Types of convergence of matrices

by

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CHAPTER 1. GENERAL INTRODUCTION

In this work properties of the spectra of matrices that play an important role in applications to stability of dynamical systems are studied.

1.1 Introduction

A (square) complex matrix is *stable* (also referred to as *Hurwitz stable* or *continuous time stable*) if all its eigenvalues lie in the open left half plane of the complex plane. It is well known (see, for example, [29, p. 55-10]) that the continuous time dynamical system

$$\frac{dx}{dt} = Ax(t) \tag{1.1}$$

is asymptotically stable at 0 if and only if the matrix A is stable. A matrix is *convergent* (also referred to as *Schur stable* or *discrete time stable*) if all its eigenvalues lie in the open unit disk of the complex plane. Analogously to the continuous time case, it is well known that the discrete time dynamical system

$$\mathbf{x}(t_{k+1}) = A\mathbf{x}(t_k) \tag{1.2}$$

is asymptotically stable at 0 if and only if the matrix A is convergent (see, for example, [8, p. 186]).

In some applications of matrix theory, the matrix of a dynamical system might not be fixed or known exactly, for example, some perturbations can occur, or there can be some uncertainty on the entries of the matrix. Moreover, when nonlinear problems are considered, errors are introduced in the process of linearization. Thus, it is important to consider not only the matrix of the system, but also some class of perturbed matrices assosiated with it. It can happen, for example, that the eigenvalues of a perturbed matrix $\widetilde{A} = A + E$ differ significantly from the eigenvalues of the matrix A even if the matrix E has a small norm. Thus, stability of the matrix A does not guarantee stability of the perturbed system. Types of stability and convergence that are stronger than just the condition on the spectrum of the matrix are useful in such situations. In the literature, several types of stability and convergence are studied that guarantee stability (respectively, convergence) not only of the matrix itself, but also of a certain set of perturbed matrices. Most often the analysis is limited to real matrices, and respectively, real perturbations (for example, [5, 21, 22, 10, 11, 25, 6]). In this thesis stronger types of convergence of matrices are studied, where the consideration is extended to complex matrices and complex perturbations. In particular, a special case of multiplicative perturbations is considered, where a perturbed matrix has the form DA with matrix D being a complex diagonal matrix with bounded diagonal entries.

One of the stronger types of stability, called qualitative or sign stability, originated from problems in ecology and economics, where it often happens that the quantitative information about the matrix is not known, or is not reliable. At the same time, the signs of the matrix entries may be known from the nature of the problem. In this case, the matrix of the system is represented by a sign pattern, and stability of a sign pattern may be studied. Analogously, in the discrete time problems, modulus convergence may be studied, where a matrix is represented by a modulus pattern. In this case, the numerical values of the entries are not specified, but it is known whether the moduli of the entries are equal to 0, between 0 and 1, equal to 1, or greater than 1. An important problem in the qualitative analysis is to characterize patterns of matrices that require certain properties of the spectrum. This problem has been completely solved in both continuous and discrete time cases with respect to stability and convergence [17, 20]. Another problem studied in qualitative analysis is to characterize patterns of matrices that allow certain properties of the spectrum, such as potentially stable or potentially convergent patterns. The difference from the study of stronger types of stability or convergence is that the question is not whether the matrix remains stable (respectively, convergent) under perturbations, but whether the matrix can be perturbed such that a stable (respectively, convergent) matrix is obtained. This problem seems to be much more difficult in both continuous time and discrete time cases, and has not been solved completely. This thesis adresses the problem of potential convergence and contains some partial results.

The following notation will be used throughout the thesis:

The *spectrum* of a matrix A (i.e. the set of the eigenvalues of A) is denoted by $\sigma(A)$.

The *spectral radius* of a matrix A is $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

The *spectral norm* of a matrix A is $\|A\| = \sqrt{\rho(A^*A)}$, where A^* is the conjugate transpose of A.

The set of all matrices with off-diagonal entries equal to 0 and positive diagonal entries is denoted by $\mathcal{D}^+ = \{\text{diag}(d_1, \ldots, d_n) : d_i > 0, i = 1, \ldots, n\}$.

The property that P is positive definite (respectively, positive semidefinite) is denoted by $P \succ 0$ (respectively, $P \succeq 0$). $A \succ B$ means that $A - B \succ 0$.

If $A = [a_{ij}]$ then $A > 0$ (respectively, $A \geq 0$) denotes entrywise inequalities, i. e. $a_{ij} > 0$ (respectively, $a_{ij} \geq 0$) for all i, j. Also, the *modulus* of the matrix is $|A| = [|a_{ij}|]$.

1.2 Thesis Organization

The thesis is organized as follows:

In the General Introduction the main ideas are described and a review of the literature on the subject is given.

Chapter 2 presents the paper "Complex D convergence and diagonal convergence of matrices" [31], submitted to *Linear Algebra and Its Applications.* The paper discusses stronger types of convergence, extending consideration to complex matrices and complex multiplicative perturbations. Some relations between the types of convergence are obtained, in particular it is shown that for complex matrices of order 3 boundary convergence is equivalent to diagonal convergence.

Chapter 3 contains the paper "Qualitative convergence of matrices" [30], accepted for publication in *Linear Algebra and Its Applications.* Although this paper was submitted before [31], it seems to be more suitable to introduce different types of convergence of matrices first, and then apply the analogous ideas to modulus patterns. The paper discusses types of potential convergence of matrix modulus patterns, shows that potential absolute convergence is equivalent to potential diagonal convergence. A complete characterization of the introduced types of potential convergence for modulus patterns of order 2 is given and some partial results on patterns of higher orders are obtained.

Chapter 4 is conclusions, where the results are summarized and directions of future research are indicated.

1.3 Literature Review

One of the first studies dedicated to stability of dynamical systems was done by A. M. Lyapunov. In his work [26] he proved an important result for systems of differential equations which provides a stability criterion for matrices:

Theorem 1.3.1 (Lyapunov Stability Theorem, [13, p. 19-4])**.** *The system (1.1) is asymptotically stable if and only there exists a positive definite matrix X such that $XA+A^*X$ is negative definite.*

An equivalent statement of the Lyapunov Stability theorem is the following (see, for example, [7]): The system (1.1) is asymptotically stable if and only if for any positive definite matrix Q, there exists a unique positive definite matrix X satisfying the Lyapunov equation

$$XA + A^*X = -Q \tag{1.3}$$

An analogous theorem for discrete time case was established by Stein [35]:

Theorem 1.3.2 (Stein Stability Theorem, [35])**.** *The system (1.2) is asymptotically stable if and only if there exists a positive definite matrix X such that $X - A^*XA$ is positive definite.*

Similarly, an equivalent version of the Stein Stability Theorem is the following (see, for example, [7]): The system (1.2) is asymptotically stable if and only if for any positive definite matrix Q, there exists a unique positive definite matrix X satisfying the Stein equation

$$X - A^*XA = Q \tag{1.4}$$

The relationship between continuous time stability and discrete time stability is described by the matrix version the Cayley transform:

$$f(A) = (I + A)(I - A)^{-1} \text{ and } g(B) = (B + I)^{-1}(B - I) \tag{1.5}$$

Note that if A is stable, then $f(A)$ is defined, and if B is convergent, then $g(B)$ is defined. Also, if $f(A)$ is defined, and if $g(f(A))$ is defined, then $g(f(A)) = A$, and similarly, if both $g(B)$ and $f(g(B))$ are defined, then $f(g(B)) = B$. The following theorem relates stable and convergent matrices.

Theorem 1.3.3. [37] *If the matrix B is convergent then $A = (B + I)^{-1}(B - I)$ is stable. Moreover, if a positive definite matrix X is such that $XA + A^*X$ is negative definite, then $X - B^*XB$ is positive definite. The converse is true as well.*

The following stronger types of stability and convergence are studied in the literature.

A real matrix A is called multiplicatively D *stable* if DA is stable for all matrices $D \in \mathcal{D}^+$.

A real matrix A is additively D *stable* (also called *strongly stable*) if $A - D$ is stable for all diagonal matrices D such that $D \geq 0$.

A real matrix A is *diagonally stable* (also called *Volterra-Lyapunov stable* or *Lyapunov diagonally stable*) if there exists a matrix $P \in \mathcal{D}^+$ such that $PA + A^*P \prec 0$.

Stronger types of stability arise in many problems of ecology and economics. The main idea is that if the matrix associated with the dynamical system possesses a certain stronger type of stability, the stability of the system is "robust," i.e. it can withstand certain perturbations.

There are many papers dedicated to characterizing stronger types of stability (see, for example, [6, 24, 25]). Simple tests for multiplicative and additive D stability and for diagonal stability were found only for matrices of small order ($n \leq 3$), or for special classes of matrices.

Theorem 1.3.4. [6] *For normal matrices and for matrices with nonnegative off diagonal entries, additive D stability, multiplicative D stability and diagonal stability are all equivalent to stability.*

Theorem 1.3.5. [6] *If A is additively D stable or multiplicatively D stable, then all principal minors of $-A$ are nonnegative and at least one minor of each order is positive. If A is diagonally stable, then all principal minors of $-A$ are positive.*

A characterization of diagonal stability for matrices of order n using Hadamard products is found in [24]:

Theorem 1.3.6. [24] *A matrix A is diagonally stable if and only if $A \circ S$ is stable for all $S = S^T = [s_{ij}] \succeq 0$, $s_{ii} = 1$, where $\circ$ denotes Hadamard product, i. e. $(A \circ S)_{ij} = a_{ij}s_{ij}$.*

Analogously, for the discrete time case, stronger types of convergence were introduced and studied in [21, 22, 10, 11]. The possible applications include control theory and the theory of asynchronous computations (see, for example, [22]). However, the discrete time case is much less studied.

A matrix A is called $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) *convergent* if DA is convergent for all real (respectively, complex) diagonal matrices D with $|D| \leq I$. A matrix A is called *vertex* (respectively, *boundary*) *convergent* if DA is convergent for all real (respectively, complex) diagonal matrices D with $|D| = I$. In the literature, where only real matrices and real perturbations are studied, $D_{\mathbb{R}}$ convergence is referred to as D convergence. Sometimes in the definitions of vertex convergence and D convergence the product AD is used instead of DA. Since $\sigma(DA) = \sigma(AD)$, these definitions define the same class of matrices.

A matrix A is called *diagonally convergent* if there exists a matrix $P \in \mathcal{D}^+$ such that $P - A^*PA \succ 0$.

Note that from Theorem 1.3.3 it follows that if a matrix A is diagonally stable, then the corresponding Cayley transform of A, $B = (A+I)(I-A)^{-1}$ is diagonally convergent and vice versa.

The characterization of the stronger types of convergence is a complicated problem that has not been solved yet. It is shown in [11] that for real matrices of order 2 and 3 $D_{\mathbb{R}}$ convergence is equivalent to vertex convergence, which provides a simple test for $D_{\mathbb{R}}$ convergence. Moreover, in [22, p. 64] it is shown that for a real 2 by 2 matrix vertex convergence is equivalent to diagonal convergence. It is also shown that for real matrices of order $n \geq 4$ vertex convergence does not imply $D_{\mathbb{R}}$ convergence. There are some partial characterizations of $D_{\mathbb{R}}$ and diagonal convergence for real matrices of order $n \geq 4$ in [22, 10, 11].

Theorem 1.3.7. [22, pp. 58-62] *For the following classes of real matrices convergence is equivalent to diagonal convergence:*

- *the class of symmetric matrices;*
- *the class of diagonally symmetrizable matrices, i. e. matrices A such that there exists a nonsingular diagonal matrix T so that $T^{-1}AT$ is symmetric;*
- *the class of nonnegative matrices, i.e. matrices $A \geq 0$;*
- *the class of checkerboard matrices, i.e. matrices A such that there exist real matrices K_1 and K_2 with $|K_1| = |K_2| = I$ so that K_1AK_2 is nonnegative.*

In some problems of ecology and economics stability can be studied qualitatively. For example, in ecological models, where interaction of several species is considered, the community matrix is composed, whose elements a_{ij} describe the effect of species j on species i. It may happen that the magnitudes of the entries of the community matrix are not available. But from the character of the interaction it is known what sign the element a_{ij} must have. It is known, for example, that increase of the population of predators affects negatively the population of the prey, and on the other hand increase of the population of the prey affects positively the population of the predators. In this situation, a matrix with entries denoted by 0, $-$ and $+$ is constructed, which is called a sign pattern; and stability of a sign pattern is studied. For a real matrix A, the associated with it sign pattern is a matrix $Z(A)$, whose entries are from the set $\{0, -, +\}$, depending on whether the corresponding entry of A is 0, negative, or positive. The qualitative class of a sign pattern Z is the set of real matrices $Q(Z) = \{A : Z(A) = Z\}$. A convenient way to visualize a matrix pattern is using directed graphs. If Z is an n by n matrix pattern, $\Gamma(Z)$ is the associated directed graph on n vertices, where the arc (i, j) is present if and only if $z_{ij} \neq 0$.

The problem of characterizing sign patterns that require stability (i. e. patterns Z, such that all matrices in $Q(Z)$ are stable) was addressed in [32], but a correct characterization was obtained later in [15] and a complete proof was presented in [17]. Here a characterization given in [4, p. 244] is presented.

Theorem 1.3.8. [4, p. 244] *Let Z be an irreducible sign patern of order n and $A \in Q(Z)$ is a $(0,\, 1,\, -1)$ matrix. Then Z requires stability if and only if each of the following properties holds:*

1. *Each entry of the main diagonal of A is nonpositive.*
2. *If $i \neq j$, then $a_{ij}a_{ji} \leq 0$.*
3. *The digraph Γ of A is a doubly directed tree (i.e. there are no cycles of length ≥ 3 and if the arc (ij) is in Γ, then the arc (ji) is in Γ).*
4. *A does not have an identically zero determinant.*
5. *There does not exist a nonempty subset β of $\{1,\, 2, \ldots,\, n\}$ such that each diagonal element of $A[\beta]$ is zero, each row of $A[\beta]$ contains at least one nonzero entry, and no row of $A[\overline{\beta}, \beta]$ contains exactly one nonzero entry. Here $A[\beta]$ denotes the principal submatrix of A, containing rows and columns with indices from β, $A[\overline{\beta}, \beta]$ denotes the submatrix of A containing columns with indices from β and rows with indices not in β.*

An analogous qualitative analysis problem for the discrete time case, formulated in [20], refers to *modulus patterns*. The *modulus pattern of a matrix* $A = [a_{ij}]$ is the matrix $Z(A) = [z_{ij}]$, whose entries are from the set $\left\{0, \boxed{1}, \boxed{<}, \boxed{>}\right\}$, depending on whether $|a_{ij}|$ is equal to 0, in the interval $(0, 1)$, equal to 1, or greater than 1 (this notation was introduced in [30]). The qualitative class of a modulus pattern Z is the set of complex matrices $Q(Z) = \{A : Z(A) = Z\}$. A modulus pattern *requires convergence* (or is *modulus convergent*) if all matrices in $Q(Z)$ are convergent. A modulus pattern Z *allows convergence*, or is *potentially convergent*, if there exists a convergent matrix in $Q(Z)$. In [20] modulus patterns that require convergence were completely characterized.

Theorem 1.3.9. [20] *An irreducible $n \times n$ $(n \geq 2)$ modulus pattern $Z = (z_{ij})$ requires convergence if and only if the following three conditions are satisfied:*

1. *there is only one cycle in the digraph of Z, $\Gamma(Z)$, and it is of length n;*

2. *there is no entry equal to $\boxed{>}$ in Z;*

3. *there exists at least one entry equal to $\boxed{<}$ in Z.*

The problem of potential stability was stated in [3], where it was pointed out that contrary to the assumptions in the study of ecological systems that negative feedback promotes the stability of the system and positive feedback destroys it, there are cases when positive feedback helps to stabilize the system, while negative feedback destabilizes it. Also, some constructions of potentially stable sign patterns are given in [3]. Potential stability of sign patterns whose undirected graphs are trees (tree sign patterns) was studied in [16, 19]. Instability tests for tree sign patterns are given in [16], where symmetric and skew symmetric factorizations of sign patterns are used, and inertia is analyzed. In [19] potentially stable tree sign patterns of order ≤ 4 are listed. In [18] nested sequences of principal minors are used to derive partial results on potential stability. An $n \times n$ sign pattern Z is said to allow a properly signed nest if there exist a matrix $B \in Q(Z)$ and a rearrangement of the indices $\alpha_1, \ldots, \alpha_n$ such that sign $(\det B[\alpha_1, \ldots, \alpha_k]) = (-1)^k$ for $k = 1, \ldots, n$.

Theorem 1.3.10. [18] *If Z is an $n \times n$ sign pattern that allows a properly signed nest, then Z is potentially stable. Moreover, Z contains a nested sequence of potentially stable sign patterns of orders $1, 2, \ldots, n$.*

Theorem 1.3.11. [18] *Suppose Z is a tree sign pattern that has exactly one nonzero diagonal entry (which is negative). Then Z is potentially stable if and only if Z allows a properly signed nest.*

In [12] a characterization of potentially stable sign patterns whose undirected graph is a star is presented.

Potential convergence was not addressed in papers before.

CHAPTER 2. COMPLEX D CONVERGENCE AND DIAGONAL CONVERGENCE OF MATRICES

Based on a paper submitted to *Linear Algebra and its Applications*

Olga Pryporova

Abstract

In this paper, we extend the results on types of convergence for real matrices to the complex case. In particular, it is proven that for complex matrices of order $n \leq 3$ diagonal convergence, $D_{\mathbb{C}}$ convergence and boundary convergence are all equivalent. An example of a 4 by 4 matrix that is $D_{\mathbb{C}}$ convergent but not diagonally convergent is constructed.

2.1 Introduction

The notion of convergence of matrices plays an important role in discrete time dynamical systems.

A complex square matrix A is called *convergent* if $\rho(A) < 1$, i.e. if all its eigenvalues lie in the open unit disk. For a linear discrete time dynamical system $\mathbf{x}(t_{k+1}) = A\mathbf{x}(t_k)$ the solution $\mathbf{x} = 0$ is asymptotically stable if and only if the matrix A is convergent (sometimes convergence of matrices is referred to as *Schur stability* or *discrete time stability*). If a system is nonlinear, or some perturbations are allowed, then it is necessary to consider stronger types of convergence. There are several types of convergence introduced in the literature that are more restrictive than just the condition $\rho(A) < 1$ (see, for example, [1, 11, 5, 21, 22]); the types of convergence studied usually concern real matrices. Here we extend these ideas to complex matrices. Unless otherwise specified, matrices are complex square.

Definition 2.1.1. A matrix A is called $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) *convergent* if DA is convergent for all real (respectively, complex) diagonal matrices D with $|D| \leq I$.

Definition 2.1.2. A matrix A is called *vertex* (respectively, *boundary*) *convergent* if DA is convergent for all real (respectively, complex) diagonal matrices D with $|D| = I$.

Definition 2.1.3. A matrix A is called *diagonally convergent* if there exists a matrix $P \in \mathcal{D}^+$ such that $P - A^*PA \succ 0$.

The following facts are well known (see [35, 10, 22]), or are easily derived analogously to the results for real matrices:

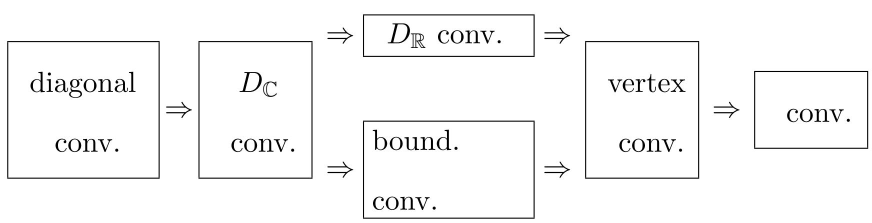
1. A complex matrix A is convergent if and only if there exists a matrix $P \succ 0$ such that $P - A^*PA \succ 0$.

2. Any principal submatrix of a $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) convergent matrix is $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) convergent.

3. If a matrix is diagonally convergent, then it is $D_{\mathbb{C}}$ convergent.

4. If $|A|$ is convergent, then A is diagonally convergent.

Therefore, the following implications are immediate:



Note that $P - A^*PA \succ 0$ with $P \in \mathcal{D}^+$, is equivalent to $I - P^{-1/2}A^*P^{1/2}P^{1/2}AP^{-1/2} \succ 0$, i.e. $\|P^{1/2}AP^{-1/2}\| < 1$ (where $\|\cdot\|$ is the spectral norm). On the other hand, $\|YAY^{-1}\| < 1$, where Y is a nonsingular diagonal matrix, is equivalent to $I - Y^{-1*}A^*Y^*YAY^{-1} \succ 0$, i.e. $Y^*Y - A^*Y^*YA \succ 0$ where $Y^*Y \in \mathcal{D}^+$, so the following observations are straightforward.

Observation 2.1.4. *A matrix A is diagonally convergent if and only if there exists $P \in \mathcal{D}^+$ such that $\|PAP^{-1}\| < 1$. In other words, A is diagonally convergent if and only if* $\inf\{\|PAP^{-1}\| : P \in \mathcal{D}^+\} < 1$.

Observation 2.1.5. *If A is diagonally similar to a diagonally convergent matrix, then A is diagonally convergent.*

One of the open problems is to identify classes of matrices for which convergence is equivalent to diagonal convergence. Suppose A is a normal matrix, then $\rho(A) = \|A\|$. So, for a normal matrix A, $\rho(A) < 1$ implies that $\|A\| < 1$. The next observation follows from Fact 4 and Observation 2.1.5

Observation 2.1.6. *The class of matrices diagonally similar to nonnegative matrices, and the class of matrices diagonally similar to normal matrices are such that convergence is equivalent to diagonal convergence.*

As follows from Fact 2, properties of $D_{\mathbb{C}}$ and $D_{\mathbb{R}}$ convergence are "hereditary", i.e. any principal submatrix also has this property. Below we show that diagonal convergence is also "hereditary". Example 2.3.1 in Section 2.3 shows that vertex convergence is not a "hereditary" property, and it is unknown whether boundary convergence is.

Proposition 2.1.7. *Any principal submatrix of a diagonally convergent matrix is diagonally convergent.*

Proof. Note that for any principal submatrix $A' = A[\alpha]$ the spectral norm $\|A[\alpha]\| \leq \|A\|$, so if there exists $P \in \mathcal{D}^+$ such that $\|PAP^{-1}\| < 1$, then for any principal submatrix of PAP^{-1}, $\|(PAP^{-1})[\alpha]\| \leq \|PAP^{-1}\| < 1$. Now note that since P is diagonal, $(PAP^{-1})[\alpha] = P[\alpha]A[\alpha](P[\alpha])^{-1}$, so there exists a positive diagonal matrix $P' = P[\alpha]$ such that $\|P'A'P'^{-1}\| < 1$, i.e. A' is diagonally convergent. □

A matrix is *reducible* if it is permutationally similar to a block-triangular matrix (with more than one diagonal block); otherwise it is *irreducible.*

The *digraph* of an $n \times n$ matrix is a directed graph on n vertices, where the arc (i, j) is present exactly when $a_{ij} \neq 0$. A matrix is irreducible if and only if its digraph is strongly connected. A *cycle* of a matrix corresponds to a cycle in the digraph.

The (simple) *graph* of a symmetric $n \times n$ matrix is a graph on n vertices, where the (undirected) edge $\{i, j\}$ $(i \neq j)$ is present if $a_{ij} \neq 0$ (note that the diagonal entries are ignored).

Since the spectrum of a reducible matrix is the union of spectra of its irreducible diagonal blocks, the following observation is clear.

Observation 2.1.8. *A reducible matrix is convergent (respectively, $D_{\mathbb{C}}$, $D_{\mathbb{R}}$, boundary, vertex convergent) if and only if each of its irreducible diagonal blocks is convergent (respectively, $D_{\mathbb{C}}$, $D_{\mathbb{R}}$, boundary, vertex convergent).*

Proposition 2.1.9. *A reducible matrix is diagonally convergent if and only if each of its irreducible diagonal blocks is diagonally convergent.*

Proof. Necessity follows from Proposition 2.1.7.

To show sufficiency we use induction on the number of irreducible diagonal blocks. Clearly, the statement is true if the number of blocks is 1. Assume that the statement holds if the number of blocks is less than m ($m \geq 2$). Suppose that A is reducible and consists of m irreducible diagonally convergent diagonal blocks. Then (up to permutation similarity) $A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} \in \mathbb{C}^{n\times n}$, where $A_1 \in \mathbb{C}^{k\times k}$ and $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ each consists of less than m diagonally convergent blocks, so both A_1 and A_2 are diagonally convergent. Then there exist positive diagonal matrices P_1 and P_2 and a number $\epsilon > 0$ such that $\|P_1A_1P_1^{-1}\| \leq 1-\epsilon$ and $\|P_2A_2P_2^{-1}\| \leq 1-\epsilon$. Let $A_0 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $P_0 = P_1 \oplus P_2$. Since $\|A \oplus B\| = \max\{\|A\|, \|B\|\}$, $\|P_0A_0P_0^{-1}\| = \|(P_1A_1P_1^{-1}) \oplus (P_2A_2P_2^{-1})\| \leq 1-\epsilon$. Let $D_\delta = (\delta I_k) \oplus I_{n-k}$ then

$$D_\delta P_0 A P_0^{-1} D_\delta^{-1} = \begin{bmatrix} P_1A_1P_1^{-1} & \delta P_1 B P_2^{-1} \\ 0 & P_2A_2P_2^{-1} \end{bmatrix} = P_0A_0P_0^{-1} + \delta \begin{bmatrix} 0 & P_1BP_2^{-1} \\ 0 & 0 \end{bmatrix}.$$

Choosing $\delta > 0$ small enough so that $\delta\|P_1BP_2^{-1}\| < \epsilon$, we obtain a positive diagonal matrix $P = D_\delta P_0$ such that $\|PAP^{-1}\| \leq \|P_0A_0P_0^{-1}\| + \delta\|P_1BP_2^{-1}\| < 1$, which implies that A is diagonally convergent. $\square$

2.2 When Boundary Convergence Implies Diagonal Convergence

Let $A \in \mathbb{C}^{n\times n}$. We consider the set of matrices

$$\mathcal{A}(A) = \mathcal{A} = \{PAP^{-1} : P \in \mathcal{D}^+\},$$

and the nonnegative number

$$s(A) = \inf\{\|B\| : B \in \mathcal{A}(A)\}.$$

By Observation 2.1.4, A is diagonally convergent if and only if $s(A) < 1$.

Theorem 2.2.1. *If $s(A)$ is not attained in $\mathcal{A}(A)$, then A is reducible.*

Proof. We denote by $\mathcal{P}$ the set of all sequences $\{P_k = \operatorname{diag}(p_1^{(k)}, \dots, p_n^{(k)})\}_{k=1}^\infty \subset \mathcal{D}^+$ such that $\lim_{k\to\infty} \|P_k A P_k^{-1}\| = s(A)$. Note that since $(PAP^{-1})_{ij} = \frac{p_i}{p_j} a_{ij}$, if for some $\{P_k\}_{k=1}^\infty \in \mathcal{P}$ and for some pair (i, j) the sequence $\left\{p_i^{(k)}/p_j^{(k)}\right\}_{k=1}^\infty$ is unbounded, then $a_{ij} = 0$. (Otherwise, if $a_{ij} \neq 0$, then $\{\|P_k A P_k^{-1}\|\}_{k=1}^\infty$ would be unbounded which is a contradiction). For $i, j \in \{1, \dots, n\}$, if the sequence $\left\{p_i^{(k)}/p_j^{(k)}\right\}_{k=1}^\infty$ is bounded for all $\{P_k\}_{k=1}^\infty \in \mathcal{P}$, we denote it $i \to j$, and otherwise, $i \nrightarrow j$. So, $i \nrightarrow j$ implies that $a_{ij} = 0$. We will show that the relation $i \to j$ is transitive. Suppose $i \to j$ and $j \to l$, i.e. for each sequence $\{P_k\}_{k=1}^\infty \in \mathcal{P}$ there exist $M_1 = M(\{P_k\}, i, j) > 0$ and $M_2 = M(\{P_k\}, j, l) > 0$ such that $\frac{p_i^{(k)}}{p_j^{(k)}} < M_1$ and $\frac{p_j^{(k)}}{p_l^{(k)}} < M_2$ for all $k \geq 1$. So $\frac{p_i^{(k)}}{p_l^{(k)}} < M_1 M_2$ for all $k \geq 1$, which implies that $i \to l$.

Now suppose that there exists $\{P_k\}_{k=1}^\infty \in \mathcal{P}$, such that for all $i, j \in \{1, \dots, n\}$ the sequence $\left\{p_i^{(k)}/p_j^{(k)}\right\}_{k=1}^\infty$ is bounded. In particular, there exists $M > 0$ such that $\frac{1}{M} < \frac{p_i^{(k)}}{p_1^{(k)}} < M$ for all $k \geq 1$ and for all $i = 1, \dots n$. Then there exists a subsequence $\{P_{k_l}\}_{l=1}^\infty \subset \{P_k\}_{k=1}^\infty$, such that for all $i = 1, \dots, n$, $\lim_{l\to\infty} \frac{p_i^{(k_l)}}{p_1^{(k_l)}} = c_i$ with $0 < \frac{1}{M} \leq c_i \leq M < \infty$. Consider $\widehat{P} = \operatorname{diag}(1, c_2, \dots, c_n) \in \mathcal{D}^+$, and a sequence $\{\widehat{P}_l\}_{l=1}^\infty \subset \mathcal{D}^+$, where $\widehat{P}_l = \frac{1}{p_1^{(k_l)}} P_{k_l}$, so $P_{k_l} A P_{k_l}^{-1} = \widehat{P}_l A \widehat{P}_l^{-1}$, and since $\widehat{P}_l \to \widehat{P}$, by continuity of the norm, $\|\widehat{P} A \widehat{P}^{-1}\| = \lim_{l\to\infty} \|\widehat{P}_l A \widehat{P}_l^{-1}\| = \lim_{l\to\infty} \|P_{k_l} A P_{k_l}^{-1}\| = s(A)$, which implies that $s(A)$ is attained in $\mathcal{A}$. Therefore, if $s(A)$ is not attained, then for any sequence $\{P_k\}_{k=1}^\infty \in \mathcal{P}$ there exists a pair (i, j), such that $\{p_i^{(k)}/p_j^{(k)}\}_{k=1}^\infty$ is unbounded (so $i \nrightarrow j$ is necessarily true for at least one pair (i, j)). Note that since the relation $i \to j$ is transitive, $i \not\to j$ implies that there is no path from i to j in the digraph of A, i.e the matrix A is reducible. □

To illustrate the situation when a matrix is reducible, consider for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
Then $\mathcal{A}(A) = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a > 0 \right\}$, so $s(A) = 1$, although no matrix in $\mathcal{A}(A)$ has spectral norm equal to 1.

In the following we will assume that matrix $A \neq 0$ is irreducible, so $s(A) > 0$ is attained in $\mathcal{A}(A)$, i.e. there is a matrix $A_0 \in \mathcal{A}(A)$ such that $\|A_0\| = s(A)$. To study properties of the matrix A_0, we need the following theorem.

Theorem 2.2.2. *(Theorem 1,* [33, p.33]*) Suppose that $B(t)$ is a matrix such that $b_{ij}(t)$, $i, j = 1, \ldots, n$ are complex power series in a real variable t convergent in a neighborhood of $t = 0$; and $b_{ij}(t) = \overline{b_{ji}(t)}$. Suppose that λ is an eigenvalue of $B = B(0)$ of multiplicity $h \geq 1$ and suppose the open interval $(\lambda - d_1, \lambda + d_2)$, where $d_1 > 0$ and $d_2 > 0$, contains no eigenvalue of B other than λ.*

Then there exist power series

$$\lambda_1(t), \ldots, \lambda_h(t),$$

and vector power series

$$\mathbf{v}^{(k)}(t) = [v_1^{(k)}(t), \ldots, v_n^{(k)}(t)]^T,\ k = 1, \ldots, h,$$

all convergent in a neighborhood of $t = 0$, which satisfy the following conditions:

1. *For all $k = 1, \ldots, h$ and for all t in a neighborhood of $t = 0$ the vector $\mathbf{v}^{(k)}(t)$ is an eigenvector of $B(t)$ belonging to the eigenvalue $\lambda_k(t)$, and $\mathbf{v}^{(1)}(t), \ldots, \mathbf{v}^{(h)}(t)$ are orthonormal. Furthermore, $\lambda_k(0) = \lambda$, $k = 1, \ldots, h$.*

2. *For each pair of positive numbers d_1', d_2' with $d_1' < d_1$, $d_2' < d_2$ there exists a positive number δ such that the spectrum of $B(t)$ in the closed interval $[\lambda - d_1', \lambda + d_2']$ consists of the points $\lambda_1(t), \ldots, \lambda_h(t)$ provided $|t| < \delta$.*

Remark. Under the additional assumption that $b_{ij}(t)$ are real, the same proof as in [33, p.33] guarantees existence of real vectors $\mathbf{v}^{(k)}(t)$.

Lemma 2.2.3. *Let $A \in \mathbb{C}^{n\times n}$. Fix a nonzero real diagonal matrix $\Delta = \operatorname{diag}(\delta_1, \dots, \delta_n)$ and consider the following parameterized family of positive semidefinite matrices:*

$$B(t) = B(t, \Delta) = (P(t)AP(t)^{-1})^* P(t)AP(t)^{-1},$$

where $P(t) = P(t, \Delta) = I_n + t\Delta$. Suppose $\lambda > 0$ is the largest eigenvalue of $B(0)$ and it has multiplicity h. Then in a neighborhood of $t = 0$ there exist continuously differentiable real-valued functions $\lambda_1(t), \dots, \lambda_h(t)$ that are the h largest eigenvalues of $B(t)$ (not necessarily distinct), $\lambda_k(0) = \lambda$ for all $k = 1, \dots, h$, and their derivatives at $t = 0$ are

$$\lambda_k'(0) = 2(\mathbf{v}^{(k)*}A^*\Delta A\mathbf{v}^{(k)} - \lambda\mathbf{v}^{(k)*}\Delta\mathbf{v}^{(k)}),$$

$k = 1, \dots, h$, where $\mathbf{v}^{(k)} = \mathbf{v}^{(k)}(0)$, $k = 1, \dots, h$ such that $\mathbf{v}^{(1)}(t), \dots, \mathbf{v}^{(h)}(t)$ are orthonormal eigenvectors of $B(t)$ corresponding to $\lambda_1(t), \dots, \lambda_h(t)$, respectively.

Proof. The entries of the matrix $B(t)$, $b_{ij} = \sum_{l=1}^{n} \frac{(1+\delta_l t)^2}{(1+\delta_i t)(1+\delta_j t)} a_{lj}\overline{a}_{li}$, are rational functions of t, so they can be expressed as power series of t convergent for $t \in (-\epsilon, \epsilon)$, where $\epsilon < \frac{1}{\max\{|\delta_i| : i=1,\dots,n\}}$. Therefore, by Theorem 2.2.2 there exist power series $\lambda_k(t)$ and $\mathbf{v}^{(k)}(t) = [v_1^{(k)}(t), \dots, v_n^{(k)}(t)]^T$, convergent in a neighborhood of $t = 0$ (so their derivatives are continuous at $t = 0$), that are the h largest eigenvalues of $B(t)$ in this neighborhood, and corresponding to them eigenvectors of $B(t)$, that are mutually orthonormal for all t in this neighborhood.

Note that since $B(t)\mathbf{v}^{(k)}(t) = \lambda_k(t)\mathbf{v}^{(k)}(t)$, $k = 1, \dots, h$, differentiating with respect to t gives

$$\begin{aligned} \tfrac{dB}{dt}\mathbf{v}^{(k)}(t) + B(t)\tfrac{d\mathbf{v}^{(k)}(t)}{dt} &= \tfrac{d\lambda_k(t)}{dt}\mathbf{v}^{(k)}(t) + \lambda_k(t)\tfrac{d\mathbf{v}^{(k)}(t)}{dt}; \\ \left(\tfrac{dB}{dt} - \tfrac{d\lambda_k}{dt}I_n\right)\mathbf{v}^{(k)}(t) &= (\lambda_k I_n - B(t))\tfrac{d\mathbf{v}^{(k)}}{dt}. \end{aligned}$$

Multiplying by $(\mathbf{v}^{(k)}(t))^*$ from the left gives:

$$(\mathbf{v}^{(k)}(t))^*\left(\frac{dB}{dt} - \frac{d\lambda_k}{dt}I_n\right)\mathbf{v}^{(k)}(t) = (\mathbf{v}^{(k)}(t))^*(\lambda_k I_n - B(t))\frac{d\mathbf{v}^{(k)}}{dt}.$$

Since $B(t)$ is Hermitian, $\left(\mathbf{v}^{(k)}(t)\right)^*(\lambda_k(t)I_n - B(t)) = \left((\lambda_k I_n - B)\mathbf{v}^{(k)}(t)\right)^* = 0$, so we have

$$\frac{d\lambda_k}{dt}(\mathbf{v}^{(k)}(t))^*\mathbf{v}^{(k)}(t) = (\mathbf{v}^{(k)}(t))^*\frac{dB}{dt}\mathbf{v}^{(k)}(t), \text{ i.e.}$$

$$\lambda_k'(t) = (\mathbf{v}^{(k)}(t))^* \frac{dB(t)}{dt} \mathbf{v}^{(k)}(t). \tag{2.1}$$

Now we compute

$$\begin{aligned}
\frac{dB}{dt} &= \frac{d}{dt}\left[(P^{-1}(t)A^*P^2(t)AP^{-1}(t)\right] = \\
&= P^{-1}(t)A^*P^2(t)A\frac{dP^{-1}(t)}{dt} + P^{-1}(t)A^*\frac{dP^2(t)}{dt}AP^{-1}(t) + \frac{dP^{-1}(t)}{dt}A^*P^2(t)AP^{-1}(t) = \\
&= -P^{-1}(t)A^*P^2(t)A\Delta P^{-2}(t) + P^{-1}(t)A^*(2\Delta + 2t\Delta^2)AP^{-1}(t) - P^{-2}(t)\Delta A^*P^2(t)AP^{-1}(t) \\
\frac{dB}{dt}\Big|_{t=0} &= -A^*A\Delta + 2A^*\Delta A - \Delta A^*A.
\end{aligned}$$

Since $A^*A\mathbf{v}^{(k)} = \lambda\mathbf{v}^{(k)}$, evaluating $\lambda_k'(t)$ at the point $t = 0$ and using (2.1) we obtain:

$$\lambda_k'(0) = 2(\mathbf{v}^{(k)})^*A^*\Delta A\mathbf{v}^{(k)} - 2\lambda(\mathbf{v}^{(k)})^*\Delta\mathbf{v}^{(k)}.$$

□

Theorem 2.2.4. *Suppose $s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$, and $s^2 = (s(A))^2$ is a simple eigenvalue of $A_0^*A_0$. Then there exists $D \in \mathbb{C}^{n\times n}$, $|D| = I$, such that $\rho(DA) = s$. If A is real, then there exists a real D, $|D| = I$ such that $\rho(DA) = s$.*

Proof. We will use notations of Lemma 2.2.3. Since $s^2 = \min\{\rho(B) : B = P^{-1}A_0^*P^2A_0P^{-1}, P \in \mathcal{D}^+\}$ (the minimum is attained when $P = I$), and s^2 is a simple eigenvalue of $A_0^*A_0$, it implies that for any real diagonal matrix Δ, if $P(t) = I + t\Delta$, $t \in (-\epsilon, \epsilon)$, the derivative of $\lambda_1(t)$ is 0 for $t = 0$ (where $\lambda_1(t)$ is the largest eigenvalue of $B(t) = P^{-1}(t)A_0^*P^2(t)A_0P^{-1}(t)$). By Lemma 2.2.3, this implies that $\mathbf{v}^*A_0^*\Delta A_0\mathbf{v} - s^2\mathbf{v}^*\Delta\mathbf{v} = 0$ (where $\mathbf{v} = \mathbf{v}^{(1)}(0)$) for any real diagonal matrix Δ, which is equivalent to $|A_0\mathbf{v}| = s|\mathbf{v}|$. This means that there exists a matrix D with $|D| = I$, such that $DA_0\mathbf{v} = s\mathbf{v}$, so that $\rho(DA_0) = s$. Since $A_0 = P_0^{-1}AP_0$ for some $P_0 \in \mathcal{D}^+$, $\rho(DA) = \rho(DP_0A_0P_0^{-1}) = \rho(P_0DA_0P_0^{-1}) = \rho(DA_0) = s$. Also, if A is real, then so are A_0 and $\mathbf{v}$, so $D = \mathrm{diag}(\pm1, \ldots, \pm1)$. □

Lemma 2.2.5. *Suppose $s(A) = \|A_0\|$ for some complex $A_0 \in \mathcal{A}(A)$, and $s^2 = (s(A))^2$ is an eigenvalue of multiplicity 2 of $A_0^*A_0$. Using notations of Lemma 2.2.3 define the vectors $\mathbf{b}^{(1)} = [b_1^{(1)} \ldots b_n^{(1)}]^T$ and $\mathbf{b}^{(2)} = [b_1^{(2)} \ldots b_n^{(2)}]^T$ in the following way: $b_j^{(k)} = |(A_0\mathbf{v}^{(k)})_j|^2 - s^2|\mathbf{v}_j^{(k)}|^2$, $k = 1, 2,\ j = 1, \ldots, n$. Then either one of $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ is equal to 0, or there exists $\alpha > 0$ such that $\mathbf{b}^{(1)} = -\alpha\mathbf{b}^{(2)}$.*

Proof. Since s^2 is an eigenvalue of multiplicity 2 of a positive semidefinite matrix $A_0^*A_0$, by Lemma 2.2.3, there exist two continuously differentiable functions $\lambda_1(t)$ and $\lambda_2(t)$, that are the two largest eigenvalues of $B(t) = P^{-1}(t)A_0^*P^2(t)A_0P^{-1}(t)$ in a neighborhood of $t = 0$, such that $\lambda_1(0) = \lambda_2(0) = s^2$, and their derivatives at 0 are $\lambda_k'(0) = 2(\mathbf{v}^{(k)*}A_0^*\Delta A_0\mathbf{v}^{(k)} - s^2\mathbf{v}^{(k)*}\Delta\mathbf{v}^{(k)})$, $k = 1, 2$, where $\mathbf{v}^{(1)} = \mathbf{v}^{(1)}(0)$ and $\mathbf{v}^{(2)} = \mathbf{v}^{(2)}(0)$ are two mutually orthogonal unit eigenvectors of $A_0^*A_0$ that correspond to the eigenvalue s^2. Denote $\mathbf{d} = [\delta_1 \ \dots \ \delta_n]^T$. Then $\lambda_k'(0) = 2\mathbf{d}^T\mathbf{b}^{(k)}$, $k = 1, 2$.

Since $s^2 = \min\{\rho(B) : B = P^{-1}A_0^*P^2A_0P^{-1}, P = \mathrm{diag}(p_1, \dots, p_n) \succ 0\} = \rho(A_0^*A_0)$, there does not exist any real diagonal matrix Δ such that both $\lambda_1'(0) < 0$ and $\lambda_2'(0) < 0$ (otherwise it would be possible to find a matrix $B(t, \Delta)$ with two largest eigenvalues both smaller than s^2). If both $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ are nonzero and there does not exist a real vector $\mathbf{d}$ such that both $\mathbf{d}^T\mathbf{b}^{(1)} < 0$ and $\mathbf{d}^T\mathbf{b}^{(2)} < 0$, this can happen only if $\mathbf{b}^{(1)} = -\alpha\mathbf{b}^{(2)}$ for some $\alpha > 0$. □

Theorem 2.2.6. *Suppose A is real and $s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$, where $s^2 = (s(A))^2$ is an eigenvalue of multiplicity 2 of $A_0^*A_0$. Then there exists $D \in \mathbb{C}^{n\times n}$, $|D| = I$, such that $\rho(DA) = s$.*

Proof. We apply Lemma 2.2.5 and take into account that $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ can be chosen real, since A is real. If for some k vector $\mathbf{b}^{(k)} = 0$, i.e. $|A_0\mathbf{v}^{(k)}| = s|\mathbf{v}^{(k)}|$, then we have the same situation as in the Theorem 2.2.4, so there exists a matrix D with $|D| = I$, such that $DA_0\mathbf{v} = s\mathbf{v}$. If both $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ are nonzero, then $\mathbf{b}^{(1)} = -\alpha\mathbf{b}^{(2)}$ for some $\alpha > 0$, i.e.

$$|(A_0\mathbf{v}^{(1)})_j|^2 - s^2|\mathbf{v}_j^{(1)}|^2 = -\alpha|(A_0\mathbf{v}^{(2)})_j|^2 + \alpha s^2|\mathbf{v}_j^{(2)}|^2 \text{ for } j = 1, \dots, n.$$

We can write this as

$$|(A_0\mathbf{v}^{(1)})_j|^2 + \alpha|(A_0\mathbf{v}^{(2)})_j|^2 = s^2|\mathbf{v}_j^{(1)}|^2 + s^2\alpha|\mathbf{v}_j^{(2)}|^2.$$

Now, let $\mathbf{x} = \mathbf{v}^{(1)} + i\sqrt{\alpha}\mathbf{v}^{(2)}$. Then

$$|(A_0\mathbf{x})_j|^2 = |(A\mathbf{v}^{(1)})_j|^2 + \alpha|(A_0\mathbf{v}^{(2)})_j|^2 = s^2|\mathbf{v}_j^{(1)}|^2 + s^2\alpha|\mathbf{v}_j^{(2)}|^2 = s^2|\mathbf{x}_j|^2, \text{ for } j = 1, \dots, n.$$

Therefore, there exists a matrix D with $|D| = I$, such that $DA_0\mathbf{x} = s\mathbf{x}$, i.e. $\rho(DA_0) = s$, so $\rho(DA) = s$. □

Corollary 2.2.7. *Suppose A is a real (respectively, complex) irreducible n by n matrix, and $s = s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$ such that s^2 is an eigenvalue of multiplicity 1 of the matrix $A_0^*A_0$. Then vertex (respectively, boundary) convergence of A implies diagonal convergence of A. Suppose $A \in \mathbb{R}^{n\times n}$ is irreducible and $s = s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$ such that s^2 is an eigenvalue of multiplicity 2 of the matrix $A_0^*A_0$. Then boundary convergence of A implies diagonal convergence of A.*

Proof. By Theorems 2.2.4 and 2.2.6 there exists a complex matrix D with $|D| = I$ such that $\rho(DA) = s$. So, if A is boundary convergent, then $s(A) < 1$, i.e A is diagonally convergent. If A is real and the multiplicity of s^2 is 1, then by Theorem 2.2.4 there exists a real matrix D with $|D| = I$ such that $\rho(DA) = s(A)$, so vertex convergence of A implies diagonal convergence of A. $\square$

Corollary 2.2.7 is not very useful in characterizing matrices for which boundary convergence implies diagonal convergence, since usually the matrix A_0 is not known, but we need to know the multiplicity of the largest eigenvalue of the matrix $A_0^*A_0$. Clearly, in the case $n = 2$, this multiplicity can be either 1 or 2. If the multiplicity is 2, then $A_0^*A_0 = s^2I_2$, so the situation is trivial, since $A_0 = sU$, where U is unitary, therefore $\rho(A) = \rho(A_0) = s$ and clearly, convergence of A implies diagonal convergence of A. If the multiplicity is 1, then we can apply Corollary 2.2.7. Thus, we have proved the following result.

Corollary 2.2.8. *Let $A \in \mathbb{C}^{2\times 2}$ be irreducible. Then A is diagonally convergent if and only if A is boundary convergent. Moreover, if $A \in \mathbb{R}^{2\times 2}$ is irreducible, then it is diagonally convergent if and only if it is vertex convergent.*

Another way to prove that for a complex (respectively, real) 2×2 matrix A boundary (respectively, vertex) convergence implies diagonal convergence is by using the following proposition. Its advantage is that for a given $A \in \mathbb{C}^{2\times 2}$, it shows how to construct matrices P and D such that $s(A) = \|P^{-1}AP\| = \rho(DA)$.

Proposition 2.2.9. *Let $A \in \mathbb{C}^{2\times 2}$ be irreducible. Then there exist matrices $P = \operatorname{diag}(1, p) \succ 0$ and $D = \operatorname{diag}(1, d)$ with $|d| = 1$, such that $C = P^{-1}DAP$ is normal; and if A is real, then so is D. Thus, the matrices P and D are such that $s(A) = \|P^{-1}AP\| = \rho(DA)$.*

Proof. Since A is irreducible, i.e. $a_{12}a_{21} \neq 0$, without loss of generality we can assume that $A = \begin{bmatrix} x_1 & y \\ \frac{1}{y} & x_2 \end{bmatrix}$ (since multiplying a matrix by a scalar does not change its normality).

So $C = \begin{bmatrix} x_1 & py \\ \frac{d}{py} & x_2 d \end{bmatrix}$, $(C^*C)_{11} = |x_1|^2 + \frac{1}{p^2|y|^2}$ and $(CC^*)_{11} = |x_1|^2 + p^2|y|^2$. If $C^*C = CC^*$, then $p = \frac{1}{|y|}$. Denote $z = py$, so $|z| = |1/z| = 1$ and $\overline{z} = 1/z$.

Then $C^*C = \begin{bmatrix} |x_1|^2 + 1 & (\overline{x_1} + x_2)z \\ (x_1 + \overline{x_2})\overline{z} & |x_2|^2 + 1 \end{bmatrix}$ and $CC^* = \begin{bmatrix} |x_1|^2 + 1 & (x_1 + \overline{x_2})z\overline{d} \\ (\overline{x_1} + x_2)\overline{z}d & |x_2|^2 + 1 \end{bmatrix}$.

Let $d = \begin{cases} 1 & \text{if } x_1 + \overline{x_2} = 0, \\ \dfrac{x_1 + \overline{x_2}}{\overline{x_1} + x_2} & \text{otherwise} \end{cases}$, then C is normal. By construction, if A is real, then so is D.

Note that $\rho(C) = \rho(DA) \leq s(DA) = s(A) \leq \|P^{-1}AP\| = \|C\|$, but since C is normal, $\|C\| = \rho(C)$, so we have the equalities $s(A) = \|P^{-1}AP\| = \rho(DA)$. □

Corollary 2.2.10. *Let $A \in \mathbb{R}^{n\times n}$ be such that its digraph consists of one cycle of length n and no other cycles, except loops. If all diagonal entries of A are nonzero, then A is diagonally convergent if and only if A is boundary convergent. If exactly one of the diagonal entries of A is 0, then A is diagonally convergent if and only if A is vertex convergent.*

Proof. Since the digraph of A is strongly connected, A is irreducible and $s(A)$ is attained in $\mathcal{A}(A)$ by some matrix A_0. Since all matrices in $\mathcal{A}(A)$ have the same digraph as A, up to permutation similarity the matrix A_0 is such that $(A_0)_{ii} = a_i$, $i = 1, \ldots n$, $(A_0)_{i,i+1} = b_i$, $i = 1, \ldots, n-1$, $(A_0)_{n1} = b_n$, where $b_i \neq 0$, $i = 1, \ldots, n$, all other entries of A_0 are 0. Then the symmetric matrix $B = A_0^* A_0$ is such that $(B)_{ii} = a_i^2 + b_i^2$, $i = 1, \ldots, n$, $(B)_{i,i+1} = (B)_{i+1,i} = a_i b_i$, $i = 1, \ldots, n-1$, $(B)_{1n} = (B)_{n1} = a_n b_n$, all other entries are 0. Consider two cases:

Case 1. All a_i are nonzero, so the graph of B is the n-cycle. Since the maximum multiplicity of any eigenvalue of a symmetric matrix whose graph is a cycle is 2 (see [9]), the result follows from Corollary 2.2.7.

Case 2. Exactly one a_i is 0. Then the graph of B is an n-path. Since the maximum multiplicity of any eigenvalue of a symmetric matrix whose graph is a path is 1 (see [9]), the result follows from Corollary 2.2.7. □

Similarly to Corollary 2.2.8 we could use Corollary 2.2.7 to show that if $A \in \mathbb{R}^{3\times 3}$ is irreducible, then it is diagonally convergent if and only if it is boundary convergent. We are going to establish a stronger result for 3×3 matrices.

Theorem 2.2.11. *Let $A \in \mathbb{C}^{3\times 3}$ be irreducible, then the following are equivalent:*

- *A is diagonally convergent;*
- *A is $D_{\mathbb{C}}$ convergent;*
- *A is boundary convergent.*

Proof. Let $A_0 = P^{-1}AP \in \mathcal{A}$ be a matrix such that $s = (s(A))^2 = \|A_0\|$. We will prove the theorem if we show that there exists a vector $\mathbf{v}$ such that $|A_0\mathbf{v}| = s|\mathbf{v}|$. If s^2 is an eigenvalue of $A_0^*A_0$ of multiplicity 1, then we use Theorem 2.2.4 and there is nothing to prove. The case when s^2 has multiplicity 3, is trivial, since then $A_0 = sU$ for some unitary matrix U.

Now suppose that s^2 is an eigenvalue of $A_0^*A_0$ of multiplicity 2. We use notations of Lemma 2.2.5, and in addition define $\mathbf{u}^{(k)} = \frac{1}{s}A_0\mathbf{v}^{(k)}$, $k = 1, 2$, so $b_j^{(k)} = s^2(|u_j^{(k)}|^2 - |v_j^{(k)}|^2)$. If for some k vector $\mathbf{b}^{(k)} = 0$, i.e. $|A_0\mathbf{v}^{(k)}| = s|\mathbf{v}^{(k)}|$, then there is nothing more to prove.

If both $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ are nonzero, then $\mathbf{b}^{(1)} = -\alpha\mathbf{b}^{(2)}$ for some $\alpha > 0$; in our new notation this is equivalent to $|\mathbf{u}_j^{(1)}|^2 - |\mathbf{v}_j^{(1)}|^2 = -\alpha|\mathbf{u}_j^{(2)}|^2 + \alpha|\mathbf{v}_j^{(2)}|^2$ for $j = 1, 2, 3$. We can also write this as $|\mathbf{u}_j^{(1)}|^2 + \alpha|\mathbf{u}_j^{(2)}|^2 = |\mathbf{v}_j^{(1)}|^2 + \alpha|\mathbf{v}_j^{(2)}|^2$. It suffices to show that there exists a complex number $\beta = re^{i\theta}$, $r > 0$, such that

$$|\mathbf{v}^{(1)} + \beta\mathbf{v}^{(2)}| = |\mathbf{u}^{(1)} + \beta\mathbf{u}^{(2)}| \tag{2.2}$$

(then we can find a complex matrix D with $|D| = I$ such that $\mathbf{v}^{(1)} + \beta\mathbf{v}^{(2)}$ is an eigenvector of DA_0 that corresponds to the eigenvalue s). Equation (2.2) is equivalent to the following system:

$$\begin{aligned} |v_1^{(1)} + re^{i\theta} v_1^{(2)}| &= |u_1^{(1)} + re^{i\theta} u_1^{(2)}|, \\ |v_2^{(1)} + re^{i\theta} v_2^{(2)}| &= |u_2^{(1)} + re^{i\theta} u_2^{(2)}|, \\ |v_3^{(1)} + re^{i\theta} v_3^{(2)}| &= |u_3^{(1)} + re^{i\theta} u_3^{(2)}|. \end{aligned} \tag{2.3}$$

Note that since $\mathbf{v}^{(1)} \perp \mathbf{v}^{(2)}$, and $\mathbf{u}^{(1)} \perp \mathbf{u}^{(2)}$, $\|\mathbf{v}^{(1)} + \beta\mathbf{v}^{(2)}\|^2 = 1 + |\beta|^2 = \|\mathbf{u}^{(1)} + \beta\mathbf{u}^{(2)}\|^2$, so any two of the equalities in (2.3) imply the third one.

We rewrite the equations (2.3) in the following way:

$$|v_j^{(1)}|^2 + r^2|v_j^{(2)}|^2 + 2rRe(e^{i\theta}\overline{v}_j^{(1)} v_j^{(2)}) = |u_j^{(1)}|^2 + r^2|u_j^{(2)}|^2 + 2rRe(e^{i\theta}\overline{u}_j^{(1)} u_j^{(2)}),\ j = 1,2,3. \tag{2.4}$$

Subtract $|v_j^{(1)}|^2 + \alpha|v_j^{(2)}|^2 = |u_j^{(1)}|^2 + \alpha|u_j^{(2)}|^2$ from both sides of the j-th equation in 2.4.

$$(r^2 - \alpha)(|v_j^{(2)}|^2 - |u_j^{(2)}|^2) + 2rRe(e^{i\theta}(\overline{v}_j^{(1)} v_j^{(2)} - \overline{u}_j^{(1)} u_j^{(2)})) = 0,\ j = 1,2,3. \tag{2.5}$$

If $|v_j^{(2)}|^2 - |u_j^{(2)}|^2 = 0$ for two values of j, then $|v_j^{(2)}|^2 - |u_j^{(2)}|^2 = 0$ for all three values of j, (since $\|\mathbf{v}^{(2)}\| = \|\mathbf{u}^{(2)}\| = 1$); this contradicts to the assumption that $\mathbf{b}^{(2)}$ is nonzero.

If $|v_j^{(2)}|^2 - |u_j^{(2)}|^2 = 0$ for exactly one value of j, say $j = 1$, then we choose θ such that $Re(e^{i\theta}(\overline{v}_1^{(1)} v_1^{(2)} - \overline{u}_1^{(1)} u_1^{(2)})) = 0$, (i.e. $\theta = \pi/2 - \text{Arg}(\overline{v}_1^{(1)} v_1^{(2)} - \overline{u}_1^{(1)} u_1^{(2)})$). Fix this θ and consider the second equation of (2.5), which is a quadratic equation with real coefficients of the variable r. Since the coefficient of r^2 and the constant term have opposite signs, there exists a unique positive root. Thus, we have found a $\beta = re^{i\theta}$ such that two equations in (2.5) are satisfied, which implies that the system (2.5) has a solution.

Now suppose $|v_j^{(2)}|^2 - |u_j^{(2)}|^2 \neq 0$ for $j = 1,2,3$. Then the system (2.5) is equivalent to

$$r^2 - 2rRe\left(e^{i\theta}\,\frac{\overline{v}_j^{(1)} v_j^{(2)} - \overline{u}_j^{(1)} u_j^{(2)}}{|v_j^{(2)}|^2 - |u_j^{(2)}|^2}\right) - \alpha = 0,\ j = 1,2,3 \tag{2.6}$$

Denote for convenience $w_j = \frac{\overline{v}_j^{(1)} v_j^{(2)} - \overline{u}_j^{(1)} u_j^{(2)}}{|v_j^{(2)}|^2 - |u_j^{(2)}|^2}$, $j = 1,2,3$. We need to find $r > 0$ and θ such that

$$r = -Re(e^{i\theta}w_j) + \sqrt{(Re(e^{i\theta}w_j))^2 + \alpha},\ j = 1,2 \tag{2.7}$$

The expression in (2.7) is always positive, so it is sufficient to show that we can always choose θ such that $Re(e^{i\theta}w_1) = Re(e^{i\theta}w_2)$, i.e. $Re(e^{i\theta}(w_1 - w_2)) = 0$, which is clearly, true (let $\theta = \pi/2 - \text{Arg}(w_1 - w_2)$). □

Proposition 2.2.9 gives explicit formulas for the matrices D_0, with $|D_0| = I_2$, such that $\rho(D_0A) = \max\{\rho(DA) : |D| \le I_2\}$ and $P_0 \in \mathcal{D}^+$ such that $\|P_0^{-1}AP_0\| = \inf\{\|P^{-1}AP\| : P \in \mathcal{D}^+\}$. It would be interesting to find such formulas for $n \ge 3$. Another open problem is to find when $\max\{\rho(DA) : |D| \le I_n\}$ is attained on the boundary, i.e. when boundary convergence implies $D_{\mathbb{C}}$ convergence. A partial result is established in the following proposition.

Proposition 2.2.12. *If $\|A\| = 1$ and A is boundary convergent, then A is $D_{\mathbb{C}}$ convergent.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ with $\|A\| = 1$ and suppose that A is not $D_{\mathbb{C}}$ convergent. Then there exist a complex diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$, $|D| \le I$ and a vector $\mathbf{v} \ne 0$ such that $DA\mathbf{v} = \mathbf{v}$. Note that $\|\mathbf{v}\| = \|DA\mathbf{v}\| \le \|D\|\|A\mathbf{v}\| \le \|A\mathbf{v}\| \le \|\mathbf{v}\|$ (since $\|D\| \le 1$ and $\|A\| = 1$), so $\|DA\mathbf{v}\| = \|A\mathbf{v}\|$. This can happen only if for each $j = 1, \dots, n$ either $|d_j| = 1$ or $(A\mathbf{v})_j = 0$. Let $D' = \operatorname{diag}(d'_1, \dots, d'_n)$, where $d'_j = 1$ whenever $(A\mathbf{v})_j = 0$, and $d'_j = d_j$ otherwise. Then $|D'| = I$ and $D'A\mathbf{v} = DA\mathbf{v} = \mathbf{v}$, i.e. A is not boundary convergent. □

Example 2.2.13. *There exists a complex 4×4 matrix A such that A is $D_{\mathbb{C}}$ convergent, but not diagonally convergent.*

Let $\mathbf{v}_1 = \frac{1}{2}[1, 1, 1, 1]^T$, $\mathbf{v}_2 = \frac{1}{3\sqrt{2}}[1, 1+i, -1+2i, -1-3i]^T$, $\mathbf{u}_1 = \frac{1}{3\sqrt{2}}[1, -1+i, 1+2i, -1-3i]^T$, $\mathbf{u}_2 = \mathbf{v}_1$, and $A = [\mathbf{u}_1, \mathbf{u}_2]\begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \end{bmatrix}$. Then A is such that $A\mathbf{v}_1 = \mathbf{u}_1$, $A\mathbf{v}_2 = \mathbf{u}_2$ and $A\mathbf{x} = 0$ for all $\mathbf{x} \perp \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Clearly, the spectral norm $\|A\| = 1$. We will show that for any positive diagonal matrix P, the spectral norm $\|PAP^{-1}\| \ge \|A\| = 1$, because $|\mathbf{v}_1| = |\mathbf{u}_2|$ and $|\mathbf{v}_2| = |\mathbf{u}_1|$. Suppose P is positive diagonal, such that $\|PAP^{-1}\| < 1$. Then $PAP^{-1}P\mathbf{v}_k = P\mathbf{u}_k$ and $\|P\mathbf{v}_k\| > \|P\mathbf{u}_k\|$, $k = 1, 2$. But this is not possible, since

$$\|P\mathbf{v}_1\| = \|P\mathbf{u}_2\| \ \text{ and } \ \|P\mathbf{v}_2\| = \|P|\mathbf{v}_2|\| = \|P|\mathbf{u}_1|\| = \|P\mathbf{u}_1\|, \tag{2.8}$$

so

$$\|P\mathbf{v}_1\| > \|P\mathbf{u}_1\| = \|P\mathbf{v}_2\| > \|P\mathbf{u}_2\| = \|P\mathbf{v}_1\|, \tag{2.9}$$

which is a contradiction. Therefore, A is not diagonally convergent.

Now we will show that A is boundary convergent, which by Proposition 2.2.12 will imply that A is $D_{\mathbb{C}}$ convergent, because $\|A\| = 1$. Clearly, $\rho(DA) \le 1$ for all D with $|D| = I$.

Suppose there exists D, with $|D| = I$ such that $\rho(DA) = 1$, i.e. there exists $\mathbf{w}$, $\|\mathbf{w}\| = 1$, such that $DA\mathbf{w} = \mathbf{w}$, so $A\mathbf{w} = D^{-1}\mathbf{w}$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be an orthonormal basis (note that we already defined unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1 \perp \mathbf{v}_2$). Then $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 + \beta_4\mathbf{v}_4$ for some complex numbers $\beta_1, \ldots, \beta_4$. Since

$$|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + |\beta_4|^2 = \|\mathbf{w}\|^2 = \|D^{-1}\mathbf{w}\|^2 = \|A\mathbf{w}\|^2 = \|\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2\|^2 = |\beta_1|^2 + |\beta_2|^2,$$

$\beta_3 = \beta_4 = 0$. So we have $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2$ such that $|\mathbf{w}| = |A\mathbf{w}|$, i.e. $|\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2| = |\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2|$. Note also, that $|\mathbf{v}_1| \neq |\mathbf{u}_1|$ and $|\mathbf{v}_2| \neq |\mathbf{u}_2|$, so $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Define $\beta = re^{i\theta} = \frac{\beta_2}{\beta_1} \neq 0$, so

$$|\mathbf{v}_1 + \beta\mathbf{v}_2| = |\mathbf{u}_1 + \beta\mathbf{u}_2|. \tag{2.10}$$

We will show that such β does not exist. Equation (2.10) is equivalent to

$$\begin{aligned} r &= 1 \\ Re(e^{i\theta}(1+i)) &= 0 \\ Re(e^{i\theta}(-1+2i)) &= 0 \\ Re(e^{i\theta}(-i)) &= 0, \end{aligned} \tag{2.11}$$

which obviously does not have a solution. Therefore, the matrix A is boundary convergent. Computations show that $\max\{\rho(DA) : |D| = I\} \approx \rho(\mathrm{diag}(1, -0.269021-0.963134i, 0.494094-0.869409i, -0.802318+0.596897i)A) \approx 0.9462$.

2.3 Examples of 2×2 and 3×3 Matrices

General characterization of diagonal, $D_{\mathbb{C}}$ and $D_{\mathbb{R}}$ convergent matrices is a complicated problem. Tests for $D_{\mathbb{R}}$ convergence for real 2×2 and 3×3 matrices are given in [10, 11], and for diagonal convergence for real 2×2 matrices in [22].

In [10, 11, 22] it was proven that for real 2×2 matrices vertex convergence implies diagonal convergence and for real 3×3 matrices vertex convergence implies $D_{\mathbb{R}}$ convergence. In [22] an example of a real 4×4 vertex convergent but not $D_{\mathbb{R}}$ convergent matrix is given. In Section 2 it was shown that diagonal convergence, $D_{\mathbb{C}}$ convergence and boundary convergence are all equivalent for 3×3 matrices and an example was given of a 4×4 matrix that is $D_{\mathbb{C}}$ convergent but not diagonally convergent. For complex matrices, diagonal convergence, $D_{\mathbb{C}}$ convergence and boundary convergence are the natural types of convergence to discuss. $D_{\mathbb{R}}$ convergence and vertex convergence can be examined for complex matrices as well, and not surprisingly, the situation is different even in 2×2 case.

Example 2.3.1. Vertex convergence of a complex 2×2 matrix does not guarantee $D_{\mathbb{R}}$-convergence.

$A = \begin{bmatrix} 1 & \frac{1+i}{2} \\ \frac{1+i}{2} & i \end{bmatrix}$ has an eigenvalue $\frac{1+i}{2}$ with multiplicity 2;

and $\operatorname{diag}(1, -1)\, A$ has an eigenvalue $\frac{1-i}{2}$ with multiplicity 2, so A is vertex convergent, but it is easy to see that A is not $D_{\mathbb{R}}$-convergent, since $a_{11} = 1$.

Moreover, for 2×2 matrices $D_{\mathbb{R}}$-convergence and $D_{\mathbb{C}}$-convergence are not equivalent. To show this we will use the following criterion.

Theorem 2.3.2. Schur-Cohn criterion (see, for example, [36])

All roots of the polynomial $p(x) = a_0x^n + \cdots + a_n$ lie inside the unit circle if and only if the Hermitian matrix $C = [c_{ij}] \in \mathbb{C}^{n\times n}$ is positive definite, where

$c_{ij} = \sum_{s=1}^{\min\{i,j\}} (\overline{a}_{i-s}a_{j-s} - \overline{a}_{n-j+s}a_{n-i+s}).$

Example 2.3.3. $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ i & \frac{i}{3} \end{bmatrix}$ is not $D_{\mathbb{C}}$-convergent (the eigenvalues of $\operatorname{diag}(1, -i)\, A$ are $\frac{2\pm 3\sqrt{2}}{6}$). To show that A is $D_{\mathbb{R}}$-convergent, it is sufficient to show that $A_1(t) = \operatorname{diag}(1, t)\, A$ and $A_2(t) = \operatorname{diag}(t, 1)\, A$ are convergent for all $t \in [-1, 1]$. To apply the Schur-Cohn criterion to the characteristic polynomial of $A_1(t)$, we compute:

$c_{11}(t) = c_{22}(t) = 1 - |\det A_1(t)|^2 = 1 - \left|\frac{7i}{18}t\right|^2.$

$c_{12}(t) = \operatorname{tr}A_1(t) - \overline{\operatorname{tr}A_1(t)}\, \det A_1(t) = \frac{1}{3}(1 + it) + \frac{7}{54}(t^2 + it) = \frac{1}{54}(18 + 7t^2 + 25it).$

In order to show that $A_1(t)$ is convergent for all $t \in [-1, 1]$, by Theorem 7.2.5 in [14, p. 404] it suffices to show that $c_{11}(t) > 0$ and $c_{11}(t)c_{22}(t) - |c_{12}(t)|^2 > 0$ for all $t \in [-1, 1]$ (i.e. leading principal minors of C are positive).

The first inequality is clearly true. To show that the second inequality holds, we define $f(t) =$

$c_{11}(t)c_{22}(t) - |c_{12}(t)|^2 = \frac{8}{9} - \frac{1759}{2916}t^2 + \frac{637}{104976}t^4$. Routine computations show that $f(1) > 0$, $f(t)$ is an even function, and it decreases on $[0, 1]$, so $f(t) > 0$ for all $t \in [-1, 1]$. By a similar argument, it can be shown that $A_2(t)$ is convergent for all $t \in [-1, 1]$. Therefore, A is $D_{\mathbb{R}}$-convergent.

Example 2.3.4. For real 3×3 matrices $D_{\mathbb{R}}$ convergence does not imply boundary convergence as the following example shows.
$A = \begin{bmatrix} \frac{1}{5} & 0 & \frac{3}{4} \\ \frac{7}{10} & 0 & \frac{9}{20} \\ \frac{2}{5} & \frac{1}{2} & -\frac{7}{10} \end{bmatrix}$ is vertex convergent (and therefore, by [11] is $D_{\mathbb{R}}$-convergent), but not boundary convergent: $\max\{\rho(DA) : D \in \mathbb{R}^{3\times3}, |D| = I\} = \rho(\text{diag}(1, 1, -1)\, A) \approx 0.96443$, while $\rho(\text{diag}(1, 1, i)\, A) \approx 1.07274$.

Example 2.3.4 also implies that for real 3×3 matrices $D_{\mathbb{R}}$ convergence is not equivalent to diagonal convergence (this was noted before and an example was provided in [22]).

CHAPTER 3. QUALITATIVE CONVERGENCE OF MATRICES

Based on a paper accepted for publication in *Linear Algebra and its Applications*

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Abstract

In this paper, we study potential convergence of modulus patterns. A modulus pattern Z is convergent if all complex matrices in $Q(Z)$ (i.e. all matrices with modulus pattern Z) are convergent. A modulus pattern is potentially (absolutely) convergent if there exists a (non-negative) convergent matrix in $Q(Z)$. We also introduce types of potential convergence that correspond to diagonal and D convergence, studied in [22]. Convergent modulus patterns have been completely characterized by E. Kaszkurewicz and A. Bhaya [20]. This paper presents some techniques that can be used to establish potential convergence. Potential absolute convergence and potential diagonal convergence are shown to be equivalent, and their complete characterization for $n \times n$ modulus patterns is given. Complete characterizations of all introduced types of potential convergence for 2×2 modulus patterns are also presented.

3.1 Introduction

A matrix A is *(negative) stable* if all its eigenvalues lie in the open left half plane of the complex plane. It is well known that the matrix A is stable if and only if the continuous time dynamical system,

$$\frac{dx}{dt} = Ax(t)$$

is asymptotically stable at 0 (see, for example, [29, p. 55-10]).

A complex matrix is *convergent* if all its eigenvalues lie in the open unit disk of the complex plane. The matrix A is convergent if and only if the discrete time dynamical system,

$$x(t_{k+1}) = Ax(t_k)$$

is asymptotically stable at 0 (see, for example, [8, p. 186]).

Qualitative analysis, where numerical values of the matrix are not specified, plays an important role in the study of stability. In fact, qualitative analysis arose from stability problems in economics and ecology (see, for example, [32, 27, 15, 17]). In the continuous time case, qualitative stability refers to the study of *sign patterns*, when only signs of the entries of the real matrix A are known. The analogous problem in the discrete time case refers to *modulus patterns*, where each entry of the (real or complex) matrix A is specified only by whether its modulus is equal to 0, in the interval $(0, 1)$, equal to 1, or greater than 1, denoted by 0, $\boxed{<}$, $\boxed{1}$ or $\boxed{>}$, respectively. Qualitative information about a matrix A, that is, the pattern $Z(A)$ (a sign pattern in the first case, and a modulus pattern in the second case), can be used to determine stability or convergence of A. Especially when the qualitative information is known from the nature of the problem but the numerical values may be unreliable, the study of qualitative stability or convergence is very useful.

In particular, two natural problems can be considered in the study of patterns:

(a) determine whether all matrices with the given pattern are stable (respectively convergent); and

(b) determine whether there exists a stable (respectively convergent) matrix with the given pattern. The qualitative class of an $n \times n$ pattern Z is the set $Q(Z) = \{A : Z(A) = Z\}$. If Z is a sign pattern, then only real matrices are considered, but in the case of modulus patterns, we assume that $A \in \mathbb{C}^{n\times n}$, and if we want to restrict the qualitative class to real matrices, we will write $Q(Z) \cap \mathbb{R}^{n\times n}$.

Many results about patterns are best described using graph theoretic terminology. If A is an $n \times n$ matrix, then $\Gamma(A)$ denotes the directed graph on n vertices, where the arc (i, j) is present if and only if $a_{ij} \neq 0$. A *cycle* in the matrix A is the list of nonzero entries of A that correspond to the arcs of a cycle in $\Gamma(A)$. The length of a cycle is the number of arcs in it. A nonzero diagonal entry, which corresponds to a loop in the digraph, is a cycle of length 1, and the arcs (i, j) and (j, i) form a cycle of length 2. The *gain* of the cycle $a_{i_1 i_2}, a_{i_2 i_3}, \ldots, a_{i_n i_1}$ in the matrix A is the product $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_n i_1}$. A *generalized cycle* is a union of vertex-disjoint cycles. The length of a generalized cycle is the sum of lengths of all cycles in it. For a sign or modulus pattern Z, $\Gamma(Z) = \Gamma(A)$ for any $A \in Q(Z)$, and the terms cycle and generalized cycle are also applied to Z.

A sign pattern Z *requires stability*, or is *sign stable* if all matrices in $Q(Z)$ are stable [17]. A sign pattern Z *allows stability*, or is *potentially stable* if there exists a stable matrix in $Q(Z)$ [16]. Sign patterns that require stability have been completely characterized in [17]. The problem of potential stability has not been completely solved yet, but a number of partial results have been obtained, for example, [3, 16, 19, 23, 18, 12].

A modulus pattern *requires convergence* (or is *modulus convergent*) if all matrices in $Q(Z)$ are convergent. E. Kaszkurewicz and A. Bhaya in [20] defined and completely characterized modulus convergence (which they call m-stability). Although they stated convergence results for modulus patterns of real matrices, their characterization is also valid for modulus patterns of complex matrices, as is noted there.

Theorem 3.1.1. [20] *An irreducible $n \times n$ ($n \geq 2$) modulus pattern $Z = (z_{ij})$ requires convergence if and only if the following three conditions are satisfied:*

1. *there is only one cycle in Z, and it is of length n;*
2. *there is no entry equal to $\boxed{>}$ in Z;*
3. *there exists at least one entry equal to $\boxed{<}$ in Z.*

There is a relation between convergence and stability that can be derived from the linear fractional transformation $w = \dfrac{z+1}{z-1}$, which maps the open unit disk onto the open left half plane [28, p. 252-253]. The relationship between stable and convergent matrices can be written in the form $A = (B + I)(B - I)^{-1}$. The matrix A is stable if and only if B is convergent. Unfortunately, this relationship cannot be applied directly to sign or modulus patterns, because, as is noted in [28], without knowing some quantitative information about a matrix, it is not possible to determine the qualitative class of its transform.

In many applications, the system being modelled is nonlinear, so stability of the equilibrium of its linear approximation guarantees only local asymptotical stability. Stronger types of matrix stability or convergence can provide more information on global behavior of certain nonlinear systems, and in some cases they can guarantee global stability (see, for example, [25] and [22]).

A matrix A is D *stable* if PA is stable for all diagonal matrices $P \succ 0$, where $B \succ 0$, (respectively $B \prec 0$) denotes that B is a positive (respectively, negative) definite matrix. A matrix A is *diagonally stable* if there exists a diagonal matrix $P \succ 0$ such that $PA + A^*P \prec 0$. In this paper, P generally denotes a positive definite diagonal matrix, and D generally denotes a diagonal matrix with $|D| \leq I$ (where $|B| = [|b_{ij}|]$, and $B \leq C$ denotes that $b_{ij} \leq c_{ij}$ for all i, j). Diagonal stability implies D stability (see, for example, [22, p. 32]). A matrix A is $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) *convergent* if DA is convergent for all real (respectively, complex) diagonal matrices D with $|D| \leq I$. A matrix A is *diagonally convergent* if there exists a diagonal matrix $P \succ 0$ such that $P - A^*PA \succ 0$ [22, p. 50]. The following proposition is proved in [22, p. 53] for real matrices, but the same proof can be applied for complex matrices.

Proposition 3.1.2. [22] *If $A \in \mathbb{C}^{n\times n}$ is diagonally convergent, then it is $D_{\mathbb{C}}$ convergent.*

In this paper we introduce several types of potential convergence (Definitions 3.2.4-3.2.10). All of them are characterized for order 2 modulus patterns. We show that potential absolute convergence is equivalent to potential diagonal convergence and present its complete characterization for order n modulus patterns. Also, partial results on potential convergence and potential D convergence are presented.

3.2 Types of Potential Convergence

Definition 3.2.1. The *modulus pattern of a matrix A* is a matrix $Z(A) = (z_{ij})$, whose entries are from the set $\{0, \boxed{1}, \boxed{<}, \boxed{>}\}$ such that

$$z_{ij} = \begin{cases} 0 & \text{if } a_{ij} = 0, \\ \boxed{1} & \text{if } |a_{ij}| = 1, \\ \boxed{<} & \text{if } 0 < |a_{ij}| < 1, \\ \boxed{>} & \text{if } |a_{ij}| > 1. \end{cases}$$

Definition 3.2.2. A modulus pattern Z has *modular determinant* equal to 0, $\boxed{1}$, $\boxed{<}$, or $\boxed{>}$ if determinants of all matrices in $Q(Z)$ belong to the same modulus class (0, $\boxed{1}$, $\boxed{<}$, or $\boxed{>}$, respectively).

Observation 3.2.3. *An $n \times n$ modulus pattern has modular determinant if and only if it has at most one generalized cycle of length n, and this generalized cycle does not contain both $\boxed{<}$ and $\boxed{>}$.*

Definition 3.2.4. A modulus pattern Z *allows convergence*, or is *potentially convergent*, if there exists a convergent matrix in $Q(Z)$.

The proof of the Main Theorem in [20] (cf. Theorem 3.1.1 here) first characterizes the case of nonnegative matrices in $Q(Z)$ and then extends the result to all real matrices A in $Q(Z)$ by considering the matrix $|A|$. Our goal is to study potential convergence, so it seems useful to introduce the following definitions.

Definition 3.2.5. A modulus pattern Z is said to *allow absolute convergence*, or is *potentially absolutely convergent*, if there exists a nonnegative convergent matrix $A \in Q(Z)$.

Definition 3.2.6. A modulus pattern Z is said to *allow real convergence* if there exists a convergent matrix $A \in Q(Z) \cap \mathbb{R}^{n \times n}$.

Definition 3.2.7. A modulus pattern Z is *said to allow* $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) *convergence*, or is *potentially* $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) *convergent* if there exists a $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) convergent matrix in $Q(Z)$.

Definition 3.2.8. A modulus pattern Z is said to *allow real* $D_{\mathbb{R}}$ (respectively, *real* $D_{\mathbb{C}}$) *convergence* if there exists a $D_{\mathbb{R}}$ (respectively $D_{\mathbb{C}}$) convergent matrix in $Q(Z) \cap \mathbb{R}^{n \times n}$.

Definition 3.2.9. A modulus pattern Z is said to *allow diagonal convergence*, or is *potentially diagonally convergent* if there exists a diagonally convergent matrix in $Q(Z)$.

Definition 3.2.10. A modulus pattern Z is said to *allow real diagonal convergence* if there exists a diagonally convergent matrix in $Q(Z) \cap \mathbb{R}^{n\times n}$.

Analogous definitions can be given for modulus patterns requiring real, absolute, diagonal or D convergence. However, from the bounds on spectral radius in [34, p. 9-10] it follows that a modulus pattern requires convergence if and only if it requires absolute convergence, and since absolute convergence implies diagonal convergence ([22, p. 62]), all these definitions are equivalent to requiring convergence.

Clearly, potential absolute convergence implies potential real convergence, and potential real convergence implies potential convergence but not vice versa as the following examples show.

Example 3.2.11. Let Z be the 2×2 modulus pattern with all entries equal to $\boxed{1}$. Then Z allows real convergence ($\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in Q(Z) \cap \mathbb{R}^{2\times 2}$), but Z is not potentially absolutely convergent, since the only nonnegative matrix in $Q(Z)$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is not convergent.

In general, for any $k \in \mathbb{N}$, if Z is a $2k \times 2k$ modulus pattern consisting of all $\boxed{1}$'s, then Z is not potentially absolutely convergent, but there exists a real nilpotent matrix $A \in Q(Z)$ (for example, let k rows of A consist of 1's and k rows consist of -1's), and hence Z allows real convergence.

Example 3.2.12. Let Z be the 3×3 modulus pattern with all entries equal to $\boxed{1}$. The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \zeta & \zeta & \zeta \\ \zeta^2 & \zeta^2 & \zeta^2 \end{bmatrix} \in Q(Z),$$

(where $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a primitive third root of unity) is nilpotent and thus convergent.

We will show that there does not exist a real convergent matrix in $Q(Z)$. Suppose $B \in Q(Z) \cap \mathbb{R}^{3\times 3}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. The characteristic polynomial of B is: $x^3 + a_2x^2 + a_1x + a_0$, where $a_0 = -\det B = -\lambda_1\lambda_2\lambda_3$. Since all entries of B are ± 1, $\det B$ is an integer, and if $|\det B| \geq 1$, then B is not convergent. Suppose $\det B = 0$. Then at least one of the eigenvalues, say λ_1, is 0, so $a_1 = \lambda_2\lambda_3$ and is an integer. If $|a_1| \geq 1$, then B is not convergent. If $a_1 = 0$, then at least two eigenvalues are 0. Since $|\mathrm{tr}B| \geq 1$, B is not convergent. Therefore, Z does not allow real convergence.

In general, for any $k \in \mathbb{N}$, if Z is a $(2k+1) \times (2k+1)$ modulus pattern consisting of all $\boxed{1}$'s, we can construct a complex nilpotent matrix $A \in Q(Z)$ having k rows equal to $\mathbf{e}$, $k-1$ rows are $-\mathbf{e}$, one row is $\zeta\mathbf{e}$ and one row is $\zeta^2\mathbf{e}$, where $\mathbf{e} = [1, \ldots, 1]$. Another way to construct a nilpotent matrix is to let its jth row be $\theta^j\mathbf{e}$, where θ is a primitive $(2k+1)$st root of unity, $j = 1, \ldots, 2k+1$. Using the same idea as in the 3×3 case, it can be shown that Z does not allow real convergence.

Among the four different types of potential D convergence (Definitions 3.2.7-3.2.8), obviously $D_{\mathbb{R}}$ convergence is the weakest and real $D_{\mathbb{C}}$ convergence is the strongest, while the relationship between $D_{\mathbb{C}}$ convergence and real $D_{\mathbb{R}}$ convergence is not clear.

The next theorem presents a technique that can be compared to the one used for sign patterns in [19], where subpatterns are considered for establishing potential stability. While Theorem 3 in [19] says that potential stability of a sign pattern is preserved if a zero is replaced by $+$ or $-$, Theorem 3.2.13 below says that potential convergence of a modulus pattern is preserved if 0 is replaced by $\boxed{<}$, or if $\boxed{1}$ is replaced by either $\boxed{<}$ or $\boxed{>}$.

Theorem 3.2.13. *Given a modulus pattern Z, let $\widehat{Z}$ be a pattern obtained from Z by replacing some $\boxed{<}$ entries of Z by 0's and some nonzero entries of Z by $\boxed{1}$'s. Then every type of convergence described in Definitions 3.2.4-3.2.10 allowed by $\widehat{Z}$ is also allowed by Z.*

Proof. Let $\widehat{A} \in Q(\widehat{Z})$ be a (real, nonnegative, diagonally, $D_{\mathbb{R}}$ or $D_{\mathbb{C}}$) convergent matrix. We construct a matrix $C(\epsilon) = [c_{ij}]$, $\epsilon \in [0, 1)$ in the following way:

$$c_{ij} = \begin{cases} 0 & \text{if } z_{ij} = \widehat{z}_{ij}, \\ \epsilon & \text{if } z_{ij} = \boxed{<}, \widehat{z}_{ij} = 0, \\ -\epsilon \widehat{a}_{ij} & \text{if } z_{ij} = \boxed{<}, \widehat{z}_{ij} = \boxed{1}, \\ \epsilon \widehat{a}_{ij} & \text{if } z_{ij} = \boxed{>}, \widehat{z}_{ij} = \boxed{1}. \end{cases}$$

Note that $A(\epsilon) = \widehat{A} + C(\epsilon) \in Q(Z)$ for all $\epsilon \in (0, 1)$ and $A(0) = \widehat{A}$.

Since the eigenvalues depend continuously on the entries of the matrix, there exists a sufficiently small $\epsilon \in (0, 1)$ such that $A(\epsilon)$ is also convergent. By construction, if $\widehat{A}$ is real (respectively, nonnegative), then A is real (respectively, nonnegative). If $\widehat{A}$ is diagonally convergent and $P \succ 0$ is a diagonal matrix such that $P - \widehat{A}^* P \widehat{A} \succ 0$, let $f(x)$ be the function equal to the smallest eigenvalue of $P - A^*(x)\, P\, A(x)$. Since $f(x)$ is continuous and $f(0) > 0$, there exists a sufficiently small $\epsilon \in (0, 1)$ such that $f(\epsilon) > 0$, i.e. $A(\epsilon)$ is diagonally convergent.

If $\widehat{A}$ is $D_{\mathbb{F}}$ convergent ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$), then $\max\{\rho(D\widehat{A}) : |D| \leq I, D \in \mathbb{F}^{n \times n}\} < 1$ (the maximum exists, since ρ is a continuous function taken over a compact set). Let $f(x) = \max\{\rho(DA(x)) : |D| \leq I, D \in \mathbb{F}^{n \times n}\}$. Note that f is continuous and $f(0) < 1$, so there exists an $\epsilon \in (0, 1)$ such that $f(\epsilon) < 1$, i.e. $A(\epsilon)$ is $D_{\mathbb{F}}$ convergent. □

3.3 Characterization of Potential Absolute and Potential Diagonal Convergence

By Theorem 7.2.5 in [14, p. 404], the diagonal entries of a positive definite matrix are positive, and since $(A^*A)_{jj} = \sum_{i=1}^{n} |a_{ij}|^2$, the next observations are clear.

Observation 3.3.1. *If $I - A^*A \succ 0$, then moduli of all entries of A are less than 1.*

Observation 3.3.2. *There exists a matrix $A \in Q(Z)$ such that $I - A^*A \succ 0$ if and only if Z does not have any entries from $\{\boxed{1}, \boxed{>}\}$.*

Lemma 3.3.3. *If A is diagonally convergent, then gains of all its cycles have moduli less than 1.*

Proof. Suppose that A is diagonally convergent, i.e. there exists a diagonal matrix $P \succ 0$ such that $P - A^*PA \succ 0$. This is equivalent to $I - B^*B \succ 0$, where $B = P^{1/2}AP^{-1/2}$. So, by

Observation 3.3.1, moduli of all entries of B are less than 1, therefore, gains of all its cycles have moduli less than 1. Since diagonal similarity preserves gains of the cycles, the gains of all cycles in A also have moduli less than 1. $\square$

Corollary 3.3.4. *A necessary condition for a modulus pattern to allow diagonal convergence is that each of its cycles has at least one entry* $\boxed{<}$.

Proposition 3.3.5. *A sufficient condition for a modulus pattern to allow absolute convergence is that each of its cycles has at least one entry* $\boxed{<}$.

Proof. Suppose that each of the cycles in Z has at least one entry $\boxed{<}$. Let $\widehat{Z}$ be the modulus pattern obtained from Z by replacing all $\boxed{<}$ entries by 0's. Then $\widehat{Z}$ does not have any cycle, so all matrices in $Q(\widehat{Z})$ are nilpotent. Therefore, from Theorem 3.2.13 it follows that Z allows absolute convergence. $\square$

Since for nonnegative matrices convergence implies diagonal convergence (see [22, p. 62]), we have established the following result.

Theorem 3.3.6. *Let Z be a modulus pattern. The following are equivalent:*

1. *Each cycle in Z has at least one entry* $\boxed{<}$*;*
2. *Z allows absolute convergence;*
3. *Z allows real diagonal convergence;*
4. *Z allows diagonal convergence.*

3.4 Results on Potential Convergence and Potential D Convergence

In this section we provide some partial characterizations of potentially convergent and potentially D convergent modulus patterns. Since the spectrum of a reducible matrix is the union of the spectra of all its diagonal components, and the determinant of a convergent matrix must be less than 1 in absolute value, the following observations are clear.

Observation 3.4.1. *A reducible modulus pattern Z allows convergence of one of the types described in Definitions 3.2.4-3.2.10 if and only if each diagonal component of Z allows convergence of this type.*

Observation 3.4.2. *If an $n \times n$ modulus pattern Z has only one generalized cycle of length n, then a necessary condition for potential convergence is that at least one entry in this generalized cycle is equal to $\boxed{<}$. Equivalently, if Z has modular determinant $\boxed{>}$ or $\boxed{1}$, then Z does not allow convergence.*

Observation 3.4.3. *If an $n \times n$ modulus pattern Z has only one cycle and this cycle is of length n, then Z is potentially convergent if and only if it is potentially absolutely convergent.*

In the next two subsections we will show that for irreducible 2×2 modulus patterns the condition in Observation 3.4.2 is also sufficient (Proposition 3.4.9), but for $n \geq 3$ it is no longer true (Example 3.4.13).

Proposition 3.4.4. *If a modulus pattern Z has no zero entries, then it is potentially convergent. If Z is of even order, then it allows real convergence.*

Proof. By Theorem 3.2.13 it is sufficient to show that the order n modulus pattern $\widehat{Z}$ that consists of all ones is potentially convergent. For $n = 2k$, consider a real nilpotent matrix $A \in Q(\widehat{Z})$ constructed as in Example 3.2.11. For $n = 2k + 1$, consider a complex nilpotent matrix $A \in Q(\widehat{Z})$, constructed as in Example 3.2.12. □

Since any principal submatrix of a $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) convergent matrix is $D_{\mathbb{R}}$ (respectively, $D_{\mathbb{C}}$) convergent, we state the following observation.

Observation 3.4.5. *If Z is a potentially D convergent modulus pattern (any of the four types described in Definitions 3.2.7-3.2.8), then any principal submatrix of Z is potentially D convergent of the same type. In particular, if Γ' is an order k induced sub-digraph of $\Gamma(Z)$ that has only one generalized cycle of length k, then at least one entry in this cycle must be $\boxed{<}$.*

Note that from the Observation 3.4.5 it follows that a potentially D convergent modulus pattern cannot have $\boxed{1}$ or $\boxed{>}$ on its diagonal.

Theorem 3.4.6. *Let A be a complex (respectively, real) $n \times n$ matrix that has one cycle of length n and no other cycles of length ≥ 2. If A is $D_{\mathbb{C}}$ (respectively, $D_{\mathbb{R}}$) convergent then all its cycles have gains less than 1 in absolute value.*

Proof. The proof is obvious for $n = 1$, so assume that $n \geq 2$. By Observation 3.4.5 if A is D convergent, then $|a_{ii}| < 1$ for all $i = 1, \dots, n$. Suppose that A has a cycle of length n with gain δ and $|\delta| \geq 1$, while $|a_{ii}| < 1$, $i = 1, \dots, n$. We will show that A is not $D_{\mathbb{C}}$ (respectively, $D_{\mathbb{R}}$) convergent. Up to permutation similarity A is of the form:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 & a_{1n} \\ a_{21} & a_{22} & \dots & 0 & 0 \\ 0 & a_{32} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n,n-1} & a_{nn} \end{bmatrix}, \text{ where } |a_{21}a_{32}\dots a_{n,n-1}a_{1n}| = |\delta| \geq 1.$$

We choose a complex (respectively, real) matrix D ($|D| = I$) such that all diagonal entries of DA are nonnegative. Also, we choose a positive diagonal matrix P such that in the matrix $B = |\delta|^{-1/n} DP^{-1}AP$ we have $|b_{21}| = |b_{32}| = \dots = |b_{n,n-1}| = |b_{1n}| = 1$, $b_{21}b_{32}\dots b_{n,n-1}b_{1n} = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$, and $b_{jj} = \alpha_j \in [0, 1)$ for $j = 1, \dots, n$. Since $\rho(DA) = \rho(P^{-1}DAP) = \rho(DP^{-1}AP) = |\delta|^{1/n}\rho(B)$ it is enough to show that B is not convergent. The characteristic polynomial of B is $p_B(x) = \prod_{j=1}^{n}(x - \alpha_j) - e^{i\theta}$. Let $\alpha \geq 0$ be the least diagonal entry of B. If $\alpha_j = \alpha$ for all $j = 1, \dots, n$, then the eigenvalues of B are $\alpha + e^{i(\theta + 2k\pi)/n}$, $k = 0, \dots, n-1$, so, obviously, B is not convergent.

Now assume that at least one diagonal entry of B is greater than α. The characteristic polynomial of $B' = B - \alpha I$ is $p_{B'}(x) = \prod_{j=1}^{n}(x - (\alpha_j - \alpha)) - e^{i\theta}$ and $|\det(B')| = 1$. So, B' has an eigenvalue λ such that $|\lambda| \geq 1$. Note that $Re(\lambda) \geq 0$, since B' is irreducible and at least one Geršgorin disk (each of them has radius 1 since every row has only one off diagonal entry and this entry has modulus 1) has center on the positive axis (see [14, p. 344-356], Theorems 6.1.1 and 6.2.8) . Therefore, $\alpha + \lambda$, which is an eigenvalue of B, has modulus $|\alpha + \lambda| = \left((\alpha + Re(\lambda))^2 + (Im(\lambda))^2\right)^{1/2} \geq |\lambda| \geq 1$, i.e. B is not convergent. $\square$

Corollary 3.4.7. *If an $n \times n$ modulus pattern Z has one cycle of length n and no other cycles of length ≥ 2, then potential $D_{\mathbb{C}}$ convergence and potential real $D_{\mathbb{R}}$ convergence are both equivalent to potential absolute convergence.*

Proof. From Theorem 3.4.6 it follows that if a complex (respectively, real) matrix A having nonzero pattern associated with Z is $D_{\mathbb{C}}$ (respectively, $D_{\mathbb{R}}$) convergent, then all its cycles must have absolute gains < 1, so if Z allows $D_{\mathbb{C}}$ or real $D_{\mathbb{R}}$ convergence, it must also allow absolute convergence. □

Corollary 3.4.8. *Suppose that all cycles in the digraph of a modulus pattern Z have the following property:*

($*$) *Any two cycles of lengths $k_1 \geq 2$ and $k_2 \geq 2$ intersect at at most $\min\{k_1, k_2\} - 1$ vertices.*

Then potential $D_{\mathbb{C}}$ convergence of Z is equivalent to potential absolute convergence. In particular, if the digraph of Z does not have any cycles of length greater than 2, then $D_{\mathbb{C}}$ convergence of Z is equivalent to potential absolute convergence.

Proof. We only need to show that potential $D_{\mathbb{C}}$ convergence implies potential diagonal convergence. Suppose that Z is potentially $D_{\mathbb{C}}$ convergent and its digraph satisfies the property ($*$). Then every cycle of length k belongs to some order k principal submatrix Z' of Z, which does not contain any other cycle of length ≥ 2. By Observation 3.4.5 Z' must be potentially $D_{\mathbb{C}}$ convergent. So, by Theorem 3.4.6 each cycle in Z' has at least one entry $\boxed{<}$, therefore each cycle in Z has at least one entry $\boxed{<}$, i.e. Z is potentially absolutely convergent. □

3.4.1 2×2 Modulus Patterns

The following proposition characterizes potentially convergent 2×2 modulus patterns.

Proposition 3.4.9. *Let Z be an irreducible 2×2 modulus pattern. Then Z is not potentially convergent if and only if Z has modular determinant equal to $\boxed{1}$ or $\boxed{>}$. Moreover, for modulus patterns of order 2, potential convergence is equivalent to real potential convergence.*

Proof. An irreducible 2×2 modulus pattern with modular determinant equal to $\boxed{>}$ or $\boxed{1}$ (which by Observation 3.4.2 does not allow convergence) has the following form, up to permutation similarity:

$$\begin{bmatrix} \boxed{?} & \sharp \\ \sharp & 0 \end{bmatrix}, \text{ where } \boxed{?} \in \{0, \boxed{1}, \boxed{<}, \boxed{>}\} \text{ and } \sharp \in \{\boxed{1}, \boxed{>}\}.$$

We will show that all other irreducible 2×2 patterns allow real convergence. By Theorem 3.2.13, it is sufficient to show that the following three patterns allow real convergence:

$$Z_1 = \begin{bmatrix} 0 & \boxed{1} \\ \boxed{<} & 0 \end{bmatrix}, \; Z_2 = \begin{bmatrix} \boxed{1} & \boxed{1} \\ \boxed{<} & 0 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} \boxed{1} & \boxed{1} \\ \boxed{1} & \boxed{1} \end{bmatrix}.$$

The pattern Z_1 requires convergence, the pattern Z_2 allows real convergence (for example $\begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}$), and the pattern Z_3 allows real convergence (by Proposition 3.4.4).

To complete the proof of the second statement, note that a reducible 2×2 modulus pattern allows convergence if and only if both diagonal entries are $\boxed{<}$ or 0, so obviously it allows real convergence. □

From Corollary 3.4.7 it follows that a 2×2 modulus pattern allows $D_{\mathbb{C}}$ convergence (or allows real $D_{\mathbb{R}}$ convergence) if and only if it allows absolute convergence. An even stronger statement is true.

Theorem 3.4.10. *A 2×2 modulus pattern allows $D_{\mathbb{R}}$ convergence if and only if each of its cycles has at least one entry equal to $\boxed{<}$.*

Proof. Sufficiency is immediate. We show necessity. By Observation 3.4.5 it is clear that if Z allows $D_{\mathbb{R}}$ convergence, its diagonal entries must be 0 or $\boxed{<}$, and it can not have a modular determinant equal to $\boxed{1}$ or $\boxed{>}$. Note also, that if $A \in \mathbb{C}^{2 \times 2}$ is such that $|a_{12}a_{21}| = \delta > 1$, then the matrix $\frac{1}{\sqrt{\delta}}A$ is diagonally similar to a matrix B with $|b_{12}| = |b_{21}| = 1$. Since diagonal similarity preserves $D_{\mathbb{R}}$ convergence, it suffices to show that the modulus pattern $Z = \begin{bmatrix} \boxed{<} & \boxed{1} \\ \boxed{1} & \boxed{<} \end{bmatrix}$ does not allow $D_{\mathbb{R}}$ convergence. Let $A = (a_{ij}) \in Q(Z)$. Without loss of generality we can assume that $a_{11} = a > 0$ and $a_{22} = b_1 + ib_2$, where $b_1 \geq 0$ (since multiplying A by a scalar c, $|c| = 1$, and by a matrix $\mathrm{diag}(1, -1)$ does not change $D_{\mathbb{R}}$ convergence). Note that $p_A(x) = (x - a)(x - b_1 - ib_2) - a_{12}a_{21}$. Suppose $\lambda = \lambda_1 + i\lambda_2$ is an eigenvalue of A and $|\lambda| < 1$, i.e. $\lambda_1^2 + \lambda_2^2 = 1 - \epsilon$, for some $\epsilon \in (0, 1)$. We will show that another eigenvalue of A, $\mu = \mu_1 + i\mu_2$, has modulus greater than 1.

Since λ is a root of p_A, it satisfies $|\lambda - a|\,|\lambda - (b_1 + ib_2)| = 1$, so $(\lambda_1 - a)^2 + \lambda_2^2 = \alpha^2$ and $(\lambda_1 - b_1)^2 + (\lambda_2 - b_2)^2 = 1/\alpha^2$ for some $\alpha > 0$. Adding these two equalities, we obtain:
$2 - 2\epsilon + a^2 + b_1^2 + b_2^2 - 2((a + b_1)\lambda_1 + b_2\lambda_2) = \alpha^2 + 1/\alpha^2$, or
$a^2 + b_1^2 + b_2^2 - 2((a + b_1)\lambda_1 + b_2\lambda_2) = (\alpha - 1/\alpha)^2 + 2\epsilon$.

Since $\lambda + \mu = a + b_1 + ib_2$, we have $\mu_1 = a + b_1 - \lambda_1$ and $\mu_2 = b_2 - \lambda_2$.
So $|\mu|^2 = (a + b_1 - \lambda_1)^2 + (b_2 - \lambda_2)^2 = (a + b_1)^2 + b_2^2 + \lambda_1^2 + \lambda_2^2 - 2((a + b_1)\lambda_1 + b_2\lambda_2) = (\alpha - 1/\alpha)^2 + 2\epsilon + 2ab_1 + 1 - \epsilon = 1 + (\alpha - 1/\alpha)^2 + \epsilon + ab_1 > 1$. □

Corollary 3.4.11. *Let Z be a modulus pattern whose digraph does not have any cycles of length greater than 2 (or, in particular, let Z be any 2×2 modulus pattern). The following are equivalent:*

- *each cycle in Z has at least one entry $\boxed{<}$;*
- *Z allows absolute convergence;*
- *Z allows real diagonal convergence;*
- *Z allows diagonal convergence;*
- *Z allows real $D_{\mathbb{C}}$ convergence;*
- *Z allows $D_{\mathbb{C}}$ convergence;*
- *Z allows real $D_{\mathbb{R}}$ convergence;*
- *Z allows $D_{\mathbb{R}}$ convergence.*

Note that although for 2×2 modulus patterns potential $D_{\mathbb{R}}$ convergence is equivalent to potential $D_{\mathbb{C}}$ convergence, for 2×2 matrices $D_{\mathbb{R}}$ convergence and $D_{\mathbb{C}}$ convergence are not equivalent (see Example 2.3.3).

3.4.2 Some Examples of 3×3 Modulus Patterns

From Proposition 3.3.5 it follows that if all cycles of a matrix have sufficiently small gains, this matrix is convergent. On the other hand, Examples 3.2.11 and 3.2.12 show that a convergent matrix can have cycles with arbitrarily large gains (if A is nilpotent, then αA is also nilpotent for any number α). Example 3.4.12 below shows that sometimes increasing a cycle gain can make a divergent matrix convergent. For an analogous result related to sign patterns see [3]. Example 3.4.13 shows that for order 3 irreducible modulus patterns modular determinant $\boxed{<}$ does not imply potential convergence.

Example 3.4.12. Consider the following modulus patterns:

$$Z_k = \begin{bmatrix} z^{(k)} & 0 & \boxed{1} \\ \boxed{1} & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 \end{bmatrix}, \text{ where } z^{(k)} = 0,\ \boxed{1},\ \boxed{<},\ \boxed{>}, \text{ respectively, for } k = 0,\ 1,\ 2,\ 3.$$

First note that by Observation 3.4.2, Z_0 does not allow convergence.

Every matrix from $Q(Z_k)$, $(k = 1, 2, 3)$, can be written as $A(\alpha, \theta_1, \ldots, \theta_5) = \begin{bmatrix} \alpha e^{i\theta_1} & 0 & e^{i\theta_4} \\ e^{i\theta_2} & 0 & e^{i\theta_5} \\ 0 & e^{i\theta_3} & 0 \end{bmatrix}$,

where $\theta_i \in (-\pi, \pi]$, $i = 1, \ldots, 5$; $\alpha = 1$ for $k = 1$, $0 < \alpha < 1$ for $k = 2$ and $\alpha > 1$ for $k = 3$. Since multiplying by $e^{-i\theta_1}$ and applying diagonal similarity does not change the spectral radius, without loss of generality we can consider the matrix

$$B = B(\alpha, \tau_1, \tau_2) = e^{-i\theta_1} D^{-1} A(\theta_1, \ldots, \theta_5) D = \begin{bmatrix} \alpha & 0 & e^{i\tau_1} \\ 1 & 0 & e^{i\tau_2} \\ 0 & 1 & 0 \end{bmatrix},$$

where $\tau_1 = \theta_2 + \theta_3 + \theta_4 - 3\theta_1$, $\tau_2 = \theta_3 + \theta_5 - 2\theta_1$, and $D = \mathrm{diag}(1, e^{i(\theta_2 - \theta_1)}, e^{i(\theta_2 + \theta_3 - 2\theta_1)})$.

The characteristic polynomial of B is $p_B(x) = x^3 - \alpha x^2 - e^{i\tau_2} x - e^{i\tau_1} + \alpha e^{i\tau_2}$. To apply the Schur-Cohn criterion (Theorem 2.3.2) to $p_B(x)$, we compute:

$c_{11} = |a_0|^2 - |a_3|^2 = \alpha(2\cos(\tau_1 - \tau_2) - \alpha)$, $c_{12} = \overline{c}_{21} = \overline{a}_0 a_1 - \overline{a}_2 a_3 = -e^{i(\tau_1-\tau_2)}$, and $c_{22} = |a_1|^2 - |a_2|^2 + |a_0|^2 - |a_3|^2 = 2\alpha\cos(\tau_1 - \tau_2) - 1$.
The 2×2 leading principal submatrix of C is $\begin{bmatrix} \alpha(2\cos\theta - \alpha) & -e^{i\theta} \\ -e^{-i\theta} & 2\alpha\cos\theta - 1 \end{bmatrix}$, where $\theta = \tau_1 - \tau_2$.
We will show that C is not positive definite if $\alpha \in (0,\, 1]$.
$c_{11} > 0$ is equivalent to

$$\cos\theta > \frac{\alpha}{2}, \tag{3.1}$$

$c_{11}c_{22} - |c_{12}|^2 > 0$ is equivalent to

$$4\alpha^2\cos^2\theta - 2\alpha(1+\alpha^2)\cos\theta + \alpha^2 - 1 > 0. \tag{3.2}$$

Note that if $\alpha \in (0,\, 1]$, then $\alpha^2 - 1 \leq 0$, so the smaller root of the quadratic equation $4\alpha^2x^2 - 2\alpha(1+\alpha^2)x + \alpha^2 - 1 = 0$ is negative or 0. Therefore, (3.1) together with (3.2) imply

$$\cos\theta > \frac{1+\alpha^2+\sqrt{(\alpha^2-1)^2+4}}{4\alpha}. \tag{3.3}$$

But for $\alpha \in (0,\, 1]$ it can be shown that $\frac{1+\alpha^2+\sqrt{(\alpha^2-1)^2+4}}{4\alpha} \geq 1$, so (3.3) is not possible. Thus, the modulus patterns Z_1 and Z_2 are not potentially convergent. However, the modulus pattern Z_3 allows real convergence, since $B = \begin{bmatrix} \frac{13}{10} & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ is convergent (with eigenvalues $\frac{1}{2}$ and $\frac{1}{5}(2 \pm i\sqrt{11})$).

Example 3.4.13. Consider a 3×3 modulus pattern Z whose digraph has only two cycles, one of length 2 and another of length 3. Note that up to permutation and diagonal similarity, a matrix $B \in Q(Z)$ is of the form: $B = \begin{bmatrix} 0 & 0 & \beta e^{i\tau_1} \\ 1 & 0 & \gamma e^{i\tau_2} \\ 0 & 1 & 0 \end{bmatrix}$, for some $\tau_1,\, \tau_2 \in (-\pi,\, \pi]$, $\beta > 0$ and $\gamma > 0$, and $p_B(x) = x^3 - \gamma e^{i\tau_2}x - \beta e^{i\tau_1}$. Applying the Schur-Cohn criterion, we obtain $c_{11} = 1 - \beta^2$ and $c_{11}c_{22} - |c_{12}|^2 = (1-\beta^2)^2 - \gamma^2$, so C is positive definite only if $\beta < 1$ and $\gamma < 1 - \beta^2 < 1$. Thus, Z is potentially convergent if and only if both of its cycles have entry

$\boxed{<}$, i.e. if and only if it is absolutely convergent. The following is an example of a modulus pattern, with modular determinant $\boxed{<}$ that is not potentially convergent: $\begin{bmatrix} 0 & 0 & \boxed{<} \\ \boxed{1} & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 \end{bmatrix}$.

CHAPTER 4. GENERAL CONCLUSIONS

4.1 General Discussion

In the Chapter 2 of this thesis several types of convergence of complex matrices are considered; some of the results established are different from the real case. The technique used in Chapter 2 is different from methods applied for real matrices in [10, 11, 22]. The main idea underlying the proofs of the results in Chapter 2 is to analyze the derivatives of the largest eigenvalues of a suitable parameterized matrix. The following results were obtained:

1. Suppose $s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$, and $s^2 = (s(A))^2$ is a simple eigenvalue of $A_0^* A_0$. Then there exists $D \in \mathbb{C}^{n\times n}$, $|D| = I$, such that $\rho(DA) = s$. If A is real, then there exists a real D, $|D| = I$ such that $\rho(DA) = s$.

2. Suppose A is real and $s(A) = \|A_0\|$ for some $A_0 \in \mathcal{A}(A)$, where $s^2 = (s(A))^2$ is an eigenvalue of multiplicity 2 of $A_0^* A_0$. Then there exists $D \in \mathbb{C}^{n\times n}$, $|D| = I$, such that $\rho(DA) = s$.

3. For complex $n \times n$ matrices with $n \leq 3$, diagonal convergence, $D_{\mathbb{C}}$ convergence and boundary convergence are all equivalent; there exists a complex 4×4 matrix that is $D_{\mathbb{C}}$ convergent but not diagonally convergent.

4. There exists a real 3×3 $D_{\mathbb{R}}$ convergent matrix that is not boundary convergent.

5. There exists a complex 2×2 vertex convergent matrix that is not $D_{\mathbb{R}}$ convergent.

6. There exists a complex 2×2 $D_{\mathbb{R}}$ convergent matrix that is not $D_{\mathbb{C}}$ convergent.

For a complex 2×2 matrix, a simple test can be derived from Proposition 2.2.9 to determine whether the matrix is boundary convergent. In particular, if $A \in \mathbb{C}^{2\times 2}$ is irreducible, i.e.

$a_{12}a_{21} \neq 0$, then an explicit formula can be obtained for a matrix $D_0 = \mathrm{diag}(1, d)$, $|d| = 1$, such that $\rho(D_0A) = \max\{\rho(DA) : |D| = I\}$. From Proposition 2.2.9 it follows that

$$d = \begin{cases} 1 & \text{if } a_{11} + e^{i\theta}\overline{a_{22}} = 0, \\ \dfrac{a_{11} + e^{i\theta}\overline{a_{22}}}{e^{i\theta}a_{11}\overline{a_{11}} + a_{22}} & \text{otherwise,} \end{cases}$$

where $\theta = \arg(a_{12}a_{21})$.

One of the open problems is to find if there exist explicit formulae for the diagonal entries of the matrix D_0, $|D_0| = I$, such that $\rho(D_0A) = \max\{\rho(DA) : |D| = I\}$ for $n \geq 3$. Such formulae would provide a complete characterization of boundary convergence and for $n = 3$ of $D_{\mathbb{C}}$ and diagonal convergence.

It is known that for real matrices of order $n \leq 3$ vertex convergence is equivalent $D_{\mathbb{R}}$ convergence, but there exists a real 4×4 vertex convergent matrix that is not $D_{\mathbb{R}}$ convergence. In the complex case it is not known whether boundary convergence is equivalent to $D_{\mathbb{C}}$ convergence for $n > 3$. So an open problem is to find the smallest order n of a boundary convergent matrix that is not $D_{\mathbb{C}}$ convergent or prove that boundary convergence is equivalent to $D_{\mathbb{C}}$ convergence for all n.

In the Chapter 3 stronger types of potential convergence were introduced and some of them completely characterized, although only partial results were obtained on potential convergence. For modulus patterns of order $n = 2$, all introduced types of convergence were characterized. Also, examples are constructed to show that for the order of the pattern $n > 2$, the situation differs significantly from the case $n = 2$. Other results include:

1. Potential diagonal convergence, potential real diagonal convergence and potential absolute convergence of a modulus pattern Z are all equivalent to the condition that each cycle in Z has at least one entry $\boxed{<}$.

2. If a modulus pattern Z has no zero entries, then it is potentially convergent. If Z is of even order, then it allows real convergence.

3. If an $n \times n$ modulus pattern Z has one cycle of length n and no other cycles of length ≥ 2, then potential $D_{\mathbb{C}}$ convergence and potential real $D_{\mathbb{R}}$ convergence are both equivalent to

potential absolute convergence.

4. If Z does not have any cycles of length greater than 2, then each of potential absolute, real diagonal, diagonal, real $D_{\mathbb{C}}$, $D_{\mathbb{C}}$, real $D_{\mathbb{R}}$ and $D_{\mathbb{R}}$ convergence, is equivalent to the condition that each cycle in Z has at least one entry $\boxed{<}$.

From the results in Chapter 2 it follows that for $n = 3$, potential $D_{\mathbb{C}}$ convergence implies potential diagonal convergence; however, no potentially $D_{\mathbb{R}}$ convergent $n \times n$ modulus pattern was found that does not allow diagonal convergence. So, the following question can be investigated: does there exist a potentially $D_{\mathbb{R}}$ convergent modulus pattern that does not allow diagonal convergence?

4.2 Recommendations for Future Research

In the future research it may be useful to consider applications of the new results on stronger types of convergence to the analysis of switched dynamical systems. A linear switched dynamical system in the continuous and discrete time cases, respectively, has the form: $\mathbf{x}'(t) = A_{\sigma(t)}\mathbf{x}(t)$, and $\mathbf{x}(t_{k+1}) = A_{\sigma(t_k)}\mathbf{x}(t_k)$, where the switching law $\sigma : \mathbb{R} \to \{1, 2, \ldots, m\}$ is a piecewise constant function of time and $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is a given family of matrices. The main problem in the study of switched systems is to find necessary and sufficient conditions on the matrices $A_1, A_2, \ldots, A_m$ for the system to be stable under any possible switching law (these conditions will require some stronger types of stability than just conditions on the spectra of individual matrices). The discrete time case of switched systems is closely related to the joint spectral radius of a set of matrices, i.e. the quantity $\rho(\mathcal{M}) = \limsup_{k\to\infty} \max_{A_1,\ldots,A_k \in \mathcal{M}} \|A_k \ldots A_1\|^{1/k}$, where $\| \cdot \|$ is any matrix norm. The results in Chapter 2 can be related to the joint spectral radius in a very special case, when the set of matrices is $\mathcal{M} = \mathcal{M}(A) = \{DA : |D| = I\}$. It may be of interest to explore whether those results can be generalized to produce characterizations of stability of some types of switched systems.

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