Eudoxus' Theory of Proportion

Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, alike are equal to, or alike are less than the latter equimultiples taken in corresponding order.

\[ \frac{A}{B} = \frac{C}{D} \text{ if for all natural numbers } m, n \]
\[ mA > nB \text{ implies } mC > nD \]
\[ mA = nB \text{ implies } mC = nD \]
\[ mA < nB \text{ implies } mC < nD \]

(Eves' *Great Moments*)

Thinking of numbers rather than ratios,

\[ x = y \text{ if for all natural numbers } m, n \]
\[ mx > n \text{ implies } my > n \quad \left( x > \frac{m}{n} \Rightarrow y > \frac{m}{n} \right) \]
\[ mx = n \text{ implies } my = n \quad \left( x = \frac{m}{n} \Rightarrow y = \frac{m}{n} \right) \]
\[ mx > n \text{ implies } my > n \quad \left( x < \frac{m}{n} \Rightarrow y < \frac{m}{n} \right) \]

**Dedekind cuts**

**Definition**

A set \( \alpha \) of rational numbers is a cut if

i) \( \alpha \) contains at least 1 rational, but not every.

ii) if \( p \in \alpha \) and \( q < p \) then \( q \in \alpha \).

iii) \( \alpha \) contains no largest rational.

Two cuts are equal iff they are identical.

**Theorem**

i) If \( p \in \alpha \) and \( q \notin \alpha \) then \( p < q \).

ii) If \( r \) is rational and \( \alpha \) is the set of all rationals \( p < r \) then \( \alpha \) is a cut.

**Definition**

The cut constructed in (ii) of the preceding theorem is called a rational cut and denoted \( \alpha = r^* \).

**Definition**

Let \( \alpha, \beta \) be cuts. \( \alpha < \beta \) if there is a rational \( p \) such that \( p \notin \alpha \) and \( p \in \beta \).

**Theorem**

Let \( \alpha, \beta \) be cuts. Let \( \gamma \) be the set of all rationals \( r \) such that \( r = p + q \) where \( p \in \alpha \) and \( q \in \beta \). Then \( \gamma \) is a cut.

(Walter Rudin, *Principles of Mathematical Analysis*)