10/4/02

In class 9/23, Aubrey asked the following question:

Can every lattice be realized as the lattice of subgroups of a group?

Definition: <= is a *partial order* on the set A if:

1) a <= a
2) a<=b and b<= a implies a=b
3) a<=b and b<=c implies a<=c

If A is partially ordered by <= and B is a nonempty subset of A then an element z of A is a *lower bound* (respectively, *upper bound*) for B if or all b in B, z <= b (b<=z). z is a *greatest lower bound, glb*, (*least upper bound, lub*) for B if z is a lower bound (upper bound) of B and for any lower bound (upper bound) x of B, x<=z (z<=x).

The set L with partial order <= is a *lattice* if every two elements of L have a glb and lub.

It is easy to answer this question in the negative. However, the answer leads to one or more interesting refinement(s) of the question. This may turn out to be easier than I think, or it may turn into a semester long investigation. Here are some questions to start with:

1) Give an example of a lattice that is not the lattice of subgroups of a group.

Let L ,<= be a lattice. If there is an element m of L such that x<= m for all x in L then m is the *maximal element* of L. The lattice of subgroups of a group G has a maximal element, namely G. Let L = Z (integers) with the usual less-than or equal to order. Since Z is linearly ordered (for any a,b in Z, a<b or a=b or b<a), Z is a lattice. Z does not have a maximal element so the lattice Z is not the lattice of subgroups for any group.

2) Use your example in (1) to revise Aubrey's question to an interesting question to which you do not know the answer.

Is there an example of a lattice that has both a maximal element and a minimal element that is not the lattice of subgroups of a group.

Since every finite lattice has maximal and minimal elements, a better version would be:

Is there an example of a finite lattice that is not the lattice of subgroups of a group.

3) If a lattice of subgroups of a group is finite, is the group necessarily finite?

Yes. If group G is infinite and has an element x of infinite order then it contains the infinite sublattice $\langle x\rangle \supsetneq \langle x^2\rangle \supsetneq \langle x^4\rangle \supsetneq \ldots$ If G has no finite elements then consider the cyclic subgroups of G. Each is finite but G is the union of all its cyclic subgroups, so there must be infinitely many distinct cyclic subgroups. In either case the lattice is infinite.

4) If a subgroup has no nontrivial subgroups (is only one level above {1}) what does that tell you about the subgroup? It is cyclic prime order.

More questions and/or hints may appear in this document over time.

This investigation offers the possibility for extra credit. You may work together; however extra credit will divided among members of the group. You may discuss it with anyone you like, but you can't submit anything based on ideas from someone outside your group. Extra credit will not be awarded for writing up someone else's idea. You may use references (including your text); cite any reference(s) used. You do not need to submit your answer all at once; as soon as you have completed part of this submit it. Questions do not need to be answered in order unless obviously necessary (e.g. (1) and (2) above.