

Haemers' minimum rank

by

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DEDICATION

To my wife Laurie, I could not have completed
this work without your love and support. I
dedicate this thesis to you and our children.

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ABSTRACT

Haemers' minimum rank, $\eta(G)$, was first defined by Willem Haemers in 1979. He created this graph parameter as an upper bound for the Shannon capacity of a graph, $\Theta(G)$, and to answer some questions asked by Lovász in his famous paper where he determined $\Theta(C_5) = \sqrt{5}$.

In this thesis, new techniques are introduced that may be helpful for calculating $\eta(G)$ for some graphs. These techniques are used to show $\eta(G)$ is equal to the vertex clique cover number of G for all graphs of order 10 or less, and also for some graph families, including all graphs with vertex clique cover number equal to 1, 2, 3, $|G| - 2$, $|G| - 1$, or $|G|$. Also, in the case of the cut-vertex reduction formula for $\eta(G)$, we show how this can be used to find the Shannon capacity of new graphs.

CHAPTER 1. Introduction to Haemers' Minimum Rank $\eta(G)$

Haemers' minimum rank was first defined by Willem Haemers [26], [27]. He created this graph parameter as an upper bound for the Shannon capacity of a graph.

1.1 Thesis Organization

In this Chapter, we give motivation for Haemers' minimum rank, and summarize some of the results he obtained, and consequences thereof. In Chapter 2, we look at what effect certain graph operations have on η , for example, edge or vertex deletions, cut-vertex reduction, and the join. In Chapter 3, we look at a few techniques that can be used to increase the lower bound from $\alpha(G)$ to $\alpha(G)+1$ and use these to show $\eta(G) = \text{vcc}(G)$ for all graphs of order 10 or less, where $\text{vcc}(G)$ represents the vertex clique cover number of G . Chapter 4 concentrates on extreme values of $\eta(G)$, including if $\text{vcc}(G)$ is 1, 2, 3, $|G|$, $|G|-1$, or $|G|-2$, then $\eta(G) = \text{vcc}(G)$. In Chapter 5, we calculate $\eta(G)$ for a few families of graphs, and in Chapter 6, we compare $\eta(G)$ to other minimum rank parameters.

1.2 Introduction and Literature Review

All graphs in this paper are simple and undirected. An **independent set** for a graph G is a set of vertices $U \subseteq V(G)$ such that for any distinct $u, v \in U$, $u \not\sim v$. The **independence number** of a graph G , denoted $\alpha(G)$, is the maximum of $|U|$ over all independent sets U . The **chromatic number** of a graph G , denoted $\chi(G)$, is the least number of distinct colors needed to color the vertices of G such that adjacent vertices are colored with different colors. The **strong product** of two graphs G and H , denoted $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, such that (g, h) is adjacent to (g', h') in $G \boxtimes H$ if and only if they are adjacent or

equal in each coordinate. That is, (g, h) is adjacent to (g', h') exactly when $g = g'$ and $h \sim h'$, or $g \sim g'$ and $h = h'$, or $g \sim g'$ and $h \sim h'$. This product is associative and commutative, so we can define G^k to be the strong product of k copies of G .

Assume I_G is an independent set of vertices in G and I_H is an independent set of vertices in H . It is not hard to see that the cartesian product, $I_G \times I_H$, is an independent set in $G \boxtimes H$. However, a maximum independent set in $G \boxtimes H$ need not be such a product, so determining $\alpha(G \boxtimes H)$ is, in general, a difficult problem.

Observation 1.1. *For any two graphs G and H ,*

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H).$$

The **Shannon capacity** of a graph, $\Theta(G)$, is

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k \rightarrow \infty} \sqrt[k]{\alpha(G^k)}$$

From this, it is clear that

$$\sqrt[k]{\alpha(G^k)} \leq \Theta(G)$$

for any given k . In particular, $\alpha(G) \leq \Theta(G)$. In [39], Shannon also showed that $\Theta(G) \leq \chi^*(\bar{G})$, where \bar{G} represents the graph compliment of G , and $\chi^*(G)$ is the fractional chromatic number. It is well known that $\chi^*(G) \leq \chi(G)$. Thus, $\Theta(G)$ is determined for all graphs satisfying $\alpha(G) = \chi(\bar{G})$. This is true for many graphs, including perfect graphs. In particular it is true for all graphs of order 5 or less, with the exception of C_5 , the 5-cycle.

The **clique number** $\omega(G)$ of a graph G is the maximum order of a clique in G . A graph G is **perfect** if $\omega(H) = \chi(H)$ for every induced subgraph H of G , or equivalently if $\alpha(H) = \chi(\bar{H})$ for every induced subgraph H of G .

Theorem 1.2 (The Strong Perfect Graph Theorem). [16, Theorem 8.3] *A graph G is perfect if and only if neither G nor \bar{G} contains an induced odd cycle of length 5 or more.*

For example, complete graphs, bipartite graphs, line graphs of bipartite graphs, chordal graphs, and the complements of any of the previously mentioned graphs are all perfect graphs.

We have $\alpha(C_5) = 2$ and $\chi(\bar{C}_5) = 3$. Now, $\alpha(C_5^2) = 5$ so that $\sqrt{5} \leq \Theta(G)$. As $\chi^*(C_5) = \frac{5}{2}$, Shannon was able to show

$$\sqrt{5} \leq \Theta(C_5) \leq \frac{5}{2}.$$

The question of determining $\Theta(C_5)$ remained open until 1979 when Lovász determined that $\Theta(G) = \sqrt{5}$. He did so by creating a new upper bound on $\Theta(G)$, now known as the Lovász theta function, and showing that the value of this function for C_5 is $\sqrt{5}$. This function is often helpful in determining $\Theta(G)$. Note, even in 2012, $\Theta(C_7)$ has yet to be determined.

Definition 1.3. [33] Let G be a graph. An **orthonormal representation** of G is a system $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of unit vectors in \mathbb{R}^n , using the standard inner product, such that $i \not\sim j$ implies \mathbf{v}_i and \mathbf{v}_j are orthogonal. We define the **value** of an orthonormal representation $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ to be

$$\min_{\mathbf{c}} \max_{1 \leq i \leq n} \frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2}$$

where the minimum is over all unit vectors, \mathbf{c} . The **Lovász function**, $\vartheta(G)$, is defined to be the minimum value over all orthonormal representations of G .

Lemma 1.4. [33] *For any graph G , $\alpha(G) \leq \vartheta(G)$*

Lemma 1.5. [33] *For any graph G , $\vartheta(G \boxtimes H) \leq \vartheta(G)\vartheta(H)$.*

Therefore, $\alpha(G^k) \leq \vartheta(G^k) \leq \vartheta(G)^k$ which gives the next theorem.

Theorem 1.6. [33] *For any graph G , $\Theta(G) \leq \vartheta(G)$.*

Lovász then showed $\vartheta(C_5) \leq \sqrt{5}$, which gives $\Theta(C_5) = \sqrt{5}$. To do this, Lovász exhibited an orthonormal representation of C_5 , $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5)$, and unit vector \mathbf{c} such that $\mathbf{c}^T \mathbf{u}_i = 5^{-1/4}$ for $i = 1, \dots, 5$, so that

$$\max_i \frac{1}{(\mathbf{c}^T \mathbf{u}_i)^2} = \sqrt{5}.$$

Theorem 1.7. [33] $\Theta(C_5) = \sqrt{5}$

Lovász went on to give several other equivalent definitions of $\vartheta(G)$ and show in fact $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$.

At the end of his paper, Lovász presents 3 problems.

Problem 1: Is the Lovász theta function always the same as the Shannon capacity for any graph?

Problem 2: Is $\Theta(G \boxtimes H) = \Theta(G)\Theta(H)$?

Problem 3: Is $\Theta(G)\Theta(\bar{G}) \geq |V(G)|$?

Not long after, Haemers wrote a short paper [26] to show that the answer to all 3 problems is no. In this paper, he considered symmetric $n \times n$ matrices with all diagonal entries equal to one. For such a matrix, $A = [a_{ij}]$, he defined $G(A)$ to be the graph with vertex set $\{1, \dots, n\}$, where $i \sim j$ if and only if $a_{ij} \neq 0$. He proved that, for such a matrix A which has $G(A) = G$, $\Theta(G) \leq \text{rank } A$. In [27], he loosened the restrictions on the matrix and used this to define a new graph parameter, which we call Haemers' minimum rank. He showed this new parameter is an upper bound for $\Theta(G)$ that is usually not as good as, but sometimes is much better than, $\vartheta(G)$.

Definition 1.8. [27] Let G be a graph with $V(G) = \{1, \dots, n\}$. We say an $n \times n$ matrix $B = [b_{ij}]$ **fits** G if $b_{ii} \neq 0$ and for distinct i, j , $b_{ij} = 0$ if i and j are not adjacent. We then define **Haemers' minimum rank**, $\eta(G)$, as the minimum rank of any matrix over any field that fits G .

Below, in Example 1.21, we will show how this definition was used to answer the problems posed by Lovász, but first we need to learn a bit about $\eta(G)$.

Theorem 1.9. [27] *For any graph G , $\alpha(G) \leq \eta(G)$.*

Proof. Let $\{v_1, \dots, v_s\}$ be any independent set of vertices. Then, in any matrix fitting G , the columns corresponding to v_1, \dots, v_s will be a linearly independent set of order s . \square

Theorem 1.10. [27] *For any graph G ,*

$$\eta(G) \leq \chi(\bar{G})$$

Proof. We label the vertices of G and \bar{G} as $\{1, 2, \dots, n\}$ and assume \bar{G} is colored using $\chi(\bar{G})$ colors, $C_1, C_2, \dots, C_{\chi(\bar{G})}$. Without loss of generality, if vertex i is colored C_{k_i} and vertex j is

colored C_{k_j} with $i \leq j$, let us assume $C_{k_i} \leq C_{k_j}$. That is, in the labeling of the vertices of the graph, the vertices colored C_1 come first, then those colored C_2 , and so on, until the vertices colored $C_{\chi(\bar{G})}$ come last.

Now, define a matrix $A = [a_{ij}]$ as follows:

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ have the same color} \\ 0, & \text{otherwise} \end{cases}$$

Since i is in the same color class as itself, the diagonal entries are nonzero. And, for $i \neq j$, a_{ij} can only be 1 if $i \not\sim j$ in \bar{G} , so that $i \sim j$ in G . Thus A fits the graph G . The form of A is a block diagonal matrix with blocks $B_1, \dots, B_{\chi(\bar{G})}$, where B_k is a $|C_k| \times |C_k|$ block of all 1s. Thus $\text{rank } A = \chi(\bar{G})$. \square

Let G be a graph. Partition the vertices of G into sets $\{V_i\}_{i=1}^k$ such that, for each i , the subgraph of G induced by the vertices in V_i forms a clique. We define the **vertex clique cover number** of G , denoted $\text{vcc}(G)$, to be the minimum number k such that a collection of k such cliques can cover all the vertices of G .

It is well known that $\text{vcc}(G) = \chi(\bar{G})$, as the color classes of \bar{G} are cliques in G , so Haemers' result could also be stated as $\eta(G) \leq \text{vcc}(G)$. We can think of this as each clique contributing 1 since $\eta(K_n) = 1$. This is a special case of a more general result.

For any set of vertices in $V(G)$, W , we define $G[W]$ to be the subgraph of G induced by the vertices in W .

Proposition 1.11. *Let G be a graph. Partition the vertices of G into sets $\{V_i\}_{i=1}^k$ and let $G_i = G[V_i]$. Then*

$$\eta(G) \leq \sum_{i=1}^k \eta(G_i).$$

Proof. Without loss of generality, label the vertices of G so that the vertices of V_1 appear first, then V_2 , and so on, until the vertices of V_k come last. Consider a block matrix, B , with blocks B_1, \dots, B_k , where B_i is a $|V_i| \times |V_i|$ matrix that fits G_i and attains $\eta(G_i)$. Then B fits G and $\text{rank } B = \sum_{i=1}^k \eta(G_i)$. Therefore, $\eta(G) \leq \text{rank } B$. \square

Since $\alpha(G) \leq \eta(G) \leq \chi(\bar{G})$, $\eta(G)$ is determined for all graphs where $\alpha(G) = \chi(\bar{G})$, just as $\Theta(G)$ is determined in these cases.

Observation 1.12. *Let E_n denote the empty graph on n vertices, P_n the path graph on n vertices, and K_n the complete graph on n vertices. Then*

$$\begin{aligned} \eta(E_n) &= n \\ \eta(P_n) &= \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases} = \left\lceil \frac{n}{2} \right\rceil \\ \eta(K_n) &= 1 \end{aligned}$$

For each graph, G , in the theorem, G is a perfect graph, so $\alpha(G) = \eta(G) = \text{vcc}(G)$. And, each formula is easy to verify, for example, for $\alpha(G)$.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, the **Kronecker product** of A and B is the $mp \times nq$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Lemma 1.13. [44, Page 120] *For any two matrices, A and B , we have $\text{rank}(A \otimes B) = \text{rank } A \text{ rank } B$, where \otimes denotes the Kronecker product.*

Theorem 1.14. [27] *For any graphs G and H , $\eta(G \boxtimes H) \leq \eta(G)\eta(H)$.*

Proof. Let M_G and M_H be matrices that fit G and H and realize $\eta(G)$ and $\eta(H)$, respectively. Then, $M_G \otimes M_H$ fits $G \boxtimes H$ and $\text{rank}(M_G \otimes M_H) = \eta(G)\eta(H)$. □

Corollary 1.15. [27] *For any graph G , $\eta(G^k) \leq \eta(G)^k$.*

Corollary 1.16. [27] *For any graph G , $\sqrt[k]{\alpha(G^k)} \leq \eta(G)$*

Proof. $\alpha(G^k) \leq \eta(G^k) \leq \eta(G)^k$ □

Since $\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}$, the next theorem follows from Corollary 1.16.

Theorem 1.17. [27] *For any graph G , $\Theta(G) \leq \eta(G)$.*

Therefore, $\Theta(G)$ is determined for any graphs where $\eta(G) = \alpha(G)$.

Theorem 1.18. *Assume $\eta(G) = \alpha(G)$ and $\eta(H) = \alpha(H)$. Then,*

$$\alpha(G \boxtimes H) = \Theta(G \boxtimes H) = \eta(G \boxtimes H).$$

Proof. From Theorem 1.14, we have that $\eta(G \boxtimes H) \leq \eta(G)\eta(H)$. From Observation 1.1, we have that $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$. Therefore, we have

$$\eta(G)\eta(H) = \alpha(G)\alpha(H) \leq \alpha(G \boxtimes H) \leq \eta(G \boxtimes H) \leq \eta(G)\eta(H),$$

which forces equality throughout. □

Therefore, if we know $\eta(G) = \alpha(G)$ and $\eta(H) = \alpha(H)$ for two graphs, not only have we determined $\Theta(G)$ and $\Theta(H)$, but we have also determined Θ for any strong products of these graphs.

Corollary 1.19. *For two paths, P_s and P_t , we have*

$$\eta(P_s \boxtimes P_t) = \alpha(P_s \boxtimes P_t) = \left\lceil \frac{s}{2} \right\rceil \left\lceil \frac{t}{2} \right\rceil.$$

Theorem 1.20. [27] *Let A be the adjacency matrix of G and let λ be a non-zero eigenvalue of A , with multiplicity m . Then $\eta(G) \leq |G| - m$.*

Proof. $A - \lambda I$ fits G and has rank $|G| - m$ over the field of real numbers. □

This result may be useful if one of the eigenvalues has a large multiplicity. Haemers mentions that strongly regular graphs are one class of graphs where this may be useful because they have only 3 distinct eigenvalues, one of which is of multiplicity 1. And, with this result, he was able to answer the problems posed by Lovász.

Example 1.21. [26] Let G be the complement of the Schlaefli graph, the unique strongly regular graph with parameters $(27, 10, 1, 5)$. This graph is shown in Figure 3.2 in Chapter 3. G has 1 as an eigenvalue of multiplicity 20 [12]. Thus, $\eta(G) \leq 27 - 20 = 7$. Note, this is a better upper bound for $\eta(G)$ than $\text{vcc}(G) = 9$ [11]. Since $\eta(G) \leq 7$, we have $\Theta(G) \leq 7$.

We have that $\vartheta(G) = 9$ so that $\Theta(G) \neq \vartheta(G)$, which answers Problem 1 in the negative. We have $\alpha(\bar{G}) = \vartheta(\bar{G}) = 3$, so that $\Theta(\bar{G}) = 3$. Therefore, $\Theta(G)\Theta(\bar{G}) \leq 21 < 27 = |V(G)|$, which shows the answer to Problem 3 is also no. Finally, we have $\Theta(G \boxtimes \bar{G}) \geq \alpha(G \boxtimes \bar{G}) \geq |V(G)|$ in general for any graph G , so here $\Theta(G \boxtimes \bar{G}) \geq 27$. Thus, $\Theta(G \boxtimes \bar{G}) \neq \Theta(G)\Theta(\bar{G})$, which shows $\Theta(G \boxtimes H) = \Theta(G)\Theta(H)$ does not always hold for graphs G and H . This says the answer to Problem 2 is no.

Later, in Example 3.8, we will see that $\eta(G) = 7$.

Example 1.22. Consider the unique strongly regular graph with parameters $(16, 5, 0, 2)$, which we denote by G . Sometimes this is called the Clebsch graph, although sometimes its complement is called the Clebsch graph. It has eigenvalue 1 of multiplicity 10 [12]. Since it is order 16, by Theorem 1.20, $\eta(G) \leq 6$. It is known that $\text{vcc}(G) = 8$ [10].

Example 1.23. [27] We define a graph G as follows. Fix positive integers n , m , and p with $n > m$ and p a prime that does not divide m . The vertices are then the m -subsets of a fixed n -set. Two vertices x and y are adjacent exactly when $|x \cap y| \not\equiv 0 \pmod{p}$. So, this graph satisfies $|V(G)| = \binom{n}{m}$. Then $\eta(G) \leq n$.

To see this, let $A = [a_{ij}]$ be the $n \times \binom{n}{m}$ incidence matrix of the m -subsets of $[n]$. In other words, if we label the m -subsets with $X_1, X_2, \dots, X_{\binom{n}{m}}$, then

$$a_{ij} = \begin{cases} 1 & \text{if } i \in X_j \\ 0 & \text{otherwise} \end{cases}$$

We define $B = A^T A$ over $GF(p)$, the finite field with p elements. B is then $\binom{n}{m} \times \binom{n}{m}$. In general, $b_{ij} = |X_i \cap X_j| \pmod{p}$. Since $p \nmid m$, this ensures all diagonal entries are nonzero. And, if $\{X_i, X_j\} \notin E(G)$, then $b_{ij} = |X_i \cap X_j| \pmod{p} = 0$, so that B fits G . Finally, we have

$$\text{rank } B \leq \text{rank } A \leq n$$

since A has n rows.

Example 1.24. [27] Let G be the graph in Example 1.23, where in addition, we let $p = 2$, $m = 3$, and $n \equiv 0 \pmod{4}$. Then $\eta(G) = n$ and $\Theta(G) = n$.

Partition the elements of $[n]$ into sets of size 4, $Y_1 = \{1, 2, 3, 4\}, \dots, Y_{\frac{n}{4}} = \{n-3, n-2, n-1, n\}$. For a given Y_i , there exist four 3-subsets of $[n]$ that are also subsets of Y_i , e.g., $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ are each subsets of $\{1, 2, 3, 4\}$. So, there are n total 3-subsets that are subsets of some Y_i . Assume X_i and X_j denote two 3-subsets of $[n]$ that are subsets of Y_r and Y_s , respectively. If $r = s$, then $|X_i \cap X_j| = 2$. If $r \neq s$, then $|X_i \cap X_j| = 0$. Therefore, the collection of all n 3-subsets that are a subset of some Y_i forms an independent set of size n . Thus, $\alpha(G) \geq n$, so that

$$n \leq \alpha(G) \leq \eta(G) \leq n$$

which implies $\eta(G) = n$. Since $\alpha(G) \leq \Theta(G) \leq \eta(G)$, we have $\Theta(G) = n$.

It turns out for the graphs in Example 1.24, if $n > 8$, the value of the Lovász theta function is $\frac{n(n-2)(2n-1)}{3(3n-14)} > n$, so that $\eta(G)$ determines $\Theta(G)$, but the Lovász theta function does not.

René Peeters gave another upper bound and some examples where $\eta(G)$ determines $\Theta(G)$ in [35]. Note, Peeters used a different definition for a matrix fitting a graph than did Haemers. If a matrix A fits G using the definition of Haemers, then that same matrix would fit \bar{G} by the definition of Peeters. Accordingly, some notation may have opposite meanings in Peeters paper than it will below. For a fixed field F and graph G , define

$$\mathcal{A}_2(G, F) = \{A \in F^{n \times n} \mid A \text{ is symmetric and fits } G\}$$

$$\mathcal{A}_1(G, F) = \{A \in \mathcal{A}_2(G, F) \mid A_{ii} = 1 \text{ for } i = 1, \dots, n\}$$

Note, what we call $\mathcal{A}_i(G, F)$, Peeters would have called $\mathcal{A}_i(\bar{G}, F)$ but we change the notation slightly to make things a bit simpler.

Then, for any class of matrices, \mathcal{A} , we define

$$R(\mathcal{A}) = \min\{\text{rank } A \mid A \in \mathcal{A}\}.$$

Therefore, for any fixed field F

$$\alpha(G) \leq \eta(G) \leq R(\mathcal{A}_2(G, F)) \leq R(\mathcal{A}_1(G, F)) \leq \text{vcc}(G).$$

Peeters gave a more general definition than Lovász for orthogonal representation. Let V be a vector space of dimension d over a field F and $B : V \times V \rightarrow F$ a bilinear form on V . A vector

v is **isotropic** if $B(v, v) = 0$. An **orthogonal representation** of a graph G , where $|G| = n$, in the inner product space (V, B) is a set $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ of non-isotropic vectors in V such that for any two distinct vertices $i, j \in V(G)$, $i \not\sim j$ implies $B(\mathbf{v}_i, \mathbf{v}_j) = 0$. Such a representation is called **orthonormal** if $B(\mathbf{v}_i, \mathbf{v}_i) = 1$ for all $i = 1, \dots, n$. Then Lovász' definition is the case where B is the standard inner product and $F = \mathbb{R}$.

Theorem 1.25. [35] *Let F be a finite field and G a graph.*

$$R(\mathcal{A}_1(G, F)) = \min\{d \mid G \text{ has an orthonormal representation in } (F^d, B) \text{ for some symmetric bilinear form } B\}$$

$$R(\mathcal{A}_2(G, F)) = \min\{d \mid G \text{ has an orthogonal representation in } (F^d, B) \text{ for some symmetric bilinear form } B\}$$

Definition 1.26. [35] We say vectors \mathbf{x} and \mathbf{y} are **equivalent** if $\mathbf{x} = a\mathbf{y}$ for some nonzero $a \in F \setminus \{0\}$. We define $\Gamma_2(F, d, B)$ by starting with as vertex set the equivalence classes of non-isotropic vectors of (V, B) , where two equivalence classes, $[\mathbf{u}]$ and $[\mathbf{v}]$, are adjacent if and only if $B(\mathbf{u}, \mathbf{v}) \neq 0$. Whether this quantity is zero or nonzero is independent of the choice of representatives of the equivalence classes involved, so this is well-defined. Let $\Gamma_1(F, d, B)$ be the subgraph of $\Gamma_2(F, d, B)$ induced by the vertices that satisfy $B(\mathbf{u}, \mathbf{u}) = 1$ for some \mathbf{u} in the equivalence class that is the vertex. Observe $[\mathbf{v}]$ is a vertex of $\Gamma_1(F, d, B)$ if and only if $B(\mathbf{v}, \mathbf{v})$ is a square in F .

Theorem 1.27. [35] *If G is any graph of the form $\Gamma_1(F, d, I)$ or $\Gamma_2(F, d, I)$, where I stands for the standard inner product, then $\alpha(G) = \eta(G) = d$, so that in fact $\Theta(G) = d$ is determined.*

Example 1.28. [35] Let G denote the graph $\Gamma_1(\mathbb{F}_2, 2m + 1, I)$, where I is the standard inner product. Then, G consists of the symplectic graph $\mathcal{Sp}(2m, 2)$ plus one more vertex, v , that is adjacent to every vertex in $\mathcal{Sp}(2m, 2)$. By Theorem 1.27, $\alpha(G) = \Theta(G) = \eta(G) = 2m + 1$. Note, by Theorems 2.9 and 2.13 below, we have $\alpha(\mathcal{Sp}(2m, 2)) = \Theta(\mathcal{Sp}(2m, 2)) = \eta(\mathcal{Sp}(2m, 2)) = 2m + 1$ [27]. And, $\vartheta(\mathcal{Sp}(2m, 2)) = 2^m + 1$, so this is another graph where Haemers' minimum rank is much better than Lovász' bound.

Example 1.29. [20] We define a graph G as follows. Let $V(G) = \{v_1, \dots, v_{13}\}$, where each vertex is a vector in \mathbb{R}^3 :

$$\begin{aligned} v_1 &= (1, 1, 1)^T, v_2 = (-1, 1, 0)^T, v_3 = (1, 0, -1)^T, v_4 = (0, -1, 1)^T \\ v_5 &= (1, 1, 0)^T, v_6 = (1, 0, 1)^T, v_7 = (0, 1, 1)^T \\ v_8 &= (-1, 1, 1)^T, v_9 = (1, -1, 1)^T, v_{10} = (1, 1, -1)^T \\ v_{11} &= (1, 0, 0)^T, v_{12} = (0, 1, 0)^T, v_{13} = (0, 0, 1)^T \end{aligned}$$

Now, we let two vertices in G be adjacent if and only if their dot product is non-zero. By this definition, and Theorem 1.25, we have that $\eta(G) \leq R(\mathcal{A}_2(G, \mathbb{R})) \leq 3$. Since $\{v_{11}, v_{12}, v_{13}\}$ is an independent set, we have $\alpha(G) \geq 3$. Thus, $\alpha(G) = \eta(G) = 3$. However, $\text{vcc}(G) = 4$.

Alternatively, we can see that $\eta(G) \leq 3$ by constructing a matrix $A = [a_{ij}]$ such that A fits G and $\text{rank } A = 3$. We start by defining $B = \begin{bmatrix} v_1 & v_2 & \dots & v_{13} \end{bmatrix}$. Then, we let $A = B^T B$. Since $\text{rank } B = 3$, we have $\text{rank } A = 3$. And, $a_{ij} = v_i^T v_j$. Thus, all diagonal entries are nonzero and entry a_{ij} , for i, j distinct, is nonzero if and only if $v_i \sim v_j$ by the definition of G . Thus, A fits G .

Remark 1.30. The maximum rank over all matrices that fit G is not interesting as it is always equal to the number of vertices. This rank is realized by a diagonal matrix with nonzero diagonal entries. And, all ranks between the maximum and minimum are also attainable. To see this, start with a matrix realizing the minimum rank. Then, change one nonzero, nondiagonal entry at a time to 0. Each change will result in a rank change of at most 1 (see Corollary 2.3 below), and the process will eventually lead to a diagonal matrix of full rank.

Observation 1.31. *Let G be a graph with connected components G_1, \dots, G_k . Then*

$$\begin{aligned} \eta(G) &= \eta(G_1) + \dots + \eta(G_k) \\ \text{vcc}(G) &= \text{vcc}(G_1) + \dots + \text{vcc}(G_k) \\ \alpha(G) &= \alpha(G_1) + \dots + \alpha(G_k) \end{aligned}$$

Thus, we are usually only concerned with connected graphs.

Corollary 1.32. *Let G be a graph with an isolated vertex v . Then $\eta(G) = \eta(G - v) + 1$.*

Remark 1.33. Based on the fact that we are dealing with ranks of $|G| \times |G|$ matrices, we have $0 \leq \eta(G) \leq |G|$ for any graph G . But, 0 is not possible because the diagonal entries of any matrix fitting G are nonzero. Thus, for any graph G , $1 \leq \eta(G) \leq |G|$. And, these bounds are realized by $G = K_n$ and $G = E_n$, respectively, where again E_n is used to denote the empty graph on n vertices.

CHAPTER 2. Operations

2.1 Edge and Vertex Deletions

For a graph G and a set of vertices $W \subseteq V(G)$, $G - W$ denotes the subgraph of G formed by deleting the vertices in W , i.e., the induced subgraph $G[V \setminus W]$. If $W = \{v\}$ contains just one vertex, we write $G - v$ in place of $G - \{v\}$. For a set of edges $F \subseteq E(G)$, we write $G - F$ to denote the subgraph of G with vertex set $V(G)$ and edge set $E(G) \setminus F$. Again, if $F = \{e\}$ contains just one edge, we write $G - e$ instead of $G - \{e\}$. A subgraph of G is called a **spanning subgraph** if its vertex set is $V(G)$.

Remark 2.1. Let G be a graph and H a spanning subgraph of G , i.e., $V(H) = V(G)$. Then $\eta(G) \leq \eta(H)$. To see this, let A be a matrix that fits H and realizes $\eta(H)$. Then A fits G so $\eta(G) \leq \text{rank } A = \eta(H)$.

Proposition 2.2. *Let G be a graph, e any edge in G , and v a vertex incident with the edge e . Then*

$$\eta(G - v) \leq \eta(G) \leq \eta(G - e) \leq \eta(G - v) + 1.$$

Proof. Take any matrix A fitting G which realizes $\eta(G) = \text{rank } A$. Let $A(v)$ denote the matrix created by starting with A and deleting the row and column corresponding to v . Then $A(v)$ fits $G - v$, so

$$\eta(G - v) \leq \text{rank } A(v) \leq \text{rank } A = \eta(G).$$

This gives the first inequality. The second inequality follows from Remark 2.1 since $G - e$ is a spanning subgraph of G .

The second inequality says if we delete one edge, η either remains the same, or increases. Thus, if we delete all edges incident with v , which includes e , η either remains the same, or

increases. The resulting graph is the disjoint union of $G - v$ and $\{v\}$. This graph is a spanning subgraph of $G - e$, so we have

$$\eta(G - e) \leq \eta(G - v) + \eta(\{v\}) = \eta(G - v) + 1.$$

This gives the third inequality. □

Corollary 2.3. *For any graph G , and edge $e \in E(G)$,*

$$0 \leq \eta(G - e) - \eta(G) \leq 1.$$

So, we know that $\eta(K_n) = 1$ and $\eta(G) = |G|$ for G the empty graph. And, as we delete edges from K_n one by one, in whatever order we choose, each deletion either leaves η unchanged, or increases it by 1. Eventually, we have deleted all the edges and η has increased to $|G|$. Or, similarly, as we add edges to the empty graph, each addition either leaves η unchanged, or decreases it by 1.

Corollary 2.4. *For any graph G , and vertex $v \in V(G)$,*

$$0 \leq \eta(G) - \eta(G - v) \leq 1.$$

Corollary 2.5. *If H is an induced subgraph of G , then $\eta(H) \leq \eta(G)$.*

Proposition 2.6. *For any graph G , and edge $e \in E(G)$,*

$$0 \leq \text{vcc}(G - e) - \text{vcc}(G) \leq 1$$

and

$$0 \leq \alpha(G - e) - \alpha(G) \leq 1$$

Proof. Assume $e = uv$.

Any vertex clique cover of $G - e$ is also a vertex clique cover of G , so $\text{vcc}(G) \leq \text{vcc}(G - e)$. Let \mathcal{C} denote a collection of disjoint clique subgraphs of G such that $|\mathcal{C}| = \text{vcc}(G)$. If there is some clique $C \in \mathcal{C}$ that contains both u and v , and thus e , then we can cover the same vertices with two cliques such that one contains u and the other contains v . Therefore, if there exists

such a clique, $\text{vcc}(G-e) \leq \text{vcc}(G) + 1$. If no such clique exists, then $\text{vcc}(G-e) \leq \text{vcc}(G)$. This finishes the proof for the vertex clique cover number.

Any independent set in G is independent in $G-e$, so $\alpha(G) \leq \alpha(G-e)$. Now, let \mathcal{I} denote an independent set of vertices in $G-e$ such that $|\mathcal{I}| = \alpha(G-e)$. If u and v are both in \mathcal{I} , then $\mathcal{I} \setminus \{u\}$ is independent in G , so that $\alpha(G) \geq \alpha(G-e) - 1$. If only one of u or v is in \mathcal{I} , then \mathcal{I} is independent in G , so that $\alpha(G) \geq \alpha(G-e)$. In any case, $\alpha(G) \geq \alpha(G-e) - 1$. \square

Proposition 2.7. *For any graph G , and vertex $v \in V(G)$,*

$$0 \leq \text{vcc}(G) - \text{vcc}(G-v) \leq 1$$

and

$$0 \leq \alpha(G) - \alpha(G-v) \leq 1.$$

Proof. It is clear that $\text{vcc}(G-v) \leq \text{vcc}(G)$. Let \mathcal{C} be a minimum vertex clique cover of $G-v$, so $\text{vcc}(G-v) = |\mathcal{C}|$. Then, all the cliques in \mathcal{C} , along with the clique $\{v\}$, cover G . So, $\text{vcc}(G) \leq |\mathcal{C}| + 1 = \text{vcc}(G-v) + 1$.

Let \mathcal{I} be any independent set in $G-v$. Then \mathcal{I} is independent in G , so $\alpha(G-v) \leq \alpha(G)$. Let \mathcal{J} be a maximum independent set in G . If \mathcal{J} contains the vertex v , then $\mathcal{J} \setminus \{v\}$ is independent in $G-v$. If $v \notin \mathcal{J}$, then \mathcal{J} is independent in $G-v$. In any case, $\alpha(G) - 1 \leq \alpha(G-v)$. \square

Corollary 2.8. *If H is an induced subgraph of G , then $\text{vcc}(H) \leq \text{vcc}(G)$ and $\alpha(H) \leq \alpha(G)$.*

In a graph G , for any vertex $u \in V(G)$, we define the **open neighborhood** of u , $N(u)$, to be the set of all neighbors of u . We define the **closed neighborhood** of u , $N[u] = N(u) \cup \{u\}$. If the graph in question is not clear from the context, we use the notation $N_G(u)$ and $N_G[u]$ to denote the open and closed neighborhoods of the vertex u in the graph G .

Theorem 2.9. *Let G be a graph of order n . Assume there exist two vertices u and v such that $N[u] \subseteq N[v]$. Then, $\eta(G-v) = \eta(G)$.*

Proof. Proposition 2.2 gives $\eta(G-v) \leq \eta(G)$. Without loss of generality, assume $u = n-1$ and $v = n$. Observe uv is an edge of G since $u \in N[u] \subseteq N[v]$. Let $A = [a_{ij}]$ be any matrix that fits $G-v$ such that $\text{rank } A = \eta(G-v)$. We create a new matrix, $B = [b_{ij}]$, that fits G in

the following way. Copy the last column of A and add it on after the last column of A . Call the matrix formed A_1 . Now, take the last row of A_1 and add a copy after the last row of A_1 to create B . Claim: B fits G . First, note that $b_{nn} = b_{n-1,n-1} = a_{n-1,n-1}$ based on the construction. Since $a_{n-1,n-1}$ is nonzero, so is b_{nn} . Also, any nonzero entry in row or column n , other than b_{nn} , corresponds to an edge incident with v . The last row and column has one nonzero entry corresponding to the edge uv , which is definitely nonzero since $b_{n,n-1} = b_{n-1,n} = a_{n-1,n-1}$. The other nonzero entries came from the nonzero entries in the last row/column of A corresponding to edges incident with u . Since $N[u] \subseteq N[v]$, we have proven the claim. Thus, B fits G and $\text{rank } B = \text{rank } A$. Thus, $\eta(G) \leq \eta(G - v)$, so $\eta(G) = \eta(G - v)$. \square

Corollary 2.10. *Let G be a graph and v any vertex of degree $|G| - 1$. Then, $\eta(G) = \eta(G - v)$.*

A vertex of degree 1 is called a **leaf** or a **pendant vertex**.

Corollary 2.11. *Let G be a graph and v a neighbor of a leaf. Then, $\eta(G) = \eta(G - v)$.*

Example 2.12. Take any graph G of order n , with vertices $\{1, 2, \dots, n\}$. To G , add n new vertices $\{v_1, v_2, \dots, v_n\}$ such that $v_i \sim i$ but v_i is not adjacent to any other vertex. Call this new graph H . Then $\eta(H) = n$, no matter what graph G we started with.

In particular, consider the extreme cases. Let H_E denote the graph formed by the above construction when starting with the empty graph. H_E has $2n$ vertices and n edges. Let H_K denote the graph formed by the above construction when starting with K_n . Then H_K has $2n$ vertices and $\binom{n}{2} + n = \frac{n^2+n}{2}$ edges.

Now, we can delete from H_K the edges of the K_n , in any order, until there are no more, arriving at H_E . Let us label the edges of the K_n by $\{e_1, \dots, e_{\binom{n}{2}}\}$, in the order they are deleted. By Corollary 2.3,

$$n = \eta(H_K) \leq \eta(H_K - e_1) \leq \dots \leq \eta(H_K - \{e_1, \dots, e_{\binom{n}{2}}\}) = \eta(H_E) = n,$$

so that all intermediate graphs, F , will satisfy $\eta(F) = n$.

Shannon showed an analogous result to that of Theorem 2.9, for the Shannon capacity of a graph.

Theorem 2.13. [39, Theorem 3] *Let G be a graph with $u, v \in V(G)$ such that $N[u] \subseteq N[v]$. Then*

$$\alpha((G - v)^k) = \alpha(G^k) \text{ for any positive integer } k, \text{ and therefore}$$

$$\Theta(G - v) = \Theta(G)$$

Proof. From Proposition 2.7, we know that $\alpha((G - v)^k) \leq \alpha(G^k)$, since $(G - v)^k$ can be formed from G^k by deleting all vertices that contain v in at least one coordinate. So, we must show $\alpha((G - v)^k) \geq \alpha(G^k)$.

Take any maximum independent set, I_1 , in G^k . If there exists a vertex $V = (v_1, \dots, v_k) \in I_1$ such that at least one coordinate of V is v , then let $U = (u_1, \dots, u_k)$ be the vertex of G^k such that $u_i = u$ if $v_i = v$ and $u_i = v_i$ otherwise. Then V is adjacent to U in G^k since v is adjacent to u in G . Since $V \in I_1$, $U \notin I_1$. Since $N_G[u] \subseteq N_G[v]$, $N_{G^k}[U] \subseteq N_{G^k}[V]$. Therefore, $I_2 = (I_1 \setminus \{V\}) \cup \{U\}$ is independent in G^k and has the same cardinality as I_1 .

Repeat this process until all vertices with at least one coordinate v are removed. That is, at step i , if I_i contains any vertex with at least one coordinate v , replace it with a vertex that does not contain v in any coordinate to get a new independent set in G^k , I_{i+1} , with the same cardinality as I_i . Eventually, we reach I_f , an independent set in G^k with the same cardinality as I_1 , such that no vertex in I_f has v in any coordinate. Thus, I_f is independent in $(G - v)^k$ so that $\alpha((G - v)^k) \geq \alpha(G^k)$. \square

Theorem 2.14. *Let G be a graph of order n . Assume there exist two vertices u and v such that $N[u] \subseteq N[v]$. Then,*

$$\text{vcc}(G - v) = \text{vcc}(G).$$

Proof. By Proposition 2.7, $\text{vcc}(G - v) \leq \text{vcc}(G)$ for any graph G and vertex $v \in V(G)$. So, we need only prove the inequality in the other direction.

Let \mathcal{C} be any minimum disjoint collection of clique subgraphs of $G - v$ that cover all the vertices of G , so $|\mathcal{C}| = \text{vcc}(G - v)$. Let $K \in \mathcal{C}$ be the clique containing u . Then, $V(K) \subseteq N[u] \subseteq N[v]$, so the subgraph of G induced by the vertices in $K \cup \{v\}$ is a clique. Thus, there exists a vertex clique cover of G of size $|\mathcal{C}|$, so that $\text{vcc}(G - v) \geq \text{vcc}(G)$. \square

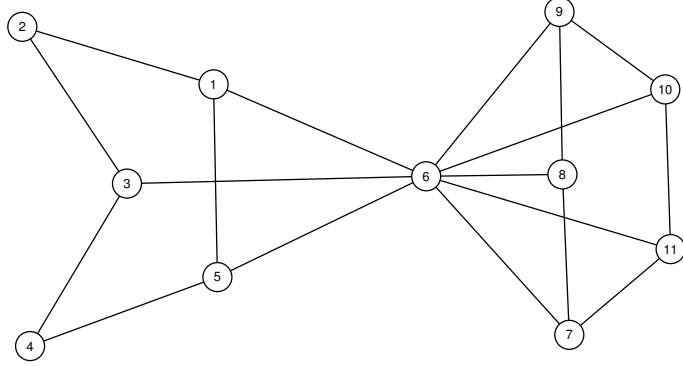


Figure 2.1 Graph for Example 2.15

Example 2.15. Let G be the graph in Figure 2.1. Since $N[7] \subseteq N[6]$, we can delete vertex 6 without affecting the independence number, Haemers' minimum rank, or the vertex clique cover number by Theorems 2.9, 2.13 and 2.14. Since $G - \{6\}$ consists of two disjoint C_5 's, we see that $\alpha(G) = 4$, $\eta(G) = 6$, and $\text{vcc}(G) = 6$.

2.2 Cut-Vertex Reduction

In a connected graph G , a vertex v is called a **cut-vertex** if $G - v$ is not connected. If $G - v$ has k components G_1, G_2, \dots, G_k , $k \geq 2$, we define $B_i = G[V(G_i) \cup \{v\}]$ for $i = 1, \dots, k$. The B_i are connected and we call them the **branches** of G at v . A connected graph is **nonseparable** if it does not have a cut-vertex. A **block** of a graph is a maximal nonseparable subgraph.

Definition 2.16. For a graph G , the η -**rank spread** at vertex v is defined as

$$r_v^\eta(G) = \eta(G) - \eta(G - v).$$

Or, more generally, for any minimal cut-set of vertices C , i.e., no subset of C is a cut-set,

$$r_C^\eta(G) = \eta(G) - \eta(G - C).$$

Proposition 2.17. For any graph G and any minimal cut-set C , let $\{H_i\}_{i=1}^k$ denote the connected components of $G - C$. We let G_i denote the subgraph of G induced by the vertices

$V(H_i) \cup C$. Then

$$r_C^\eta(G) \leq \min_j r_C^\eta(G_j).$$

Proof. For any j , the subgraphs G_j and $\{G_i - C\}_{i \neq j}$ are disjoint and cover all the vertices of G , so that $\eta(G) \leq \eta(G_j) + \sum_{i \neq j} \eta(G_i - C)$ by Theorem 1.11. Since $G - C$ is the disjoint union of the graphs $G_i - C$, we have $\sum_i \eta(G_i - C) = \eta(G - C)$. Also, $\eta(G_j) = r_C^\eta(G_j) + \eta(G_j - C)$. Putting these all together gives

$$\eta(G) \leq \eta(G_j) + \sum_{i \neq j} \eta(G_i - C) = r_C^\eta(G_j) + \sum_i \eta(G_i - C) = r_C^\eta(G_j) + \eta(G - C).$$

Taking the minimum over all j gives

$$\eta(G) \leq \min_j r_C^\eta(G_j) + \eta(G - C).$$

Therefore,

$$r_C^\eta(G) \leq \min_j r_C^\eta(G_j).$$

□

In the specific case of a cut-set of size 1, i.e., a cut-vertex, we can say more. We know that $r_v^\eta(G)$ is either 0 or 1 from Corollary 2.4. The proof of the following theorem uses a technique similar to that found in the proof of [3, Theorem 2.3].

Theorem 2.18. *Let G be a graph with cut-vertex v . Let $\{G_i\}_{i=1}^k$, denote the branches of G at v . Then,*

$$r_v^\eta(G) = \min_i \{r_v^\eta(G_i)\}.$$

Proof. We prove $r_v^\eta(G) = 0$ if and only if there exists some j , $1 \leq j \leq k$, such that $r_v^\eta(G_j) = 0$. If $r_v^\eta(G_j) = 0$ for some j , then $r_v^\eta(G) = 0$ since $0 \leq r_v^\eta(G) \leq \min_i r_v^\eta(G_i) = 0$ by Proposition 2.2 and Proposition 2.17.

Thus, assume $r_v^\eta(G) = 0$. If necessary, relabel the vertices so that $v = 1$, the vertices of $G_1 - v$ come next, then those of $G_2 - v$, and so on, with those from $G_k - v$ last. Then there

exists a matrix of the form

$$A = \begin{bmatrix} \alpha & \mathbf{c}_1^T & \mathbf{c}_2^T & \cdots & \mathbf{c}_k^T \\ \mathbf{b}_1 & A_1 & 0 & \cdots & 0 \\ \mathbf{b}_2 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_k & 0 & 0 & \cdots & A_k \end{bmatrix}$$

such that A fits G with $\text{rank } A = \eta(G)$, and A_i fits $G_i - v$ with $\text{rank } A_i = \eta(G_i - v)$. Since $r_v^\eta(G) = 0$, this implies there exist \mathbf{x}_i 's such that $A\mathbf{x}_i = \mathbf{b}_i$, and there exist \mathbf{y}_i 's such that $\mathbf{y}_i^T A_i = \mathbf{c}_i^T$. Further, we have $\alpha = \sum_{i=1}^n \mathbf{y}_i^T A_i \mathbf{x}_i$ and $\alpha \neq 0$, since A fits G . Since this sum is not 0, there exists at least one j such that $\alpha_j = \mathbf{y}_j^T A_j \mathbf{x}_j \neq 0$. Then, the matrix

$$A' = \begin{bmatrix} \alpha_j & \mathbf{c}_j^T \\ \mathbf{b}_j & A_j \end{bmatrix}$$

fits G_j so that $\eta(G_j) \leq \text{rank } A'$. Now, the submatrix A_j fits $G_j - v$ and has the same rank as A' . Since $\text{rank } A_j = \eta(G_j - v)$, we have

$$\eta(G_j) \leq \text{rank } A' = \text{rank } A_j = \eta(G_j - v) \leq \eta(G_j),$$

so that

$$r_v^\eta(G_j) = 0.$$

Therefore, we have shown $r_v^\eta(G) = 0$ if and only if there exists some j , $1 \leq j \leq k$, such that $r_v^\eta(G_j) = 0$. Since the rank spread of G is either 0 or 1, the fact that $r_v^\eta(G) = 1$ if and only if $r_v^\eta(G_i) = 1$ for all i is immediate. This finishes the proof that $r_v^\eta(G) = \min_i \{r_v^\eta(G_i)\}$. □

Corollary 2.19. *If a graph G has a cut-vertex v and $\{G_i\}_{i=1}^k$ are the branches of G at v , then*

$$\eta(G) = \min_i \{r_v^\eta(G_i)\} + \sum_{i=1}^k \eta(G_i - v).$$

Example 2.20. Let G be the graph in Figure 2.2. To use Theorem 2.18, we start by noting that vertex 6 is a cut-vertex. Let G_1 denote the subgraph of G induced by the vertices $\{1, 2, 3, 4, 5, 6\}$.

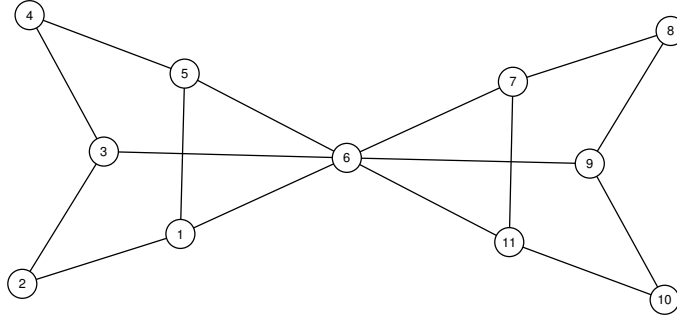


Figure 2.2 Graph for Example 2.20

We have $\eta(G_1) = 3 = \eta(G_1 - \{6\})$ so $r_6^\eta(G_1) = 0$. By Theorem 2.18, $r_6^\eta(G) = 0$. Thus, $\eta(G) = \eta(G - \{6\}) = 2\eta(C_5) = 6$.

Note, $I = \{2, 4, 6, 8, 10\}$ is the unique maximum independent set, so that $\alpha(G) = 5$. It is not hard to see that $\text{vcc}(G) = 6$. For example, if the clique induced by $\{1, 5, 6\}$ is used, then it would take an additional two cliques to cover the vertices in $\{2, 3, 4\}$ and the subgraph induced by the vertices in $\{7, 8, 9, 10, 11\}$, is a 5-cycle, which requires three cliques, for a total of six. Starting with the clique induced by $\{6, 7, 11\}$ is the same by symmetry. And, there are no other triangles. Thus, $\eta(G)$ is not determined by these bounds.

Note, Theorem 2.18 is not true for $|C| > 1$.

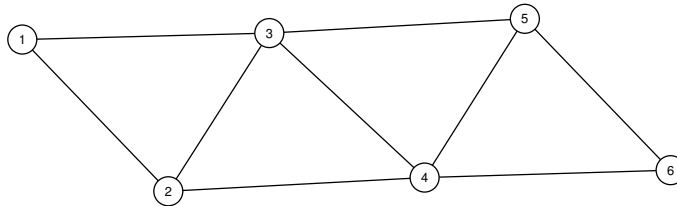


Figure 2.3 Graph for Example 2.21

Example 2.21. Let G be the graph in Figure 2.3. Then G is 2-connected and a cut-set of size 2 is $C = \{3, 4\}$. The subgraph of G induced by the vertices $\{1, 2, 3, 4\}$, say G_1 , satisfies $\eta(G_1) = 2$ and $\eta(G_1 - C) = 1$. The subgraph of G induced by the vertices $\{3, 4, 5, 6\}$, G_2 , is isomorphic to G_1 . So, $\min_i \{r_C^\eta(G_i)\} = 1$.

On the other hand, $\eta(G) = 2$ because $\text{vcc}(G) = 2$, and $\eta(G - C) = 2$, so $r_C^\eta(G) = 0$. This shows that the result of Theorem 2.18 does not hold for cut-sets of size bigger than 1.

A theorem analogous to Theorem 2.18 is true for the independence number and vertex clique cover number. For simplicity, let us define $r_v^{\text{vcc}}(G) = \text{vcc}(G) - \text{vcc}(G - v)$ and $r_v^\alpha(G) = \alpha(G) - \alpha(G - v)$.

Theorem 2.22. *Let G be a graph with cut-vertex v . Let $\{G_i\}_{i=1}^k$, denote the branches of G at v . Then,*

$$r_v^\alpha(G) = \min_i \{r_v^\alpha(G_i)\}$$

$$r_v^{\text{vcc}}(G) = \min_i \{r_v^{\text{vcc}}(G_i)\}.$$

Proof. We start with the proof for the rank spread of α . As in the proof of Theorem 2.18, we first prove $r_v^\alpha(G) = 0$ if and only if there exists at least one i such that $r_v^\alpha(G_i) = 0$.

To begin, assume that $r_v^\alpha(G_j) = 0$ for some j . Let I be a maximum independent set of vertices in G . If $v \notin I$, then I is independent in $G - v$. Since $\alpha(G) \geq \alpha(G - v)$, this forces $r_v^\alpha(G) = 0$.

Now, consider the case when $v \in I$. Let $I_{j,1} = I \cap V(G_j)$. Note $v \in I_{j,1}$. Since $r_v^\alpha(G_j) = 0$, there exists $I_{j,2}$, an independent set of vertices in G_j such that $v \notin I_{j,2}$ and $|I_{j,2}| \geq |I_{j,1}|$. Then

$$J = (I \setminus I_{j,1}) \cup I_{j,2}$$

has the same cardinality as I by the maximality of I , and $v \notin J$. Since v is a cut-vertex and $v \notin J$, it is clear J is independent in G . Thus, in this case also we have $r_v^\alpha(G) = 0$.

So, now assume $r_v^\alpha(G) = 0$, so that $\alpha(G) = \alpha(G - v)$, and we want to show that $r_v^\alpha(G_j) = 0$ for some j . Let us assume $r_v^\alpha(G_i) = 1$ for all $i = 1, \dots, k$ and we will arrive at a contradiction. Since $\alpha(G) = \alpha(G - v)$, there exists a maximum independent set in G that does not contain v ,

say I . For each i , let $I_{i,1} = I \cap V(G_i)$. Then $I_{i,1}$ is independent in $G_i - v$. Since $r_v^\alpha(G_i) = 1$, there exists an independent set of vertices $I_{i,2}$ in $G_i - v$ such that $|I_{i,2}| \geq |I_{i,1}|$ and $I_{i,2} \cup \{v\}$ is independent in G_i . Then $\{v\} \cup \left(\bigcup_{i=1}^k I_{i,2}\right)$ is independent in G and has cardinality $|I| + 1$. This contradicts the maximality of I so $r_v^\alpha(G_j) = 0$ for some j .

Thus, we have shown $r_v^\alpha(G) = 0$ if and only if there exists at least one i such that $r_v^\alpha(G_i)$. Since $r_v^\alpha(G)$ can only be 0 or 1 by Proposition 2.7, this is all we need to show.

Now, we move on to prove the statement involving the rank spread of vcc. We have that any clique subgraph of G is wholly contained within one G_j since deleting a vertex of a clique does not split it into multiple connected components. Therefore, for any j ,

$$\text{vcc}(G) \leq \text{vcc}(G_j) + \sum_{i \neq j} \text{vcc}(G_i - v).$$

Without loss of generality, we assume the cliques are disjoint. Thus, v will be in exactly one clique for any vertex clique cover of G . Therefore for any vertex clique cover of G , there is some j such that the clique cover is made up of some cliques that cover $\bigcup_{i \neq j} (G_i - v)$ and some cliques that cover G_j . Thus, for some j , we actually have

$$\text{vcc}(G) = \text{vcc}(G_j) + \sum_{i \neq j} \text{vcc}(G_i - v).$$

Since $G - v$ is equal to the disjoint union of the graphs $G_i - v$, we have that $\sum_{i=1}^k \text{vcc}(G_i - v) = \text{vcc}(G - v)$. And, by Proposition 2.7, we have $\text{vcc}(G - v) \leq \text{vcc}(G)$. Putting all these together gives, for some j , we have

$$\sum_{i=1}^k \text{vcc}(G_i - v) = \text{vcc}(G - v) \leq \text{vcc}(G) = \text{vcc}(G_j) + \sum_{i \neq j} \text{vcc}(G_i - v).$$

It is now immediate that $r_v^{\text{vcc}}(G) = 0$ if and only if $r_v^{\text{vcc}}(G_j) = 0$.

Since $r_v^{\text{vcc}}(G)$ is 0 or 1 for any graph G , we have and $r_v^{\text{vcc}}(G) = 1$ if and only if $\min_i \{r_v^{\text{vcc}}(G_i)\} = 1$, so that in any case $r_v^{\text{vcc}}(G) = \min_i \{r_v^{\text{vcc}}(G_i)\}$. \square

Theorem 2.23. *Let H be a graph with a cut-vertex v , where the branches at v are $\{G_i\}_{i=1}^k$. If, for each i , we have*

$$\begin{aligned}\eta(G_i) &= \alpha(G_i) \\ \eta(G_i - v) &= \alpha(G_i - v).\end{aligned}$$

Then

$$\eta(H) = \alpha(H)$$

so that $\Theta(H) = \alpha(H)$ is determined.

Proof. By Corollary 2.19, we have

$$\eta(H) = \min_i r_v^\eta(G_i) + \sum_{i=1}^k \eta(G_i - v)$$

and by Theorem 2.22, we have

$$\alpha(H) = \min_i r_v^\alpha(G_i) + \sum_{i=1}^k \alpha(G_i - v).$$

The conditions in the theorem guarantee that $\eta(G_i - v) = \alpha(G_i - v)$ and $r_v^\eta(G_i) = r_v^\alpha(G_i)$ for each $i = 1, \dots, k$. Therefore, $\eta(H) = \alpha(H)$. \square

We can use this theorem to construct infinitely many new graphs where $\Theta(H)$ is determined by putting together smaller graphs that satisfy $\eta(G) = \alpha(G)$ and have some vertex v such that $\eta(G - v) = \alpha(G - v)$.

Example 2.24. Let G_i , for $i = 1, \dots, k$, be the graph in Example 1.29. Recall $\alpha(G_i) = \eta(G_i) = 3$ and $\text{vcc}(G_i) = 4$, so that $\Theta(G_i) = 3$ is determined by η but not by vcc . For any vertex $v \in V(G_i)$, we have $\alpha(G_i - v) = 3$, and since $3 = \alpha(G_i - v) \leq \eta(G_i - v) \leq \eta(G_i) = 3$, we have $\eta(G_i - v) = 3$ as well. So, let u_i be any vertex in G_i , for each i , i.e., they are not necessarily all the same vertex.

Create the graph H as the union of the graphs, G_i , such that we identify the vertices $u_1 = \dots = u_k$ in H , and call this vertex u , but the other vertices in G_i are not connected in any other way. Then u is a cut-vertex in H and the branches for u are $\{G_i\}_{i=1}^k$. Theorem 2.23

applies to give $\eta(H) = \alpha(H)$. In particular, by Corollary 2.19

$$\alpha(H) = \eta(H) = \min_i r_u^\eta(G_i) + \sum_{i=1}^k \eta(G_i - u) = 3k.$$

And, therefore, $\Theta(H) = 3k$.

2.3 Join

Definition 2.25. The **join** $G \vee H$ of graphs G and H is a new graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Theorem 2.26. $\eta(G_1 \vee G_2) = \max\{\eta(G_1), \eta(G_2)\}$

Proof. Since any matrix fitting $G_1 \vee G_2$ must contain submatrices fitting G_1 and G_2 , it is clear that

$$\eta(G_1 \vee G_2) \geq \max\{\eta(G_1), \eta(G_2)\}.$$

We will prove the other inequality by constructing a matrix, A , that fits $G_1 \vee G_2$ and satisfies

$$\text{rank } A \leq \max\{\eta(G_1), \eta(G_2)\}.$$

Let A_1 and A_2 be any matrices that fit G_1 and G_2 and attain $\eta(G_1)$ and $\eta(G_2)$, respectively. These will be used in the construction of A .

Let $\mathbf{c}_1, \dots, \mathbf{c}_m$ be any maximum independent set of columns of A_1 , and let $\mathbf{d}_1, \dots, \mathbf{d}_n$ be any maximum independent set of columns of A_2 . Without loss of generality, assume $\text{rank } A_1 \leq \text{rank } A_2$, so $m \leq n = \max\{\eta(G_1), \eta(G_2)\}$. If $m < n$, for $m < j \leq n$, we define $\mathbf{c}_j = \mathbf{0}$, the $\mathbf{0}$ matrix of size $|G_1| \times 1$.

Since $\mathbf{c}_1, \dots, \mathbf{c}_m$ is a maximum independent set of columns of A_1 , there exist constants $x_{i,j}$ such that column i , $1 \leq i \leq |G_1|$, of A_1 is

$$x_{i,1}\mathbf{c}_1 + \dots + x_{i,m}\mathbf{c}_m.$$

If $m < n$, we define $x_{i,j} = 0$ for $m < j \leq n$ and all i so that column i of A_1 can be written as

$$x_{i,1}\mathbf{c}_1 + \dots + x_{i,n}\mathbf{c}_n.$$

Similarly, there exist constants $y_{i,j}$ such that column i , $1 \leq i \leq |G_2|$, of A_2 is

$$y_{i,1}\mathbf{d}_1 + \cdots + y_{i,n}\mathbf{d}_n.$$

Using these, we define matrices $X = [x_{i,j}]^T$ and $Y = [y_{i,j}]^T$. Note, X is $n \times |G_1|$ and Y is $n \times |G_2|$.

Now, let $M = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ \mathbf{d}_1 & \cdots & \mathbf{d}_n \end{bmatrix}$. By this construction, we have M is $(|G_1| + |G_2|) \times n =$

$|G| \times n$ and $\text{rank } M = n$. Consider the matrix $A = M \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} MX & MY \end{bmatrix}$ and note that $\text{rank } A \leq \text{rank } M = n = \max\{\eta(G_1), \eta(G_2)\}$. We have

$$MX = \begin{bmatrix} (x_{1,1}\mathbf{c}_1 + \cdots + x_{1,n}\mathbf{c}_n) & \cdots & (x_{|G_1|,1}\mathbf{c}_1 + \cdots + x_{|G_1|,n}\mathbf{c}_n) \\ (x_{1,1}\mathbf{d}_1 + \cdots + x_{1,n}\mathbf{d}_n) & \cdots & (x_{|G_1|,1}\mathbf{d}_1 + \cdots + x_{|G_1|,n}\mathbf{d}_n) \end{bmatrix}$$

This shows that MX is a block matrix of the form $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$, where A_1 is the matrix fitting G_1 with $\text{rank } A_1 = \eta(G_1)$. Similarly, we have

$$MY = \begin{bmatrix} (y_{1,1}\mathbf{c}_1 + \cdots + y_{1,n}\mathbf{c}_n) & \cdots & (y_{|G_2|,1}\mathbf{c}_1 + \cdots + y_{|G_2|,n}\mathbf{c}_n) \\ (y_{1,1}\mathbf{d}_1 + \cdots + y_{1,n}\mathbf{d}_n) & \cdots & (y_{|G_2|,1}\mathbf{d}_1 + \cdots + y_{|G_2|,n}\mathbf{d}_n) \end{bmatrix},$$

and MY is a block matrix of the form $\begin{bmatrix} B_2 \\ A_2 \end{bmatrix}$, where A_2 is the matrix fitting G_2 with $\text{rank } A_2 =$

$\eta(G_2)$. Thus, the matrix $A = \begin{bmatrix} A_1 & B_2 \\ B_1 & A_2 \end{bmatrix}$. Note, in $G = G_1 \vee G_2$, all vertices in $V(G_1)$ are adjacent to all vertices in $V(G_2)$, so that A fits G , no matter the structure of B_1 and B_2 .

Since $\text{rank } A \leq \max\{\eta(G_1), \eta(G_2)\}$, we have $\eta(G) \leq \max\{\eta(G_1), \eta(G_2)\}$, which finishes the proof. \square

Theorem 2.27. $\text{vcc}(G_1 \vee G_2) = \max\{\text{vcc}(G_1), \text{vcc}(G_2)\}$ and $\alpha(G_1 \vee G_2) = \max\{\alpha(G_1), \alpha(G_2)\}$

Proof. Since $G_1 \vee G_2$ contains as induced subgraphs both G_1 and G_2 , we have that $\text{vcc}(G_1 \vee G_2) \geq \max\{\text{vcc}(G_1), \text{vcc}(G_2)\}$ and $\alpha(G_1 \vee G_2) \geq \max\{\alpha(G_1), \alpha(G_2)\}$ by Corollary 2.8.

Let $\mathcal{C} = \{C_1, \dots, C_i\}$ be a vertex clique cover of G_1 such that $|\mathcal{C}| = \text{vcc}(G_1)$, and $\mathcal{D} = \{D_1, \dots, D_j\}$ be a vertex clique cover of G_2 such that $|\mathcal{D}| = \text{vcc}(G_2)$. Without loss of generality,

let us assume $i \leq j$, i.e., $\text{vcc}(G_1) \leq \text{vcc}(G_2)$. Then, since $K_a \vee K_b = K_{a+b}$, we have that $\{C_1 \vee D_1, C_2 \vee D_2, \dots, C_i \vee D_i, D_{i+1}, \dots, D_j\}$ is a vertex clique cover of $G_1 \vee G_2$ containing $j = \max\{\text{vcc}(G_1), \text{vcc}(G_2)\}$ cliques. Therefore, $\text{vcc}(G_1 \vee G_2) \leq \max\{\text{vcc}(G_1), \text{vcc}(G_2)\}$.

Since in $G_1 \vee G_2$, all vertices of G_1 are adjacent to all those of G_2 , a maximum independent set of vertices must be wholly contained in the subgraph G_1 , or wholly contained in the subgraph G_2 . Thus, $\alpha(G_1 \vee G_2) \leq \max\{\alpha(G_1), \alpha(G_2)\}$. \square

Corollary 2.28. *Let K_{m_1, m_2, \dots, m_r} denote the complete multipartite graph on partite sets of size m_1, m_2, \dots, m_r . Then*

$$\alpha(K_{m_1, m_2, \dots, m_r}) = \eta(K_{m_1, m_2, \dots, m_r}) = \text{vcc}(K_{m_1, m_2, \dots, m_r}) = \max(m_1, \dots, m_r).$$

Proof. A complete multipartite graph can be written as the join of empty graphs, $K_{m_1, m_2, \dots, m_r} = \overline{K_{m_1}} \vee \overline{K_{m_2}} \vee \dots \vee \overline{K_{m_r}}$. Since $\eta(\overline{K_i}) = i$, Theorems 2.26 and 2.27 prove the assertion of the theorem. \square

Assume we have graphs G and H such that $\alpha(G) = \eta(G)$ and $\alpha(H) = \eta(H)$. By Theorems 2.26 and 2.27, we have $\eta(G \vee H) = \max\{\eta(G), \eta(H)\}$ and $\alpha(G \vee H) = \max\{\alpha(G), \alpha(H)\}$, so that in fact $\eta(G \vee H) = \alpha(G \vee H)$. Thus, $\Theta(G \vee H)$ is determined. In fact, all we need is that $\max\{\eta(G), \eta(H)\} = \max\{\alpha(G), \alpha(H)\}$ for $\Theta(G \vee H)$ to be determined.

CHAPTER 3. Techniques for Increasing the Lower Bound

In this section, a few techniques will be given that are sometimes helpful for graphs where $\alpha(G) < \text{vcc}(G)$. If successful, a technique will increase the lower bound to $\alpha(G) + 1$. The basic idea is to assume $\eta(G) = \alpha(G)$, look at a general matrix that fits G and attains rank $A = \eta(G)$, and try to find a contradiction. If we are working directly with a general matrix fitting a graph G , we let $*$ denote an entry that must be non-zero, 0 represent an entry that must be 0, and $?$ represent an entry that is free.

Theorem 3.1. For $n \geq 3$, $\eta(\overline{C_n}) = \text{vcc}(\overline{C_n}) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$

Proof. If n is even, $\overline{C_n}$ is perfect as it is the complement of a bipartite graph. And, $\eta(\overline{C_n}) = \text{vcc}(\overline{C_n}) = \chi(C_n) = 2$.

If $n = 3$, then $\overline{C_3}$ is the empty graph on 3 vertices so $\eta(\overline{C_3}) = \text{vcc}(\overline{C_3}) = 3$.

So, assume $n \geq 5$ is odd. We have $\alpha(\overline{C_n}) = \omega(C_n) = 2$ and $\text{vcc}(\overline{C_n}) = \chi(C_n) = 3$. Let us assume $\eta(\overline{C_n}) = 2$. Let $A = [a_{ij}]$, with j th column $\mathbf{c}_j = [a_{1j} \cdots a_{nj}]^T$, be a matrix fitting $\overline{C_n}$ and realizing $\eta(\overline{C_n}) = 2$.

Since the vertices $\{1, 2\}$ are an independent set, to realize a rank of 2, \mathbf{c}_3 must be a linear combination of $\{\mathbf{c}_1, \mathbf{c}_2\}$. Thus, $\mathbf{c}_3 = b_1\mathbf{c}_1 + b_2\mathbf{c}_2$ for some b_1, b_2 . In particular, $a_{2,3} = b_1a_{2,1} + b_2a_{2,2}$. Now, $2 \not\sim 1$ and $2 \not\sim 3$ so $a_{2,1} = a_{2,3} = 0$. Since $a_{2,2}$ is nonzero, we must have $b_2 = 0$. Thus, \mathbf{c}_3 must be a multiple of \mathbf{c}_1 . Using the same argument, using the independent set of vertices $\{k, k+1\}$, we see that \mathbf{c}_{k+2} is a multiple \mathbf{c}_k . Therefore, columns $\{\mathbf{c}_3, \mathbf{c}_5, \dots, \mathbf{c}_n\}$ are all multiples of \mathbf{c}_1 . But, $1 \not\sim n$ so $a_{1,n} = 0$. Since $a_{1,1}$ is nonzero, this is not possible. Therefore, $\eta(\overline{C_n}) \geq 3$. □

Theorem 3.2. *Let G be a graph and $I = \{v_1, \dots, v_k\}$ a maximum independent set of vertices in G . Suppose there exists $u \in V(G) \setminus I$ such that $J = N(u) \cap I = \{v_j\}$ for some $1 \leq j \leq k$. Also, suppose there exists a vertex $w \in V(G) \setminus (I \cup \{u\})$ that is adjacent to exactly one of u or v_j . For all such vertices w , delete the corresponding edge wu or wv_j . Let H be the resulting spanning subgraph of G . Then*

$$\begin{aligned} \eta(G) = \alpha(G) & \quad \text{if and only if} & \quad \eta(H) = \alpha(G) \\ \eta(G) \geq \alpha(G) + 1 & \quad \text{if and only if} & \quad \eta(H) \geq \alpha(G) + 1 \end{aligned}$$

Proof. Let everything be as in the statement of the theorem. We label the vertices of G by $\{1, 2, \dots, n\}$. Without loss of generality, assume the maximum independent set of vertices is $I = \{1, 2, \dots, k\}$, $u = k + 1$, and $N(u) \cap I = \{k\}$, i.e., $v_j = k$.

For any graph, G , we have $\alpha(G) \leq \eta(G)$. Since H is a spanning subgraph of G , $\eta(G) \leq \eta(H)$. Therefore, if $\eta(H) = \alpha(G)$, these inequalities imply $\eta(G) = \alpha(G)$.

So, assume $\eta(G) = \alpha(G)$. We will prove $\eta(H) = \alpha(G)$. In any matrix, $A = [a_{ij}]$ with j th column $\mathbf{c}_j = [a_{1j} \cdots a_{nj}]^T$, fitting G , columns $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ form a linearly independent set. Since $\eta(G) = \alpha(G) = k$, it is a maximum linearly independent set of columns and therefore all remaining columns, $\mathbf{c}_{k+1}, \dots, \mathbf{c}_n$ are all linear combinations of columns $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$. In particular, column \mathbf{c}_{k+1} is

Thus, there exist constants b_i such that $\mathbf{c}_{k+1} = b_1 \mathbf{c}_1 + \cdots + b_k \mathbf{c}_k$. Since, out of the independent set $I = \{1, \dots, j, \dots, k\}$, $k + 1$ is only adjacent to k , $\mathbf{c}_{k+1} = b_k \mathbf{c}_k$. Columns \mathbf{c}_k and \mathbf{c}_{k+1} are shown below.

$$\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ * & ? \\ ? & * \\ \vdots & \vdots \\ a_{w,k} & a_{w,k+1} \\ \vdots & \vdots \end{bmatrix}$$

Since columns \mathbf{c}_k and \mathbf{c}_{k+1} are multiples of each other, $a_{w,k} = 0$ if and only if $a_{w,k+1} = 0$. Since, under the assumptions of this theorem, only one of the edges $\{w, k\}$ and $\{w, k+1\}$ is in $E(G)$, any matrix fitting G and realizing $\eta(G) = \alpha(G)$ must have $a_{w,k} = a_{w,k+1} = 0$. And, the same argument applies to the rows, giving $a_{k,w} = a_{k+1,w} = 0$.

Now, any matrix fitting H can be created by starting with a matrix fitting G and then making $a_{w,k} = a_{w,k+1} = a_{k,w} = a_{k+1,w} = 0$. Therefore, any matrix fitting G and attaining $\text{rank } A = \alpha(G)$ must also fit H , so that $\eta(H) \leq \alpha(G)$. Combining this with the inequality $\eta(H) \geq \eta(G)$ that comes from H being a spanning subgraph of G gives $\eta(H) = \eta(G) = \alpha(G)$. This finishes the proof that $\eta(G) = \alpha(G)$ if and only if $\eta(H) = \alpha(G)$. The contrapositive is $\eta(G) \neq \alpha(G)$ if and only if $\eta(H) \neq \alpha(G)$. Since $\eta(G), \eta(H) \geq \alpha(G)$, this is equivalent to $\eta(G) \geq \alpha(G) + 1$ if and only if $\eta(H) \geq \alpha(G) + 1$. \square

Theorem 3.3. *Consider the cycle on n vertices, C_n . We have*

$$\eta(C_n) = \text{vcc}(C_n) = \begin{cases} 1 & \text{if } n = 3 \\ \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } n \geq 5 \end{cases}$$

Proof. If $n = 2k$, we have $\alpha(C_{2k}) = k = \text{vcc}(C_{2k})$ so that $\eta(C_{2k}) = k = \frac{n}{2}$.

Let $G = C_{2k+1}$, the $(2k+1)$ -cycle, with vertices labeled in order 1 through $2k+1$. To find $\eta(C_{2k+1})$, we will use Theorem 3.2. We have $\alpha(G) = k$ and $I = \{1, 3, 5, \dots, 2k-1\}$ is a maximum independent set of vertices. $N(2k+1) \cap I = \{1\}$. Then, out of $2k+1$ and 1, vertex $2k$

is adjacent only to $2k + 1$ and vertex 2 is adjacent only to 1. So, we let H be the graph formed by deleting edges $\{1, 2\}$ and $\{2k, 2k + 1\}$ from G . Now, $\{1, 2, 4, 6, \dots, 2k\}$ is independent in H and thus $\eta(H) \geq \alpha(H) \geq k + 1 = \alpha(G) + 1$. As a result, $\eta(G) \geq k + 1$. Since $\text{vcc}(G) = k + 1$, we have $\eta(G) = k + 1$. \square

Proposition 3.4. *Let n denote the largest integer such that a graph G contains an induced n -cycle, C_n . Then*

$$\eta(G) \geq \begin{cases} 1 & \text{if } n = 3 \\ \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } n \geq 5 \end{cases}$$

Proof. This is an immediate consequence of Corollary 2.5 on induced subgraphs and Theorem 3.3 on $\eta(C_n)$. \square

Theorem 3.5. *Let W_n denote the wheel on n vertices. Then*

$$\eta(W_n) = \text{vcc}(W_n) = \begin{cases} 1 & \text{if } n = 4 \\ \frac{n}{2} & \text{if } n \text{ is even and } n \geq 5 \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Proof. W_n is formed as the join of a C_{n-1} and a single vertex. So, $\eta(W_n) = \max\{\eta(C_{n-1}), 1\} = \eta(C_{n-1})$ by Theorem 2.26. \square

Note, W_n , with $n \geq 6$ even, is an example of a graph where the bound based on an induced cycle, given in Proposition 3.4, is better than $\alpha(W_n)$. That is, $\alpha(W_n) = \frac{n-2}{2}$, but the bound based on Proposition 3.4 is $\frac{n}{2}$ since W_n contains an induced cycle C_{n-1} , with $n - 1 \geq 5$ and $n - 1$ odd.

The following theorem is somewhat of a generalization of Theorem 3.2, as here $J = N(u) \cap I$ is allowed to contain more than one vertex. However, if J contains exactly one vertex, then Theorem 3.2 is stronger than this theorem.

Theorem 3.6. *Let G be a graph and $I = \{v_1, \dots, v_k\}$ a maximum independent set of vertices of G . Let u be any vertex in $V(G) \setminus I$. Since I is maximum, u is adjacent to some nonempty*

subset of I , which we call $J = N(u) \cap I$. Suppose there exists $w \in V(G) \setminus (I \cup \{u\})$ such that $u \sim w$ but $N(w) \cap J = \emptyset$. Let $H = G - uw$. Then

$$\begin{aligned} \eta(G) = \alpha(G) & \quad \text{if and only if} & \quad \eta(H) = \alpha(G) \\ \eta(G) \geq \alpha(G) + 1 & \quad \text{if and only if} & \quad \eta(H) \geq \alpha(G) + 1 \end{aligned}$$

Proof. For any graph, we have $\alpha(G) \leq \eta(G)$. Since H is a spanning subgraph of G , $\eta(G) \leq \eta(H)$. Therefore, if $\eta(H) = \alpha(G)$, these inequalities imply $\eta(G) = \alpha(G)$.

Assume $\eta(G) = \alpha(G)$. We label the vertices of G by $\{1, 2, \dots, n\}$. Without loss of generality, assume $I = \{1, 2, \dots, k\}$, $J = \{1, 2, \dots, m\}$, $u = k + 1$, and $w = k + 2$. In any matrix, $A = [a_{ij}]$ with j th column $\mathbf{c}_j = [a_{1j} \cdots a_{nj}]^T$, fitting G , columns $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ form a linearly independent set. Since $\eta(G) = \alpha(G) = k$, it is a maximum linearly independent set of columns and therefore all remaining columns, $\mathbf{c}_{k+1}, \dots, \mathbf{c}_n$ are all linear combinations of columns $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$.

Based on the fact that vertex $k + 1$ is not adjacent to vertices $m + 1, \dots, k$, we have $a_{k+1, m+1} = a_{k+1, m+2} = \cdots = a_{k+1, k} = 0$. Since $N(k + 2) \cap J = \emptyset$, column \mathbf{c}_{k+2} is a linear combination of only those in $\{\mathbf{c}_{m+1}, \dots, \mathbf{c}_k\}$ and this forces $a_{k+1, k+2} = 0$. Repeating this with the rows forces $a_{k+2, k+1} = 0$. Therefore, any matrix fitting G and attaining rank $A = \alpha(G)$ must also fit H , so that $\eta(H) \leq \alpha(G)$. Thus, $\eta(H) = \alpha(G)$. This finishes the proof that $\eta(G) = \alpha(G)$ if and only if $\eta(H) = \alpha(G)$, and thus the entire proof. \square

Example 3.7. Consider the Petersen graph, P , shown in Figure 3.1. We have $\alpha(P) = 4$ and $\text{vcc}(P) = 5$. The set $I = \{1, 3, 9, 10\}$ forms a maximum independent set of vertices. Let $u = 2$. The set of vertices in I that are adjacent to u is $J = \{1, 3\}$. Now $2 \sim 7$ but $N(7) \cap J = \emptyset$. Let $H_1 = P - \{\{2, 7\}\}$. $\alpha(H_1) = 4$, so we are not yet finished. We repeat the same sort of process on H_1 .

We use the same independent set, I . Now, the subset of I that $u = 6$ is adjacent to is $J = \{1, 9\}$. $8 \sim 6$ but $N(8) \cap J = \emptyset$. So, we can delete the edge $\{6, 8\}$ and form a new graph, $H_2 = H_1 - \{\{6, 8\}\}$. Now $\alpha(H_2) = 5$ since the vertices $\{2, 5, 6, 7, 8\}$ form an independent

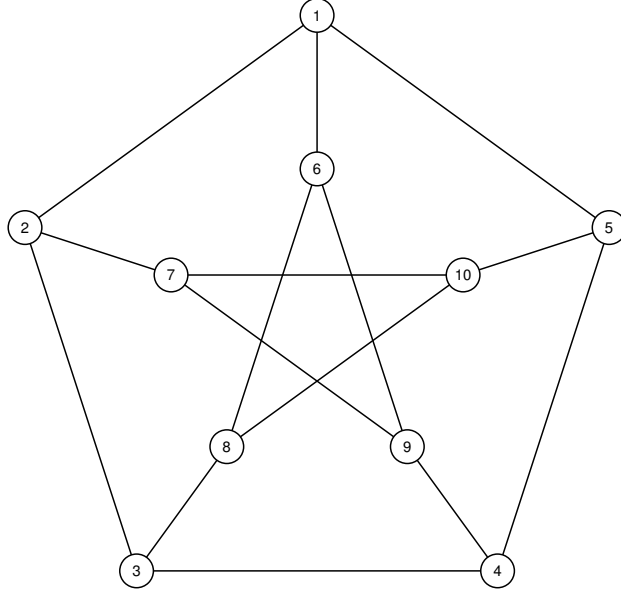


Figure 3.1 Petersen graph

set. Therefore, Theorem 3.6 gives $\eta(H_1) \geq 5$ and thus $\eta(P) \geq 5$. Since $\text{vcc}(P) = 5$, we have $\eta(P) = 5$.

Example 3.8. The complement of the Schlaefli graph, G , is the unique strongly regular graph with parameters $(27, 10, 1, 5)$. It is shown in Figure 3.2. In Example 1.21, we saw that $\eta(G) \leq 7$. Since $\alpha(G) = 6$ [11], we have that $\eta(G)$ is either 6 or 7. Using Theorems 3.2 and 3.6, we can show that $\eta(G) = 7$.

We start with Theorem 3.6. With vertices labeled as in Figure 3.2, it is clear that $I = \{1, 2, 3, 4, 5, 6\}$ is an independent set of vertices in G . Using the notation from Theorem 3.6, we let $u = 7$. Then $J = N(7) \cap I = \{2, 3\}$. We let $x = 17$ and we have $17 \sim 7$ and $N(17) \cap J = \emptyset$ since 17 is not adjacent to 2 or 3. Therefore, by Theorem 3.6, we can look at the graph H_1 , obtained from deleting the edge between 7 and 17, and if $\eta(H_1) \geq \alpha(G) + 1 = 7$, then $\eta(G) \geq 7$.

To see that $\eta(H_1) \geq 7$, we use Theorem 3.2. We can see that $I = \{4, 5, 6, 7, 8, 9\}$ is an independent set of vertices in G and therefore H_1 . Let $u = 17$. Then $N(17) \cap I = \{4\}$. Since, in H_1 , $N(17) = \{1, 4, 16, 20, 21, 22, 23, 25, 27\}$ and $N(4) = \{i\}_{i=10}^{19}$, we have $N(17) \cap N(4) = \{16\}$.

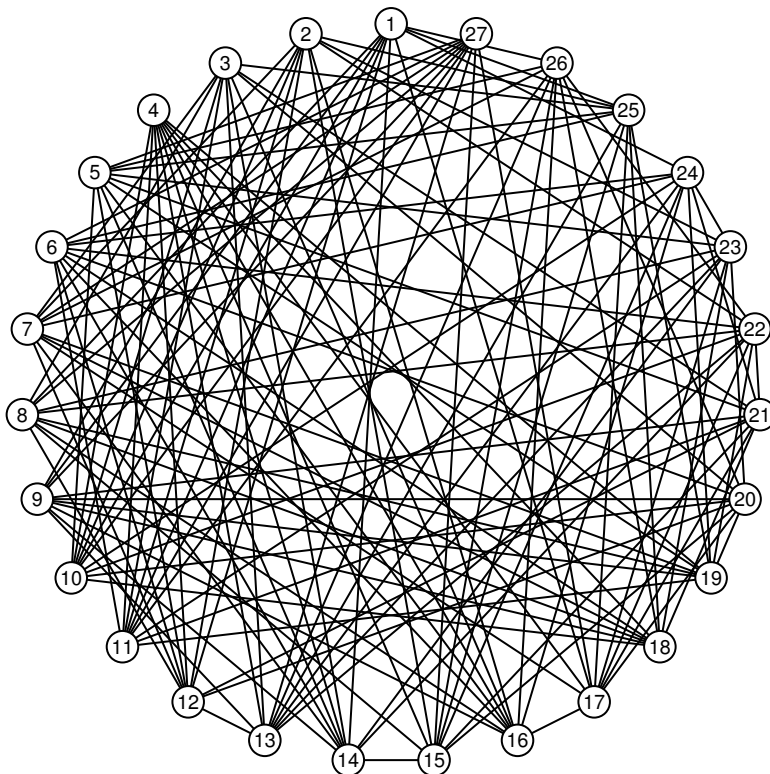


Figure 3.2 Complement of the Schlegel graph, the unique strongly regular graph with parameters $(27, 10, 1, 5)$

Thus, we form H_2 from H_1 by deleting $\{17, i\}$ for $i = 1, 4, 20, 21, 22, 23, 25, 27$ and $\{4, j\}$ for $j = 10, \dots, 15, 17, 18, 19$. In H_2 , there are many independent sets with 8 vertices, including $\{1, 2, 3, 4, 17, 18, 19, 27\}$. Thus, $\eta(H_2) \geq 7 = \alpha(G) + 1$ which implies $\eta(H_1) \geq 7$, so that $\eta(G) \geq 7$. So, $\eta(G) = 7$.

Theorems 3.2 and 3.6 are all we need to calculate η for most graphs of order 10 or less when $\alpha(G) < \text{vcc}(G)$. But, there are a few for which these theorems do not apply. The following example shows another technique which can be used to increase the lower bound from $\alpha(G)$ to $\alpha(G) + 1$.

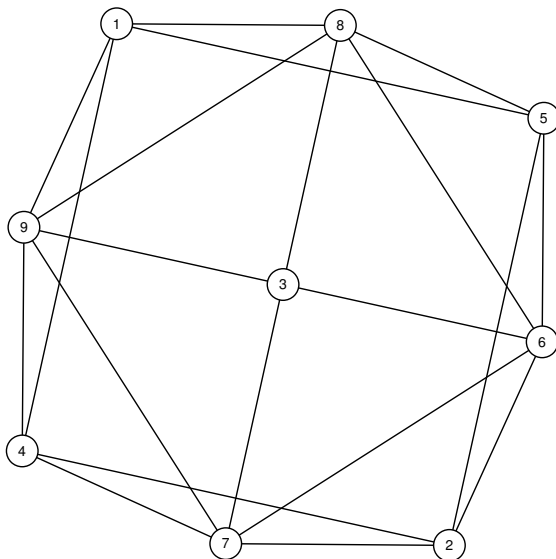


Figure 3.3 Graph with graph6 string “HEpjlYr”

Proposition 3.9. *Let G be the graph with graph6 string “HEpjlYr”, shown in Figure 3.3. Then $\alpha(G) = 3$, $\text{vcc}(G) = 4$, and $\eta(G) = 4$.*

Proof. We have $\text{vcc}(G) = 4$, because a set of cliques that cover the vertices of G is given by $\{\{1, 5, 8\}, \{2, 4, 7\}, \{3, 6\}, \{9\}\}$. Since the largest clique subgraph of G is a K_3 and since vertex 3 is not in any triangle, we could not possibly have $\text{vcc}(G) = 3$. We have $\alpha(G) = 3$, where $\{1, 2, 3\}$ is a maximum independent set. Thus, $\eta(G)$ is 3 or 4. Under the assumption that $\eta(G) = 3$, there exists a matrix A that fits G such that the rank is 3. Since columns 1, 2, and 3 form a linear independent set, all other columns can be written as a linear combination of these columns. Similarly, all rows must be linear combinations of rows 1, 2, and 3. The general form of a matrix fitting G is the first matrix below, labeled (1).

In particular, column 4 is a linear combination of columns 1 and 2, i.e., $s_1\mathbf{c}_1 + s_2\mathbf{c}_2 = \mathbf{c}_4$ for some constants s_1, s_2 . Now, looking in row 6, we have $s_1a_{6,1} + s_2a_{6,2} = a_{6,4}$. As $a_{6,1} = a_{6,4} = 0$, we have either that $s_2 = 0$ or $a_{6,2} = 0$.

We assume first that $a_{6,2} = 0$ and arrive at a contradiction. Since $a_{6,1} = a_{6,2} = 0$, we conclude that row 6 must be a multiple of row 3, which gives us matrix (2) above.

Since $a_{2,9} = a_{3,9} = 0$, we have that column 9 is a multiple of column 1, which leads to matrix (3).

As $a_{5,1} = a_{5,3} = 0$, we see that row 5 is a multiple of row 2. This gives matrix (4).

At this point, we see that column 4 must be a multiple of column 1, and also that column 7 is a multiple of column 3. Recall also that column 9 is a multiple of column 1, so any changes made to column 1 in this process should also be made to column 9. Therefore we have matrix (5).

And, now we see that $a_{8,1} = a_{8,2} = a_{8,3} = 0$, which means any matrix of this form must have rank at least 4. This contradicts our assumption that such a matrix has rank 3.

Therefore, if our graph has $\eta(G) = 3$, the matrix that fits G and realizes this rank must have $a_{6,2} \neq 0$, and column 4 must be a multiple of column 1. The exact same argument on the rows says $a_{2,6} \neq 0$ and row 4 is a multiple of row 1. Therefore, any matrix fitting G and realizing $\eta(G) = 3$ must be in the general form:

$$\begin{bmatrix} * & 0 & 0 & * & 0 & 0 & 0 & 0 & ? \\ 0 & * & 0 & 0 & ? & ? & ? & 0 & 0 \\ 0 & 0 & * & 0 & 0 & ? & ? & ? & ? \\ * & 0 & 0 & * & 0 & 0 & 0 & 0 & ? \\ 0 & ? & 0 & 0 & * & ? & 0 & ? & 0 \\ 0 & ? & ? & 0 & ? & * & ? & ? & 0 \\ 0 & ? & ? & 0 & 0 & ? & * & 0 & ? \\ 0 & 0 & ? & 0 & ? & ? & 0 & * & ? \\ ? & 0 & ? & ? & 0 & 0 & ? & ? & * \end{bmatrix}$$

Let H be the graph with $V(H) = V(G)$ and $E(H) = E(G) \setminus \{\{1, 5\}, \{1, 8\}, \{4, 2\}, \{4, 7\}\}$.

This matrix fits H . So, this implies $\eta(H) \leq 3$.

Now, the subgraph of H induced by the vertices $\{1, 2, 3, 4, 5, 7, 8\}$ is the disjoint union of a K_2 and a C_5 , where the K_2 is made from the vertices $\{1, 4\}$, and the C_5 from $\{2, 3, 5, 7, 8\}$. So $\eta(H) \geq \eta(K_2) + \eta(C_5) = 4$. This contradicts our previous implication, that $\eta(H) \leq 3$. Since this was implied by our assumption that $\eta(G) = 3$, we see that this is also not true. Therefore, $\eta(G) \geq 4$, so that $\eta(G) = 4$. \square

There are actually two other graphs of order 9 satisfying $\alpha(G) < \text{vcc}(G)$ for which Theorems 3.2 and 3.6 do not apply. Both satisfy $\alpha(G) = 3$ and $\eta(G) = \text{vcc}(G) = 4$, just as the graph in Proposition 3.9. The method used in Proposition 3.9 can be used to show $\eta(G) = 4$, but the work we have already done gives us a much simpler way.

Example 3.10. Let G be the graph with graph 6 string “HEpjlYr”, shown in Figure 3.3. In proposition 3.9, we saw $\alpha(G) = 3$, $\eta(G) = \text{vcc}(G) = 4$. Let $H_1 = G - \{\{6, 7\}\}$ and $H_2 = H_1 - \{\{8, 9\}\}$. We have $\text{vcc}(H_1) = \text{vcc}(H_2) = 4$. To see this, note that $4 = \text{vcc}(G) \leq \text{vcc}(H_1) \leq \text{vcc}(H_2)$ by Theorem 2.6. Since the set of cliques $\{\{1, 5, 8\}, \{2, 4, 7\}, \{3, 6\}, \{9\}\}$ cover the vertices of H_2 , we have $\text{vcc}(H_2) \leq 4$. Also, we have $\alpha(H_1) = \alpha(H_2) = 3$. But, by Proposition 2.2,

$$4 = \eta(G) \leq \eta(H_1) \leq \eta(H_2) \leq \text{vcc}(H_2) = 4,$$

so that $\eta(H_1) = \eta(H_2) = 4$.

The proof of the next theorem uses a computer program [41] written in *Sage* [40]. This program contains the techniques on increasing the lower bound in Theorem 3.2, Theorem 3.6, and the matrix technique of Proposition 3.9, among other things. We know $\alpha(G) \leq \eta(G) \leq \text{vcc}(G)$ for any graph G . For all graphs of order 10 or less, it turns out the gap is at most 1, i.e., $\text{vcc}(G)$ is $\alpha(G)$ or $\alpha(G) + 1$. For those with $\text{vcc}(G) = \alpha(G)$, of course $\eta(G) = \text{vcc}(G) = \alpha(G)$. For the graphs where $\text{vcc}(G) = \alpha(G) + 1$, the *Sage* program [41] succeeded in increasing the lower bound by 1 on all such graphs of order 10 or less. Thus, $\eta(G) = \text{vcc}(G)$ in all cases. The technique of Proposition 3.9 is needed for only 3 graphs of order 9 and 50 graphs of order 10.

The program [41] can optionally print out all the detail of what it does. This output is available for inspection [42] and has been checked by hand for all the graphs needing the technique of Proposition 3.9. For the 50 graphs of order 10, as with the 3 graphs of order 9, it

is only necessary to use the technique of Proposition 3.9 on a small number of them. That is, assume $\text{vcc}(G) = \text{vcc}(H)$, H is a spanning subgraph of G , and we determine $\eta(G) = \text{vcc}(G)$. Then, using the edge deletion bound of Proposition 2.2 we have $\text{vcc}(H) = \eta(G) \leq \eta(H) \leq \text{vcc}(H)$, which determines $\eta(H) = \text{vcc}(H)$.

Note that when there exist two vertices u and v with $N[u] \subseteq N[v]$, the program does delete vertex v . Theorem 2.9 says that $\eta(G) = \eta(G - v)$ in this case and Theorem 2.14 shows that $\text{vcc}(G) = \text{vcc}(G - v)$ in this case. Therefore, if the program deletes such a vertex v and determines that $\eta(G - v) = \text{vcc}(G - v)$, this proves $\eta(G) = \text{vcc}(G)$ for that graph.

Also, the program was run only on connected graphs, but Observation 1.31, which tells us that $\eta(G)$ and $\text{vcc}(G)$ sum over connected components, extends the result from connected graphs of order 10 or less to all graphs of order 10 or less.

Theorem 3.11. *For all graphs of order 10 or less, $\eta(G) = \text{vcc}(G)$.*

In fact, the program was run on all connected graphs of order 11 as well. The program was able to determine $\eta(G)$ for all but 213 of these, and in every case when it was determined, $\eta(G) = \text{vcc}(G)$. The cut-vertex reduction technique in Theorem 2.18 determines $\eta(G) = \text{vcc}(G)$ for one more graph of order 11, see Example 2.20, so that $\eta(G) = \text{vcc}(G)$ is known to be true for all but 212 graphs of order 11. Of these there are 44 satisfying $\text{vcc}(G) = \alpha(G) + 1$ and 168 satisfying $\text{vcc}(G) = \alpha(G) + 2$. In all of them, it is known that $\eta(G)$ is either $\text{vcc}(G)$ or $\text{vcc}(G) - 1$. And, in all cases, it is known that the upper bound of $\text{vcc}(G)$ is at least as good as the bound given by $|G|$ minus the highest multiplicity of a nonzero eigenvalue of Theorem 1.20.

CHAPTER 4. Extreme Values of $\eta(G)$

Proposition 4.1. *Assume G is a graph with order $n \geq 1$. Then the following are equivalent:*

1. $\eta(G) = 1$
2. $\text{vcc}(G) = 1$
3. $\alpha(G) = 1$
4. $G = K_n$

Proof. It is clear that (4) implies (2). Since $1 \leq \alpha(G) \leq \eta(G) \leq \text{vcc}(G)$, we have that (2) implies (1), which implies (3). If the largest independent set of vertices is of size 1, any two vertices are connected, so (3) implies (4). □

That (1) and (2) are equivalent in the following proposition was mentioned in [35], noting that $\eta(\overline{C_n}) = 3$ for odd $n \geq 3$.

Proposition 4.2. *Assume G is a graph with order $n \geq 1$. Then the following are equivalent:*

1. $\eta(G) = 2$
2. $\text{vcc}(G) = 2$
3. $\alpha(G) = 2$ and G is a perfect graph

Proof. Assume $\eta(G) = 2$. Note, $\eta(C_n) \geq 3$ for any odd $n \geq 5$, by Theorem 3.3, and $\eta(\overline{C_n}) = 3$ for any odd $n \geq 5$, by Theorem 3.1. By Theorem 2.5, if G contained an induced C_n , for odd $n \geq 5$, or $\overline{C_n}$ for $n \geq 5$, then $\eta(G) \geq 3$. Since $\eta(G) = 2$, G does not. Therefore, by the Strong

Perfect Graph Theorem, G is perfect. This implies $\alpha(G) = \text{vcc}(G) = 2$. So, (1) implies both (2) and (3).

If $\text{vcc}(G) = 2$, then G is not a complete graph by Proposition 4.1, so $\eta(G) \geq 2$. Since $\eta(G) \leq \text{vcc}(G)$, $\eta(G) = 2$. Thus, (2) implies (1).

If G is perfect and $\alpha(G) = 2$, then $2 = \alpha(G) = \eta(G) = \text{vcc}(G)$, so (3) implies (1). \square

Proposition 4.3. *Assume G is a graph with order $n \geq 1$. Then $\text{vcc}(G) = 3$ implies $\eta(G) = 3$.*

Proof. If $\text{vcc}(G) = 3$, then $\eta(G) \leq 3$. But, by Propositions 4.1 and 4.2, if $\eta(G)$ were 1 or 2, then $\text{vcc}(G)$ would be equal to $\eta(G)$, and thus would not be 3. So, $\eta(G) = 3$. \square

Example 4.4. If $G = C_5$ or $G = \overline{C_n}$ for odd $n \geq 5$, then $\alpha(G) = 2$ and $\eta(G) = \text{vcc}(G) = 3$. Thus, we do not have $\eta(G) = 2$ if and only if $\alpha(G) = 2$, i.e., the condition that G is perfect in item (3) of Proposition 4.2 is necessary. This example also shows we do not have $\eta(G) = 3$ if and only if $\alpha(G) = 3$.

Example 4.5. Let G be the graph from Example 1.29. Then $\eta(G) = 3$ and $\text{vcc}(G) = 4$, which shows we do not have $\eta(G) = 3$ if and only if $\text{vcc}(G) = 3$.

Proposition 4.6. *Assume G is a graph with order $n \geq 1$. Then the following are equivalent:*

1. $\eta(G) = n$
2. $\text{vcc}(G) = n$
3. $\alpha(G) = n$
4. G is the empty graph.

Proof. (4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (4) are all clear.

\square

Proposition 4.7. *Assume G is a graph with order $n \geq 2$. Then the following are equivalent:*

1. $\eta(G) = n - 1$
2. $\text{vcc}(G) = n - 1$

3. $\alpha(G) = n - 1$

4. G is the disjoint union of k isolated vertices and a $K_{1,n-(k+1)}$, where $n \geq k + 2$.

Proof. It is clear that (4) implies (3). Since $\alpha(G) \leq \eta(G) \leq \text{vcc}(G)$, by Proposition 4.6 it is clear that (3) implies (1) and (1) implies (2). So, we need only show that (2) implies (4).

Assume $\text{vcc}(G) = n - 1$. If $n = 2$, then $\text{vcc}(G) = 1$, which means $G = K_2$ by Proposition 4.1. Since $K_2 = K_{1,1}$, we see that when $n = 2$, (4) implies (2). So, assume $n \geq 3$.

The only way to cover the vertices of G by $n - 1$ disjoint cliques is with a K_2 and $(n - 2)$ K_1 's. Let u_1 and u_2 denote the vertices in the K_2 and let $\{v_i\}_{i=1}^{n-2}$ denote the $n - 2$ vertices in the K_1 's. We have $v_i \not\sim v_j$ for all $1 \leq i, j \leq n - 2$, for otherwise we would have another K_2 . Thus, $\bigcup_i N(v_i) \subseteq \{u_1, u_2\}$.

If any v_i were adjacent to u_1 and u_2 , we would have a K_3 , so that is not possible. If there exist distinct v_i, v_j such that $v_i \sim u_1$ and $v_j \sim u_2$, we would have a two K_2 's, so that is not possible. Thus, $|\bigcup_i N(v_i)| = 1$. Without loss of generality, assume $\bigcup_i N(v_i) = \{u_1\}$. Thus, every v_i is either isolated or adjacent to u_1 . This gives (4). \square

Theorem 4.8. *Assume G is a connected graph with order $n \geq 3$ such that $\text{vcc}(G) = n - 2$. Then either $G = C_5$ or G is a spanning subgraph of $K_{1,1,n-2}$ but not a spanning subgraph of $K_{1,n-1}$.*

Proof. First, if $|G| = 3$, then $\text{vcc}(G) = 1$ implies $G = K_3 = K_{1,1,1}$ by Theorem 4.1. If $|G| = 4$, $G \neq K_4$. But, any proper spanning subgraph of K_4 will be a subgraph of $K_{1,1,2}$. And, if G were the spanning subgraph of $K_{1,3}$, we would have $\text{vcc}(G) \geq 3$ by Proposition 4.7. So the result holds for $|G| = 4$. Therefore, assume $|G| \geq 5$. Since $\text{vcc}(G) = |G| - 2$, we have a very limited choice of cliques with which to cover the vertices of G . The only possibilities are a K_3 and $(|G| - 3)$ K_1 's, or two K_2 's and $(|G| - 4)$ K_1 's. Note, some graphs with $\text{vcc}(G) = |G| - 2$ have vertex clique covers of both kinds, e.g., $K_{1,1,|G|-2}$, so the cases do overlap. But, more importantly, all graphs with $\text{vcc}(G) = |G| - 2$ fall into at least one of the two cases.

Case 1: Assume we have a graph G with $\text{vcc}(G) = |G| - 2$ and a specific vertex clique cover made of one K_3 and $(|G| - 3)$ K_1 's. Label the single vertices $v_1, \dots, v_{|G|-3}$ and the 3 vertices

of K_3 as t_1, t_2, t_3 . First, note that for all $1 \leq i, j \leq |G| - 3$, $v_i \not\sim v_j$ because, if $v_i \sim v_j$, then we would be able to cover the vertices of G with $|G| - 3$ cliques. Therefore, $\bigcup_i N(v_i) \subseteq \{t_1, t_2, t_3\}$.

We will show $\bigcup_i N(v_i) = \{t_1, t_2, t_3\}$ is not possible. Assume $v_i \sim t_1$, $v_j \sim t_2$, and $v_k \sim t_3$. If $i = j = k$, then $G[\{v_i, t_1, t_2, t_3\}]$ would be a K_4 . If $i = j \neq k$, then $G[\{v_i, t_1, t_2\}]$ would be a K_3 and $G[\{v_k, t_3\}]$ would be a K_2 , so this is not possible. If i, j, k are all distinct, then we would have 3 K_2 's, $G[\{v_i, t_1\}]$, $G[\{v_j, t_2\}]$, and $G[\{v_k, t_3\}]$, so this is not possible.

Therefore, $|\bigcup_i N(v_i)| \leq 2$, and whether it is 1 or 2, it is clear that G is a spanning subgraph of $K_{1,1,|G|-2}$. Since G contains a triangle, G is not a spanning subgraph of $K_{1,|G|-1}$.

Case 2: Assume we have a graph G with $\text{vcc}(G) = |G| - 2$ and a specific vertex clique cover made of two K_2 's and $(|G| - 4)$ K_1 's. Label the vertices of the K_1 's with $v_1, \dots, v_{|G|-4}$, the two vertices of one K_2 with a_1, a_2 , and the two vertices of the other K_2 by b_1, b_2 . Since $|G| \geq 5$, we have at least one v_i , namely v_1 . And, since G is connected, we have a P_4 subgraph in any case, so that G is not a subgraph of $K_{1,n-1}$. As in the previous case, for $1 \leq i, j \leq |G| - 4$, $v_i \not\sim v_j$ as that would lead to three disjoint K_2 's and $\text{vcc}(G) \leq |G| - 3$.

If $\{a_1, a_2\} \subseteq \bigcup_i N(v_i)$, then either there exists one v_i that is adjacent to both a_1 and a_2 , or there exist distinct v_i, v_j such that $v_i \sim a_1$ and $v_j \sim a_2$. In the first case, we could replace the K_2 on a_1 and a_2 by a K_3 . In the second case, we could replace the K_2 on a_1 and a_2 with two K_2 's, on v_i, a_1 and v_j, a_2 . Either way, this would lead to $\text{vcc}(G) \leq |G| - 3$. Thus, at most one of $\{a_1, a_2\}$ can be in $\bigcup_i N(v_i)$, and similarly at most one of $\{b_1, b_2\}$. Without loss of generality, let us assume $\bigcup_i N(v_i) \subseteq \{a_1, b_1\}$. We finish the proof with 3 subcases.

Subcase i: Consider the subcase where there exists v_i such that v_i is adjacent to both a_1 and b_1 . Without loss of generality, let it be v_1 . If $a_2 \sim b_2$, we have a 5-cycle subgraph using the vertices a_1, a_2, b_2, b_1, v_1 . If any other edge is present between these vertices, then we would have a K_3 and a K_2 as subgraphs, so this is not possible. And, if there exists v_2 , since the graph is connected, it would need to be adjacent to at least one of the vertices in the 5-cycle. Therefore, we would have three K_2 's, which is also not possible. Thus, if $a_2 \sim b_2$, we must have $G = C_5$. So, assume $a_2 \not\sim b_2$. Even if all three edges $(a_1, b_1), (a_1, b_2), (a_2, b_1)$ are present, the graph is still a spanning subgraph of $K_{1,1,n-2}$. For example, if all v_i are adjacent to both a_1 and b_1 , then $G = K_{1,1,n-2}$.

Subcase ii: Consider the case where there exists at least one i such that $v_i \sim a_1$ and $v_i \not\sim b_1$ and there exists at least one j such that $v_j \not\sim a_1$ and $v_j \sim b_1$, but no k such that v_k is adjacent to both a_1 and b_1 . Without loss of generality, assume $v_1 \sim a_1$ and $v_2 \sim b_1$. We can not have $a_2 \sim b_2$ or we would have a P_6 subgraph and thus three disjoint K_2 subgraphs. If $a_1 \sim b_1$, $a_1 \sim b_2$, and $a_2 \sim b_1$, the resulting graph would be a spanning subgraph of $K_{1,1,n-2}$, so any graph in this subcase would be as well.

Subcase iii: Lastly, consider the subcase where $|\bigcup_i N(v_i)| = 1$. Without loss of generality, assume $\bigcup_i N(v_i) = \{a_1\}$. Note, a_2 can not be adjacent to both b_1 and b_2 or we would have a K_3 and a K_2 . If $a_1 \sim b_1$, $a_1 \sim b_2$, $a_2 \sim b_1$, but $a_2 \not\sim b_2$, the graph is a spanning subgraph of $K_{1,1,n-2}$. Similarly, if $a_1 \sim b_1$, $a_1 \sim b_2$, $a_2 \sim b_2$, but $a_2 \not\sim b_1$, the graph is also a spanning subgraph of $K_{1,1,n-2}$. Thus, this subcase is finished.

This finishes the proof of Case 2 and thus the theorem. \square

Theorem 4.9. *Assume G is a connected graph with order $n \geq 3$. Then the following are equivalent:*

1. $\eta(G) = n - 2$
2. $\text{vcc}(G) = n - 2$
3. *Either $G = C_5$ or G is a spanning subgraph of $K_{1,1,n-2}$ but not a spanning subgraph of $K_{1,n-1}$.*

Proof. (1) implies (2) by Proposition 4.7 since $\eta(G) \leq \text{vcc}(G)$. (2) implies (3) is exactly Theorem 4.8. Thus, we are left to prove (3) implies (1).

If $G = C_5$, then Theorem 3.3 gives that $\eta(G) = 3 = 5 - 2$. So assume G is a spanning subgraph of $K_{1,1,n-2}$ but not a spanning subgraph of $K_{1,n-1}$. Corollary 2.28 says that $\eta(K_{1,1,n-2}) = n - 2$ and Remark 2.1 says if G is a spanning subgraph of $K_{1,1,n-2}$, then $\eta(G) \geq n - 2$. Theorem 4.7 says $\eta(G) = n - 1$ if and only if G is the disjoint union of k isolated vertices and a $K_{1,n-(k+1)}$. Any such graph would be a spanning subgraph of $K_{1,n-1}$. Since G is not a spanning subgraph of $K_{1,n-1}$, we have $\eta(G) \leq n - 2$. Thus, $\eta(G) = n - 2$, which finishes the proof. \square

Observation 4.10. *Assume G is a connected graph with order $n \geq 4$. Then $\eta(G) = n - 3$ implies $\text{vcc}(G) = n - 3$.*

Proposition 4.11. *Let E denote a subset of the edges of a K_n with $1 \leq |E| \leq 4$. Let $G = K_n - E$. Then $\alpha(G) = \eta(G) = \text{vcc}(G)$. If the subgraph of G induced by E contains a triangle, then $\eta(G) = 3$, otherwise $\eta(G) = 2$.*

Proof. First, a C_5 is self-complementary so it would take at least $|E| = 5$ for G to contain an induced C_n or $\overline{C_n}$, $n \geq 5$. Since $|E| \leq 4$, G is perfect, and thus $\alpha(G) = \eta(G) = \text{vcc}(G) = \chi(\overline{G})$.

\overline{G} consists of the subgraph of K_n induced by the edges in E , as well as some isolated vertices. Since \overline{G} has at most 4 edges, 3 colors will be required for a proper coloring if and only if \overline{G} contains a triangle. On the other hand, since \overline{G} contains at least 1 edge, it will take at least 2 colors in any case. □

CHAPTER 5. Determination of η for various graphs and families of graphs

Proposition 5.1. *Let G be a graph constructed by starting with a cycle on $n \geq 4$ vertices, C_n , and adding one extra edge. Then,*

$$\eta(G) = \text{vcc}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd and the extra edge forms a triangle} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and the extra edge does not form a triangle} \end{cases}$$

Proof. Label the vertices of the cycle in order $1, \dots, n$. Without loss of generality, the extra edge is $\{1, j\}$, where $3 \leq j \leq \frac{n+1}{2}$.

If n is even, then $I = \{2, 4, \dots, n\}$ is an independent set of vertices. Thus, $\alpha(G) \geq \frac{n}{2}$. If $j \neq 3$, then there are no triangles, so $\text{vcc}(G) = \frac{n}{2}$. If $j = 3$, then the cliques induced by the sets of vertices $\{1, 2, 3\}, \{4, 5\}, \dots, \{n-2, n-1\}, \{n\}$ contains $\frac{n}{2}$ cliques, so that $\text{vcc}(G) \leq \frac{n}{2}$. So, in either case, we have $\alpha(G) = \eta(G) = \text{vcc}(G) = \frac{n}{2}$.

So, assume n is odd. Since $\alpha(C_n) = \frac{n-1}{2}$ and we added an extra edge, $\alpha(G) \leq \frac{n-1}{2}$. But, $I = \{2, 4, \dots, n-1\}$ is an independent set of vertices, so $\alpha(G) = \frac{n-1}{2}$. If $j = 3$, then the cliques $\{1, 2, 3\}, \{4, 5\}, \dots, \{n-1, n\}$ cover G , so $\text{vcc}(G) = \frac{n-1}{2}$. Thus, in this case, $\eta(G) = \text{vcc}(G)$. So, assume $j \neq 3$.

Using Theorem 3.2, we have $u = 1$ is not in I and $N(1) \cap I = \{2\}$ contains only one vertex. Let $w = j$. Then $j \sim 1$ but $j \not\sim 2$. So, we let $H = G - \{1, j\} = C_n$. We have $\eta(C_n) = \alpha(C_n) + 1 = \alpha(G) + 1$, so we can conclude that $\eta(G) \geq \alpha(G) + 1 = \frac{n+1}{2}$. Since there are no triangles, $\text{vcc}(G) = \frac{n+1}{2}$, which forces $\eta(G) = \text{vcc}(G)$. \square

Definition 5.2. A **cactus graph** is a connected graph such that every block is an edge or a cycle. Here, we call a connected graph a **generalized cactus graph** if every block is an edge or a cycle or a cycle with an extra edge, as in Proposition 5.1.

Theorem 5.3. *Let G be a generalized cactus graph. Then $\eta(G) = \text{vcc}(G)$.*

Proof. The proof will be by induction on $|V(G)|$. The base case will be on K_1 , in which case $\eta(K_1) = \text{vcc}(K_1) = 1$. Assume for any generalized cactus graph with $|V(G)| \leq k$, we have $\eta(G) = \text{vcc}(G)$. Now, consider a generalized cactus graph with $|V(G)| = k + 1$.

If G happens to be just a K_2 , a cycle, or a cycle with an edge, then we have already shown $\eta(G) = \text{vcc}(G)$ in Theorem 3.3 and Proposition 5.1. So, assume G is not an edge, a cycle, or a cycle plus an edge. Then, G has at least one cut-vertex, v . Let $\{G_i\}_{i=1}^k$ denote the branches of G at v . From Theorems 2.18 and 2.22, we have

$$\begin{aligned} r_v^\eta(G) &= \min_i r_v^\eta(G_i) \\ r_v^{\text{vcc}}(G) &= \min_i r_v^{\text{vcc}}(G_i) \end{aligned}$$

Since, for each i , $G_i - v$ and G_i are connected subgraphs of a generalized cactus, we have that $G_i - v$ and G_i are themselves generalized cacti, and $|V(G_i - v)|, |V(G_i)| \leq k$. So, by the induction hypothesis, we have $\eta(G_i) = \text{vcc}(G_i)$ and $\eta(G_i - v) = \text{vcc}(G_i - v)$. Therefore, $r_v^\eta(G_i) = r_v^{\text{vcc}}(G_i)$ for each i , and thus $r_v^\eta(G) = r_v^{\text{vcc}}(G)$. Now, this is equivalent to $\eta(G) - \eta(G - v) = \text{vcc}(G) - \text{vcc}(G - v)$. Since $G - v$ is a generalized cactus with $|V(G - v)| = k$, by the induction hypothesis, $\eta(G - v) = \text{vcc}(G - v)$. Therefore, we have $\eta(G) = \text{vcc}(G)$. \square

For $n \geq 3$, the **Moebius ladder** on $2n$ vertices, M_{2n} , is formed by starting with a cycle on $2n$ vertices and adding all edges of the form $(k, n + k)$ for $1 \leq k \leq n$.

Theorem 5.4. $\eta(M_{2n}) = \text{vcc}(M_{2n}) = n$

Proof. If n is odd, the graph is bipartite with partite sets the odd vertices and the even vertices. Therefore, M_{2n} is perfect when n is odd. Since M_{2n} contains no triangles, $\text{vcc}(M_{2n}) = n$ and thus $\eta(M_{2n}) = n$.

If n is even, we still have $\text{vcc}(M_{2n}) = n$. However, $\alpha(M_{2n}) = n - 1$. To see this, note that $\alpha(C_{2n}) = n$ and the only maximum independent sets correspond to all odd vertices or all even vertices. When n is even, these sets are not independent in M_{2n} because, for example, $1 \sim n + 1$ and $2 \sim n + 2$. Thus, $\alpha(M_{2n}) \leq n - 1$. On the other hand, the set

$\{1, 3, 5, \dots, n-1, n+2, n+4, \dots, 2n-2\}$ is independent. Since the order of this set is $n-1$, $\alpha(M_{2n}) = n-1$.

Thus, we know $n-1 \leq \eta(M_{2n}) \leq n$. Assume $\eta(M_{2n}) = n-1$. Then, any maximum independent set of vertices leads to a maximum linearly independent set of columns in any matrix fitting M_{2n} . In particular, columns $\{1, 3, 5, \dots, n-1, n+2, n+4, \dots, 2n-2\}$ are linearly independent, and all other columns must be linear combinations of these. Since the neighborhood of vertex $2n$ is $\{1, n, 2n-1\}$, we must have that column $2n$ is a multiple of column 1. Since the neighborhood of vertex 1 is $\{2, n+1, 2n\}$, this requires that both columns have nonzero entries in rows 1 and $2n$ and 0 entries in all other spots. Similarly, we can do the same with rows.

This says, if $\eta(M_{2n}) = n-1$, we can find a matrix of this form that fits M_{2n} and has rank $n-1$. But, this matrix fits the graph created from M_{2n} by deleting edges $(1, 2)$, $(1, n+1)$, $(2n, n)$, and $(2n, 2n-1)$. And, in this graph, the vertices $\{1, 2, 4, 6, \dots, n, n+3, n+5, \dots, 2n-1\}$ are independent and this set is size n . This implies the minimum rank of any matrix in this form is at least n . This contradicts the assumption that $\eta(M_{2n}) = n-1$. Therefore, $\eta(M_{2n}) = n$. \square

CHAPTER 6. Relationship of η to Other Minimum Rank Problems

Let \mathcal{S}_n denote the set of $n \times n$ symmetric matrices over \mathbb{R} . For any $A = [a_{ij}] \in \mathcal{S}_n$, the **graph** of A , denoted $G(A)$, is the graph having vertex set $\{1, \dots, n\}$ and edges $\{(i, j) \mid a_{ij} \neq 0 \text{ and } i \neq j\}$. For a fixed graph, G , we let $\mathcal{S}(G) = \{A \in \mathcal{S}_n \mid G(A) = G\}$, the set of **symmetric matrices associated with G** . Finally, we define the **minimum rank** of G to be

$$\text{mr}(G) = \min\{\text{rank } A \mid A \in \mathcal{S}(G)\}.$$

Note, in this definition, diagonal entries are always free, i.e., allowed to be 0 or nonzero. And, if $uv \in E(G)$, then any matrix $A = [a_{ij}]$ in $\mathcal{S}(G)$ must satisfy $a_{uv} = a_{vu} \neq 0$. We have $0 \leq \text{mr}(G) \leq |G| - 1$ for any graph G , and if G contains at least one edge then $1 \leq \text{mr}(G) \leq |G| - 1$.

We also define $\mathcal{S}_+(G)$ to be the subset of $\mathcal{S}(G)$ consisting of all real positive semidefinite matrices. Then, for any graph G , we define the **minimum positive semidefinite rank** of G to be

$$\text{mr}_+(G) = \min\{\text{rank } A \mid A \in \mathcal{S}_+(G)\}.$$

We have $0 \leq \text{mr}_+(G) \leq |G| - 1$ for any graph G , and if G contains at least one edge then $1 \leq \text{mr}_+(G) \leq |G| - 1$.

Since $\text{mr}(G)$ is the minimum over a larger set of matrices than $\text{mr}_+(G)$, for any graph G we have

$$\text{mr}(G) \leq \text{mr}_+(G).$$

There are many other variations, such as finding the minimum rank over fields other than \mathbb{R} , including over the field \mathbb{C} in the case of positive semidefinite minimum rank, finding the minimum rank over skew-symmetric matrices, or allowing directed graphs and/or loops. There has been quite a lot of interest in these parameters recently. For a survey of the known results

on $\text{mr}(G)$, see [22] or [23]. The latter of those two, as well as [36] or [8] contain a lot of information about $\text{mr}_+(G)$.

6.1 Bounds Related to η or Related to Bounds for η

If G has no isolated vertices, all diagonal entries of any matrix in $\mathcal{S}_+(G)$ must be nonzero since all principal submatrices of a positive semidefinite matrix are also positive semidefinite. So, if G has no isolated vertices, any independent set of vertices leads to a diagonal submatrix with nonzero diagonals, and any matrix in $\mathcal{S}_+(G)$ fits G . This leads to the next two observations.

Observation 6.1. [8] *For any graph G without isolated vertices, $\alpha(G) \leq \text{mr}_+(G)$.*

Observation 6.2. *For any graph G without isolated vertices, $\eta(G) \leq \text{mr}_+(G)$.*

The empty graph, E_n , which satisfies $\alpha(E_n) = \eta(E_n) = n$ and $\text{mr}_+(G) = 0$ shows these bound do not necessarily hold for graphs with isolated vertices.

Now, $\text{vcc}(G)$ is not an upper bound for $\text{mr}_+(G)$ or even for $\text{mr}(G)$, as a path shows. But, there is a similar bound that works. Let $\text{cc}(G)$ denote the **edge clique cover number** of G , that is, the number of clique subgraphs (not necessarily disjoint) needed to cover all the edges of G . If a graph has no isolated vertices, such a clique cover will cover all the vertices as well. So, if G has no isolated vertices, we have $\text{vcc}(G) \leq \text{cc}(G)$. The gap between these two can be arbitrarily large, as long as $|G|$ is large enough. For example, for a path, P_n , we have $\text{vcc}(P_n) = \lceil \frac{n}{2} \rceil$ and $\text{cc}(P_n) = n - 1$.

Proposition 6.3. [22, Observation 3.14] *Let G be any graph without isolated vertices. Then*

$$\text{mr}_+(G) \leq \text{cc}(G)$$

There is also a bound based on the highest multiplicity of an eigenvalue, similar to the bound for $\eta(G)$ in Theorem 1.20.

Observation 6.4. [22] *Let G be a graph. For a given matrix $A \in \mathcal{S}(G)$, let m be the maximum multiplicity of any eigenvalue of A . Then*

$$\text{mr}(G) \leq |G| - m.$$

6.2 Zero Forcing

For the standard minimum rank, $\text{mr}(G)$, the concept of zero forcing was used to establish a new lower bound. A similar concept can be defined for other minimum rank parameters as well. We start by coloring some subset, Z , of the vertices of G black, and all those vertices not in Z are colored white. This is called a **coloring of G** . We then apply a color change rule, which depends on which minimum rank parameter we are dealing with, repeatedly until no more changes can be made. The **derived set** or **final coloring** is the set of vertices that are colored black at the end of this process. A **zero forcing set** for G is a subset of vertices Z such that if we start the process with exactly the vertices of Z colored black, then the final coloring is equal to $V(G)$, i.e., all vertices are colored black. The minimum of $|Z|$ over all zero forcing sets Z is called the **zero forcing number** for the graph G .

Color change rule for zero forcing associated with mr : At each step of the process, we change the color of a vertex u from white to black exactly when there exists a black vertex v such that the only white neighbor of v is u . In this case, we denote the zero forcing number by $Z(G)$. Of course, we have the trivial bounds $1 \leq Z(G) \leq n$ for any graph of order n .

There is an explanation for this color change rule that deals with a general matrix in $\mathcal{S}(G)$ and a null vector of that general matrix. That is, assume we know $A \in \mathcal{S}(G)$ but that is all we know. And, assume we have a vector \mathbf{x} such that $A\mathbf{x} = 0$. If we initially set some entries of \mathbf{x} to zero, some other entries of \mathbf{x} may be forced zero based on the fact that some entries of A must be zero and others must be nonzero. In the color change rule, a black vertex represents a zero entry of \mathbf{x} and forcing a white vertex to black means that based on the current configuration of zero entries in \mathbf{x} , some others must be zero as well for $A\mathbf{x} = 0$ to hold. We give an example to make this clearer.

Example 6.5. Let G be the graph in Figure 6.1. Recall, when writing a general matrix for $\mathcal{S}(G)$, * means the entry must be nonzero, 0 means the entry must be 0, and ? means the entry

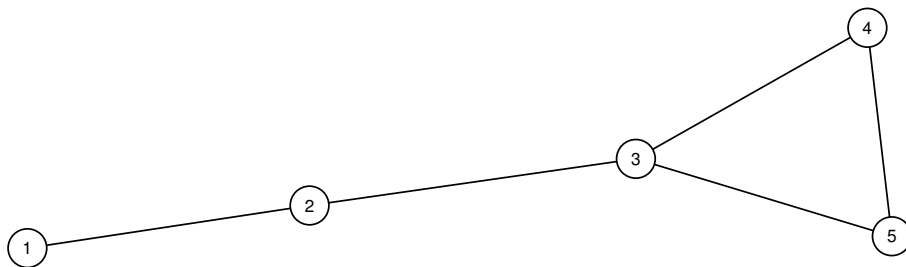


Figure 6.1 Graph for Example 6.5

is free to be zero or nonzero. Any matrix in $\mathcal{S}(G)$ for this graph is of the form

$$\begin{bmatrix} ? & * & 0 & 0 & 0 \\ * & ? & * & 0 & 0 \\ 0 & * & ? & * & * \\ 0 & 0 & * & ? & * \\ 0 & 0 & * & * & ? \end{bmatrix}$$

So, assume $A\mathbf{x} = 0$ where $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4 \ x_5]^T$. Further, assume $x_1 = 0$. Then, the first entry in the product $A\mathbf{x}$ is $0 \cdot ? + x_2 \cdot *$. This entry must be 0, but since $*$ indicates a nonzero entry, x_2 is thus forced to be 0 as well. At this point, we have $x_1 = x_2 = 0$. We look at the second entry of $A\mathbf{x}$, which is now $0 \cdot * + 0 \cdot ? + x_3 \cdot *$. Again, since $*$ is nonzero, this forces $x_3 = 0$.

At this point, even with $x_1 = x_2 = x_3 = 0$, we can not force x_4 or x_5 to be 0. For example, if we look at the third entry of $A\mathbf{x}$, we have $0 \cdot ? + x_4 \cdot * + x_5 \cdot *$. Since this could be 0 without x_4 or x_5 being 0, nothing is forced. Looking in the fourth and fifth entries of $A\mathbf{x}$ will have a similar result.

Looking back at the graph and thinking of the color change rule, this is analogous to starting with vertex 1 colored black. It has only one white neighbor, 2, so this is forced black. Now, vertex 2 is black and has only one white neighbor, 3, so this is forced black. At this point, 3

has two white neighbors so it doesn't force anything further.

Example 6.6. Consider the path on n vertices, P_n , labeled $\{1, \dots, n\}$ in order, so that one endpoint is labeled 1 and the other is labeled n . If we start with the vertices of $Z_1 = \{2\}$ colored black, then the neighborhood of the only black vertex is $N(2) = \{1, 3\}$, consisting of two vertices that are colored white. So, the derived set is simply $\{2\}$ and Z_1 is not a zero forcing set.

Let $Z_2 = \{2, 3\}$. At step 1 of the process, the vertex 2 contains exactly one white neighbor, 1, so we color 1 black. Similarly, vertex 3 contains exactly one white neighbor, 4, so we color 4 black. Repeating the process, we color 5, then 6, and so on, until finally we color vertex n . So, Z_2 is a zero forcing set and $Z(G) \leq 2$.

Let $Z_3 = \{1\}$. Since 1 has one white neighbor, namely 2, in the first step we color 2 black. We then color 3, and then 4, and so on, until all the vertices are colored black. Therefore, Z_3 is a zero forcing set and $Z(G) = 1$.

Color change rule for zero forcing associated with mr_+ : At each step of the process, let B denote the vertices that are currently colored black. Let W_1, \dots, W_k denote the connected components of $G - B$. Of course, it is possible that $k = 1$. Consider the graphs $G_i = G[W_i \cup B]$. If $u \in B$ and w is the only white neighbor of u in G_i , then color w black. In this case, we denote the positive semidefinite zero forcing number by $Z_+(G)$. Again, we have $1 \leq Z_+(G) \leq n$

Note, if during some step of the process $G - B$ consists of only one connected component, then the color change rule during that step is the same in both the standard zero forcing and positive semidefinite zero forcing. Therefore, any zero forcing set is a positive semidefinite zero forcing set, so that $Z_+(G) \leq Z(G)$.

Just as in the standard minimum rank case, this color change rule follows from $A\mathbf{x} = 0$ for a general $A \in \mathcal{S}_+(G)$. However, here the forcing is stronger because of the column inclusion property of positive semidefinite matrices.

If A is an $n \times n$ matrix and α is a subset of $\{1, \dots, n\}$, we denote by $A[\alpha]$ the principal submatrix of A lying in rows and columns α . Also, we denote by $A[\alpha, \beta]$ the submatrix lying in rows α and columns β . We say a matrix A satisfies the **column inclusion property** if

$A[\alpha, \{j\}]$ lies in the column space of $A[\alpha]$ for each $j = 1, \dots, n$ and for each $\alpha \subseteq \{1, \dots, n\}$. It is well known that positive semidefinite matrices satisfy the column inclusion property and this leads to the stronger zero forcing [2]. We illustrate the idea with an example.

Example 6.7. Consider P_5 , the path on 5 vertices. We could start with $x_1 = 0$, and would eventually force $x_2 = x_3 = x_4 = x_5 = 0$, just as in regular zero forcing. However, to illustrate the stronger zero forcing in the positive semidefinite case, let us start with $x_2 = 0$. With regular zero forcing, nothing would be forced. The first entry of $A\mathbf{x}$ is simply $x_1 \cdot a_{1,1}$ since $x_2 = a_{1,3} = a_{1,4} = a_{1,5} = 0$. The column inclusion principle says there exists \mathbf{y} such that $A[\{1\}]\mathbf{y} = A[\{1\}, \{2\}]$. Taking the transpose of both sides and multiplying by $\begin{bmatrix} x_1 \end{bmatrix}$, noting that A is symmetric, gives

$$A[\{2\}, \{1\}] \begin{bmatrix} x_1 \end{bmatrix} = \mathbf{y}^T A[\{1\}] \begin{bmatrix} x_1 \end{bmatrix} = 0$$

But, this just says $a_{2,1}x_1 = 0$. Since $a_{2,1}$ is represented by $*$, it can not be 0. Therefore, we must have $x_1 = 0$.

We do the same for x_3 . Since $x_2 = 0$ and since $a_{3,1} = a_{4,1} = a_{5,1} = 0$, we see that $A[\{3, 4, 5\}] \begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. Now, the column inclusion principle says there exists \mathbf{z} such that $A[\{3, 4, 5\}]\mathbf{z} = A[\{3, 4, 5\}, \{2\}]$. Taking the transpose of both sides and multiplying by $\begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T$, noting that A is symmetric, gives

$$A[\{2\}, \{3, 4, 5\}] \begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T = \mathbf{z}^T A[\{3, 4, 5\}] \begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T = 0$$

But, $A[\{2\}, \{3, 4, 5\}] \begin{bmatrix} x_3 & x_4 & x_5 \end{bmatrix}^T = x_3 \cdot *$. Therefore, x_3 is forced to be 0.

At this point, we can use regular zero forcing and force $x_4 = 0$ and then $x_5 = 0$. So, we see that $Z = \{2\}$ is a positive semidefinite zero forcing set for P_5 , but not a zero forcing set.

Based on the color change rule for positive semidefinite zero forcing, we can do this entire example without looking at the matrix. Again, start with $Z = \{2\}$. At step 1, we have $B = \{2\}$ and thus $G - B$ consists of two connected components, $G[\{1\}]$ and $G[\{3, 4, 5\}]$. So, we consider $G_1 = G[\{1, 2\}]$ and $G_2 = G[\{2, 3, 4, 5\}]$. In G_1 , 2 is colored black and it has only one neighbor colored white, 1. So, we change the color of 1 from white to black. In G_2 , 2 is colored black

and has a unique neighbor colored white, 3. So, we change the color of 3 from white to black. Thus, at the start of step 2, we have $B = \{1, 2, 3\}$. In the next two steps, when we delete the vertices of B , we end up with one connected component so we just proceed using standard zero forcing and color vertex 4, and then vertex 5, black.

Theorem 6.8. [1], [2] *Let G be a graph. Then*

$$\text{mr}(G) \geq |G| - Z(G)$$

$$\text{mr}_+(G) \geq |G| - Z_+(G)$$

Example 6.9. Consider P_n . Since $Z(P_n) = Z_+(P_n) = 1$, we have $\text{mr}(P_n) \geq n - 1$ and $\text{mr}_+(P_n) \geq n - 1$. Since, for any graph G , we have $\text{mr}(G) \leq |G| - 1$ and $\text{mr}_+(G) \leq |G| - 1$, we know that $\text{mr}(P_n) = \text{mr}_+(P_n) = n - 1$. Note, this can also be observed without zero forcing.

Now, we had hoped to be able to carry out a similar process for η to get a new lower bound for η . It does provide a lower bound for $\eta(G)$, but unfortunately it is not new, as $|G| - Z_\eta(G) = \alpha(G)$. We show the process now. Note, the process used below is simply the analogous process already used in determining the color change rules for zero forcing associated with $\text{mr}(G)$ and $\text{mr}_+(G)$ with the obvious changes.

We have a graph G with an associated matrix $A = [a_{ij}]$ that fits G (A is not fixed). We multiply A on the right by some vector \mathbf{x} . The question is, if we fix some entries of \mathbf{x} to be 0, which others are forced to be 0 based on the fact that $A\mathbf{x} = 0$? Again, the only knowledge we have for A is that A fits G .

Since A fits G , $a_{ii} \neq 0$ and $a_{ij} = 0$ whenever vertices i and j are not adjacent. If we want to force entry x_i to be 0, which other entries must be 0? If vertex i is adjacent to the vertices i_1, \dots, i_k , then entries $a_{i,i_1}, \dots, a_{i,i_k}$ can be nonzero or zero. $(A\mathbf{x})_i = a_{i,1}x_1 + \dots + a_{i,n}x_n = a_{i,i}x_i + \sum_{v \sim i} a_{i,v}x_v$ since all other terms are 0. Now, $a_{ii} \neq 0$ but all a_{iv} are free. So, for any choice of $\{x_v : v \sim i\}$ with at least one x_v nonzero, we can choose a_{iv} such that $\sum_{v \sim i} a_{iv}x_v \neq 0$. That is, the only way to force $x_i = 0$ is for $x_v = 0$ for all $v \sim i$.

Color change rule for zero forcing associated with η : For a graph, G , color some vertices black and let B denote the set of black vertices. Color any vertex in $G - B$ black if it is an isolated vertex in $G - B$. Nothing else is forced.

Therefore, if a vertex, v , is not an isolated vertex in the subgraph $G - B$, none of its white neighbors can be forced (since they are not isolated either). So, v can never be forced. Therefore, $Z_\eta(G) = |G| - \alpha(G)$, or $\alpha(G) = |G| - Z_\eta(G)$. So, in this case, zero forcing gives no new information.

Now that we have introduced $Z_+(G)$, we can talk more about the bound $\eta(G) \leq mr_+(G)$. This bound is usually not very useful, but may be useful in rare cases. For example, for all graphs of order 10 or less, we have $\eta(G) \leq vcc(G) \leq |G| - Z_+(G) \leq mr_+(G)$. Thus, for most graphs, it appears $vcc(G)$ is a better upper bound on $\eta(G)$, and $|G| - Z_+(G)$ is a better lower bound on $mr_+(G)$. However, it is not true for all graphs that $vcc(G) \leq |G| - Z_+(G)$, or even that $vcc(G) \leq mr_+(G)$.

Example 6.10. Observation 6.2, along with Theorem 3.11 shows that $vcc(G) \leq mr_+(G)$ for all graphs with $|G| \leq 10$. In fact, using *Sage* it was verified that $vcc(G) \leq |G| - Z_+(G)$ for all graphs of order 10 or less. However, neither inequality holds for all graphs. The graph G in Example 1.29, satisfies $vcc(G) = 4$ and $mr_+(G) = 3$ and $|G| - Z_+(G) = 3$.

To see that $mr_+(G) = 3$, note a matrix A was constructed in Example 1.29 that fit G and satisfied $\text{rank } A = 3$. By the construction, we also have $A \in \mathcal{S}(G)$. Since A can be written as a product $B^T B$, it is positive semidefinite. Therefore, $3 = \eta(G) \leq mr_+(G) \leq 3$, so that $mr_+(G) = 3$.

Question 6.11. *For any graph G with no isolated vertices, is it true that $\eta(G) \leq |G| - Z_+(G)$?*

6.3 Comparison of η , mr , and mr_+

So far we have seen that $\eta(G) \leq mr_+(G)$ for all graphs without isolated vertices and $mr(G) \leq mr_+(G)$ for all graphs. We have also seen the counterexample to $\eta(G) \leq mr_+(G)$ in the case of isolated vertices being allowed, namely G is the empty graph on n vertices. It turns out that $\eta(G)$ and $mr(G)$ are not comparable.

For $n \geq 2$, we have $\eta(C_n) = \lceil \frac{n}{2} \rceil$ and $mr(C_n) = mr_+(C_n) = n - 2$. For the complete bipartite graph $K_{s,t}$, Corollary 2.28 shows $\eta(K_{s,t}) = \max\{s, t\}$. It turns out that $mr_+(K_{s,t}) = \max\{s, t\}$ as well. And, as long as one of s or t is greater than 1, avoiding $K_{1,1} = K_2$, we have $mr(K_{s,t}) = 2$.

These examples show that $\eta(G)$ and $\text{mr}(G)$ are not comparable, and moreover we see that $\eta(G) - \text{mr}(G)$ and $\text{mr}(G) - \eta(G)$ can both be arbitrarily large if we choose the correct graph and allow $|G|$ to be big enough. Since $\text{mr}_+(G)$ is an upper bound to both $\eta(G)$ and $\text{mr}(G)$, these examples also show the difference between $\eta(G)$ and $\text{mr}_+(G)$ can be arbitrarily large, and the difference between $\text{mr}(G)$ and $\text{mr}_+(G)$ can be arbitrarily large.

Table 6.1 below shows a comparison of these three minimum rank parameters for several infinite families of graphs. First, for inclusion in the table, we calculate $\eta(G)$ for some additional infinite families of graphs.

Let Q_n denote the n th hypercube. Q_n can be constructed inductively as a cartesian product of graphs, $Q_n = Q_{n-1} \square K_2$. Then, $Q_1 = K_2$, and $Q_2 = C_4$, and so on. Q_n is well known to be bipartite for all n , and thus perfect. And, $\alpha(Q_n) = 2^{n-1}$ so that $\eta(Q_n) = 2^{n-1}$.

Let N_s , $s \geq 3$, denote the necklace with s diamonds. This graph is a 3-regular graph with $4s$ vertices. To construct it, we start with a cycle on $3s$ vertices. These $3s$ vertices can be split up into s disjoint sets of three sequential vertices, B_1, \dots, B_s . Then, for $i = 1, \dots, s$, we add a new vertex and attach it to all vertices in B_i .

To calculate $\eta(N_s)$, we use Theorem 2.9. Let us label the vertices in B_i as $\{b_{i,1}, b_{i,2}, b_{i,3}\}$, labeled in order. Let v_i denote the extra vertex that was added and attached to the vertices in B_i . Then, $N[b_{i,2}] = N[v_i] = B_i \cup \{v_i\}$. So, Theorem 2.9 says we can delete vertex v_i without changing η . We delete v_i for each i , leaving a $3s$ -cycle. Therefore, $\eta(N_s) = \eta(C_{3s}) = \lceil \frac{3s}{2} \rceil$.

G	$ G $	$\eta(G)$	$\text{mr}(G)$	$\text{mr}_+(G)$
E_n	n	n	0	0
P_n	n	$\lceil \frac{n}{2} \rceil$	$n - 1$	$n - 1$
K_n	n	1	$\begin{cases} 0 & n = 1 \\ 1 & n \geq 2 \end{cases}$	$\begin{cases} 0 & n = 1 \\ 1 & n \geq 2 \end{cases}$
C_n	n	$\begin{cases} 1 & n = 3 \\ \lceil \frac{n}{2} \rceil & \text{else} \end{cases}$	$n - 2$	$n - 2$
$\overline{C_n}$	n	$\begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} 0 & n = 3 \\ 2 & n = 4 \\ 3 & n \geq 5 \end{cases}$	$\begin{cases} 0 & n = 3 \\ 2 & n = 4 \\ 3 & n \geq 5 \end{cases}$
$K_{s,t}$	$s + t$	$\max\{s, t\}$	$\begin{cases} 1 & s = t = 1 \\ 2 & \text{else} \end{cases}$	$\max\{s, t\}$
K_{n_1, \dots, n_k} $n_1 \geq \dots \geq n_k$ $n_1 \geq 2, k \geq 3$	$n_1 + \dots + n_k$	n_1	$\begin{cases} 2 & n_3 < 3 \\ 3 & n_3 \geq 3 \end{cases}$	n_1
Q_n	2^n	2^{n-1}	2^{n-1}	2^{n-1}
N_s	$4s$	$\lceil \frac{3s}{2} \rceil$	$3s - 2$	$3s - 2$
$P_s \boxtimes P_t$	st	$\lceil \frac{s}{2} \rceil \lceil \frac{t}{2} \rceil$	$(s - 1)(t - 1)$	$(s - 1)(t - 1)$
M_{2n}	$2n$	n	$2n - 4$	$\begin{cases} 3 & \text{if } n = 3 \\ 5 & \text{if } n = 4 \\ 2n - 4 & \text{else} \end{cases}$
W_n	n	$\begin{cases} 1 & n = 4 \\ \lceil \frac{n}{2} \rceil & \text{else} \end{cases}$	$n - 3$	$n - 3$

Table 6.1 A comparison of $\eta(G)$ with $\text{mr}(G)$ and $\text{mr}_+(G)$ for some graphs.

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