)$ and $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$.

Consider $\sigma \in P_3$ with $\sigma_1 = (k_1k_2k_3\cdots k_r)$ where $k_1 = 1$ and $r \geq 3$. If $s_\sigma \neq 0$, σ corresponds to the cycle $Z = \{v_{k_1}, v_{k_2}, \ldots, v_{k_r}\}$ in G. The permutation $(k_1k_rk_{r-1}\cdots k_2)\sigma_2\cdots\sigma_\ell$ also corresponds to the cycle Z. Therefore, fix a cycle in Z in $\mathcal{C}(v_1)$. Consider the set of all permutations in P_3 which correspond to $Z = \{v_{k_1}, v_{k_2}, \ldots, v_{k_r}\}$, $P_3(Z)$. Then

$$\begin{aligned}\sum_{\sigma\in P_3(Z)}\operatorname{sgn}(\sigma)\prod_{i=1}^n m_{i\sigma(i)} &= \sum_{\sigma\in P_3(Z)}\operatorname{sgn}(\sigma)m_{k_1k_2}\ldots m_{k_rk_1}\prod_{i\notin\{k_1,\ldots,k_r\}} m_{i\sigma(i)} \\ &= \sum_{\sigma\in P_3(Z)}\operatorname{sgn}(\sigma)(-1)^r\prod_{i\notin\{k_1,\ldots,k_r\}} m_{i\sigma(i)} = -2\det(M_Z(G))\end{aligned}$$

as each $\sigma \in P_3$ corresponds to two $\sigma' \in S_{[n]\setminus\{k_1,\ldots,k_r\}}$ that preserve Z such that $\sigma(j) = \sigma'(j)$ for $j \notin \{k_1, \ldots, k_r\}$ and $\mathrm{sgn}(\sigma') = (-1)^r \mathrm{sgn}(\sigma)$. Thus

$$\begin{aligned}\phi(G) &= \sum_{i=1}^{3} \sum_{\sigma \in P_i} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} m_{i\sigma(i)} \\ &= (\lambda + d(v_1)t) \det(M_{v_1}(G)) - \sum_{v_j \sim v_1} \det(M_{\{v_1,v_j\}}(G)) - 2 \sum_{Z \in \mathcal{C}(v_1)} \det(M_Z(G)). \ \square\end{aligned}$$

We say two rooted graphs (G, u) and (H, v) are *generalized cospectrally rooted* if $\phi(G) = \phi(H)$, $\det(M_u(G)) = \det(M_v(H))$, and $\deg_u(G) = \deg_v(H)$. Given a rooted graph (K, w) and generalized cospectrally rooted graphs (G, u) and (H, v) we may follow the same argument as Schwenk and Guo et al., and apply Theorem 3.3.2 to $G \cdot K$ and $H \cdot K$ to derive the following corollary.

Corollary 3.3.4 *If (G, u) and (H, v) are generalized cospectrally rooted then $\phi(G \cdot K) = \phi(H \cdot K)$ for any rooted graph (K, w).*

The rooted graphs (H_1, v_1) and (H_2, v_2) as shown in Figures 3.7 and 3.8 are generalized cospectrally rooted. Thus we derive a stronger version of Theorem 3.3.2.

Theorem 3.3.5 *Almost every tree T has a mate T' such that $\phi(\lambda, t, T) = \phi(\lambda, t, T')$.*

Theorem 3.3.1 is asymptotic, therefore we include computational results for small trees in Table 3.1. Note that there are no trees which are adjacency (Laplacian, normalized Laplacian) cospectral with another tree on 7 or fewer vertices.

Table 3.1 The number of trees cospectral with another tree with respect to the adjacency, (signless) Laplacian, and normalized Laplacian matrices.

number of vertices	8	9	10	11	12	13	14	15	16	17	18	19
number of trees	23	47	106	235	551	1301	3159	7741	19320	48629	123867	317955
adjacency	2	10	8	54	119	415	826	2470	5246	14944	32347	84118
(signless) Laplacian	0	0	0	6	6	18	30	48	68	221	230	440
normalized Laplacian	0	2	4	12	28	56	122	242	464	815	1776	2442

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CHAPTER 4. General Conclusions

4.1 General Discussion

Previously known results of spectral graph theory and cospectral graphs was presented in Section 1.2. Results in this dissertation examine the implications of those results. We presented an infinite example of trees cospectral with respect to the normalized Laplacian and a use of the weighted normalized Laplacian to find cospectral pairs of unweighted bipartite graphs with respect to the normalized Laplacian in Chapter 2.

In Chapter 3, we showed that almost all trees are cospectral with respect to the normalized Laplacian. Further, we showed that almost all trees are cospectral with respect to the generalized characteristic polynomial. This would imply that almost all trees T have a mate T' such that T and T' are simultaneously cospectral with respect to all of the adjacency, (signless) Laplacian, and normalized Laplacian matrices.

4.2 Recommendations for Future Research

There are many open problems to be answered in the area of cospectral graphs. A number of them were discussed in Section 2.5. In Chapter 3, we showed almost all trees are cospectral with respect to the normalized Laplacian. Can this be extended to general bipartite graphs? It would be of interest to generate data to inform an answer. We have examined many examples of cospectral graphs. What are some characteristics that these graphs have in common? Are there common recurring subgraphs?

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