Zero forcing number: Results for computation and comparison with other graph parameters

by

Darren Daniel Row

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Leslie Hogben, Major Professor
Maria Axenovich
Ryan Martin
Alexander Roitershtein
Sung-Yell Song

Iowa State University

Ames, Iowa

2011

DEDICATION

This dissertation is dedicated to my family and friends who have supported, encouraged, and made tremendous sacrifices for me, especially my wife Jill and our children Thomas, Rita, Mary, Joseph, and Michael.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1. GENERAL INTRODUCTION

1.1 Introduction

A *graph* $G = (V_G, E_G)$ consists of a non-empty finite set of vertices V_G and a set of edges E_G. An *edge* is a two element subset of V_G. The number of vertices of a graph is called its *order* and is denoted $|G|$. If $\{x, y\} \in E_G$, we say x and y are *adjacent* or x and y are *neighbors*, and write $x \sim y$. A vertex is called a *leaf* if it has only one neighbor. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{x, y\} \in E_G : y, y \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For a graph $G = (V_G, E_G)$ and $e \in E_G$, the graph $(V_G, E_G \setminus \{e\})$ will be denoted $G - e$. A *complete graph*, K_n, is the graph on n vertices with $\{x, y\} \in E_G$ for all $x, y \in V_G$. A *path*, P_n, is a the graph with $V_{P_n} = \{x_1, \ldots, x_n\}$ (where the listed vertices are distinct) and $E_{P_n} = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$. The *length* of a path is the number of edges, $n - 1$ for P_n. A *cycle*, C_n, is a the graph with $V_{C_n} = \{x_1, \ldots, x_n\}$ (where the listed vertices are distinct) and $E_{C_n} = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$. The *length* of a cycle is the number of edges, n for C_n. A complete graph, a path, and a cycle are shown in Figure 1.1. A graph is called *connected* if any two vertices are linked by a path. If a graph is not connected, we say it is *disconnected*. The maximal connected subgraphs of a graph are called the *components* of the graph. If the deletion of a vertex (and the edges having it as an endpoint) in a graph G result in an increase in the number of connected components, the vertex is called a *cut-vertex* of G. Similarly, a *cut-edge* of a graph is one such that its deletion increases the number of connected components. A graph without any cycles is called a *forest*, and a connected forest is called a *tree*. A graph with one cycle is referred to as *unicyclic*. A connected graph in which any two cycles share at most one vertex is called a *cactus*. A tree and a cactus are shown in Figure 1.2.

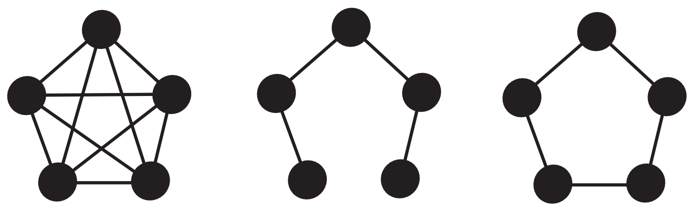

Figure 1.1 The graphs K_5, P_5, and C_5, respectively.

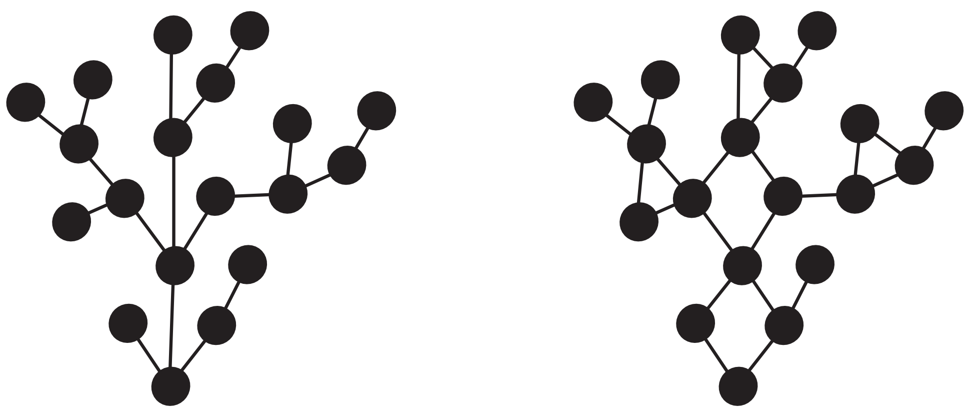

Figure 1.2 A tree and a cactus, respectively.

All matrices discussed are real and square. The *transpose* of a matrix $A = [a_{ij}]$ is $A^T = [a_{ji}]$. When a square matrix satisfies $A^T = A$, it is called *symmetric*. The *rank* of a matrix A, rank A, is the number of linearly independent rows or columns of A. The *nullity* of a matrix A, null A, is the dimension of the null space of A. The rank of a matrix added to the nullity of the matrix equals the number of columns of the matrix.

An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1,\dots,n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Given a graph G, the *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *minimum rank of* G is $\operatorname{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}$ and the *maximum nullity of* G is $\operatorname{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$. Since $\operatorname{rank} A + \operatorname{null} A$ is equal to the number of columns of A, $\operatorname{mr}(G) + \operatorname{M}(G) = |G|$. Because of this relationship, finding the value of one of these two parameters for a graph is equivalent to finding the value for both. The study of these parameters is referred to as the minimum rank/maximum nullity problem. Motivation for this problem can found in [23] and [24].

Given an initial black and white coloring of the vertices of a graph, vertices change color according to the following rule: if a black vertex is adjacent to exactly one white vertex, then the white vertex changes to black. If, after applying the rule until no more color changes are possible, all the vertices are black then the set of vertices initially colored black is called a *zero forcing set* for the graph. The *zero forcing number* of a graph G, $\operatorname{Z}(G)$, is the minimum size of a zero forcing set. The terminology and notation were formalized in [1], where justification for the term comes from the original mathematical motivation for its study. Specifically, the zero forcing number of a graph is an upper bound for the maximum nullity of the graph [1]. Computing $\operatorname{M}(G)$ requires consideration of an infinite family of matrices, whereas computing $\operatorname{Z}(G)$ requires consideration of subsets of the finite vertex set of the graph. The zero forcing number of a graph is also of interest to physicists studying quantum systems control [16], [17], and [42].

Prior to the development of zero forcing number, another graph parameter was studied in conjunction with maximum nullity. A *path cover* of a graph G is a set of vertex disjoint induced paths that cover all the vertices of G. The *path cover number* of a graph G, $\operatorname{P}(G)$, is the minimum size of a path cover. For certain families of graphs, path cover number is an upper bound for maximum nullity, but not in general. However, path cover number is a lower bound for zero forcing number [27].

Studying the effects on parameters from the deletion of a single vertex or a single edge, especially a cut-vertex or cut-edge, can be of great assistance in calculating values of the parameters for a graph. For a graph G, a vertex $v \in V_G$, and an edge $e \in E_G$, the following terminology and notation are used:

- The *rank spread* of v in G is $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$ and the *rank edge spread* of e in G is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G - e)$.
- The *null spread* of v in G is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G - v)$ and the *null edge spread* of e in G is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G - e)$
- The *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$ and the *zero edge spread* of e in G is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G - e)$.
- The *path spread* of v in G is $\mathrm{p}_v(G) = \mathrm{P}(G) - \mathrm{P}(G - v)$ and the *path edge spread* of e in G is $\mathrm{p}_e(G) = \mathrm{P}(G) - \mathrm{P}(G - e)$.

1.2 Dissertation Organization

This dissertation is organized in the format of a dissertation containing journal papers. In the general introduction, the research problem and related pertinent background information are presented. Additionally, a literature review is included.

Chapter 2 contains the paper "A technique for computing the zero forcing number of a graph with a cut-vertex" [40], submitted to *Linear Algebra and its Applications*. The paper includes results for characterizations of graphs with very high or very low zero forcing numbers, computing the zero forcing number of a graph using cut-vertex reduction, and an algorithm for computing the zero forcing number of a unicyclic graph. It is also shown that the zero forcing number of a unicyclic graph is equal to the path cover number of the graph.

Chapter 3 contains the paper "Zero forcing number, path cover number, and maximum nullity of cacti" [41], in preparation for submission to *Involve, a Journal of Mathematics*. The paper includes results related to path edge spread and cut-edge reduction for zero forcing number. It is shown that zero forcing number equals path cover number for any cactus, and zero forcing number equals maximum nullity for a restricted family of cacti.

Chapter 4 is for general conclusions. Results are summarized and related to the problem described in the introduction. Recommendations for future research are also presented.

1.3 Literature Review

The study of the minimum rank/maximum nullity problem was initiated in 1996 by Nylen [38]. There has been interest in graphs having very high or very low parameter values.

Theorem 1.3.1. [25] *Let G be graph. Then* $\mathrm{mr}(G) = |G| - 1$ *(hence* $\mathrm{M}(G) = 1$*) if and only if* $G = P_{|G|}$.

A graph G is a *graph of two parallel paths* if there exist two independent induced paths of G that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that the paths are parallel and edges (drawn as segments, not curves) between the two paths do not cross [36]. A simple path is not considered to be such a graph. A graph that consists of two connected components, each of which is a path, is considered to be such a graph.

Theorem 1.3.2. [36] *Let G be a graph. Then* $\mathrm{M}(G) = 2$ *if and only if G is a graph of two parallel paths or G is one of the types shown in Figure 1.3.*

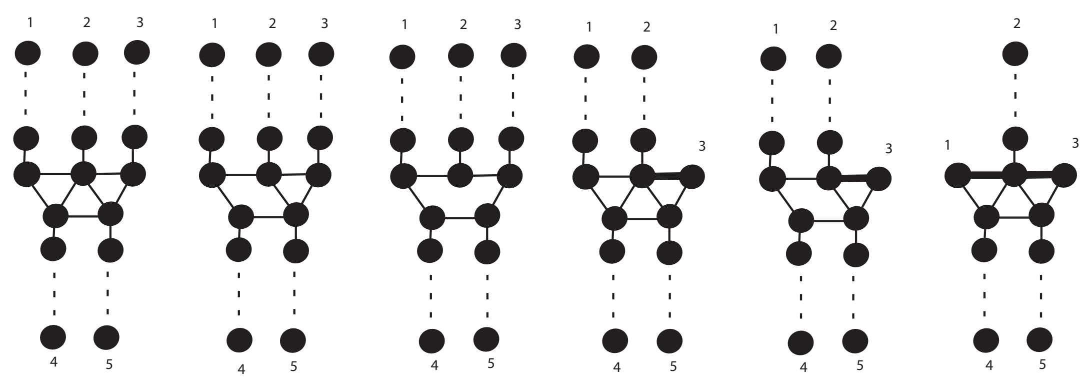

Figure 1.3 Graphs that have maximum nullity 2 but are not graphs of two parallel paths. The bold lines indicate a path of length at least one. The dotted lines indicate (possibly nonexistent) paths of arbitrary length.

The following result is well known and obvious.

Observation 1.3.3. *Let G be a connected graph. Then* $\operatorname{mr}(G) = 1$ *if and only if* $G = K_{|G|}$.

Theorem 1.3.4. [10] *Let G be a connected graph. Then* $\operatorname{mr}(G) \leq 2$ *if and only if G contains none of P_4, fish, dart, or $K_{3,3,3}$ (all shown in Figure 1.4) as an induced subgraph.*

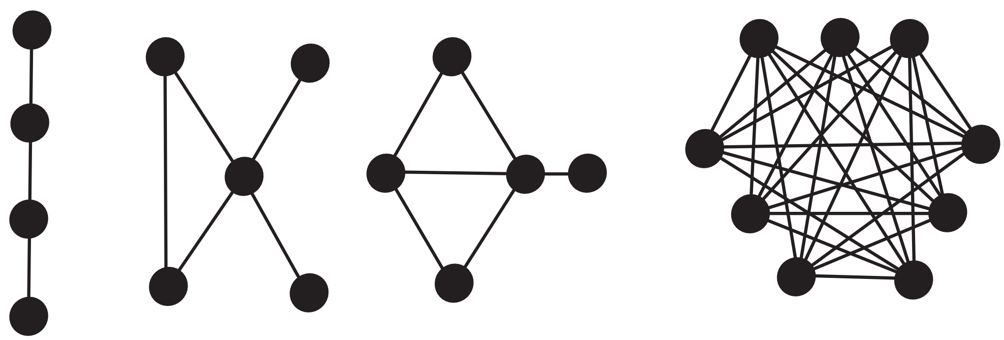

Figure 1.4 Forbidden induced subgraphs for $\operatorname{mr}(G) \leq 2$: P_4, fish, dart, and $K_{3,3,3}$, respectively.

Later, maximum nullity and path cover number were studied together [34], [7], and [8]. Maximum nullity and path cover number are not generally comparable. However, in some cases they are.

Theorem 1.3.5. [34] *Let T be a tree. Then* $\mathrm{M}(T) = \mathrm{P}(T)$.

An *outerplanar graph* is one which can be drawn in the plane in such a way that its edges intersect only at their endpoints and all the vertices belong to the unbounded face of the drawing.

Theorem 1.3.6. [43] *Let G be an outerplanar graph. Then* $\mathrm{M}(G) \leq \mathrm{P}(G)$.

The bound does not hold in general. For example, $\mathrm{M}(K_4) = 3 > 2 = \mathrm{P}(K_4)$.

The zero forcing number was introduced in [1] to be a parameter to bound maximum nullity.

Theorem 1.3.7. [1] *Let G be a graph. Then* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

Zero forcing number also bounds path cover number.

Theorem 1.3.8. [27] *Let G be a graph. Then* $\mathrm{P}(G) \le \mathrm{Z}(G)$.

The effect on parameter values of deleting a vertex or an edge has been considered.

Theorem 1.3.9. [38] *Let G be a graph and v a vertex of G. Then* $0 \le \mathrm{r}_v(G) \le 2$.

Corollary 1.3.10. *Let G be a graph and v a vertex of G. Then* $-1 \le \mathrm{n}_v(G) \le 1$.

Theorem 1.3.11. [38] *Let G be a graph and e an edge of G. Then* $-1 \le \mathrm{r}_e(G) \le 1$.

Corollary 1.3.12. *Let G be a graph and e an edge of G. Then* $-1 \le \mathrm{n}_e(G) \le 1$.

Theorem 1.3.13. [22], [31] *Let G be a graph and v a vertex of G. Then* $-1 \le \mathrm{z}_v(G) \le 1$.

Theorem 1.3.14. [22] *Let G be a graph and e an edge of G. Then* $-1 \le \mathrm{z}_e(G) \le 1$.

Theorem 1.3.15. [7], [8] *Let G be a graph and v a vertex of G. Then* $-1 \le \mathrm{p}_v(G) \le 1$.

A vertex v is called *terminal* if it is the endpoint of a path in a minimal path cover. It is called *doubly terminal* if it appears in a path by itself in a minimal path cover. A vertex that is terminal but not doubly terminal is called *simply terminal*.

Theorem 1.3.16. [8] *Let G be a graph and v a vertex of G. Then* $\mathrm{p}_v(G) = 1$ *if and only if v is doubly terminal.*

Theorem 1.3.17. [22] *Let G be a graph and v a vertex of G. Then* $\mathrm{z}_v(G) = 1$ *if and only if there exists an minimal zero forcing set that contains v in which v does not perform a force.*

Formulas are known for computing the minimum rank of a graph with a cut-vertex (or a cut-edge) in terms of the minimum ranks of the connected components of the graph after deleting the cut-vertex (or cut-edge). Such formulas are also known for path cover number considering a cut-vertex or a cut-edge. These formulas are given in terms of the vertex spread or edge spread of the parameter.

Theorem 1.3.18. [7] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Then*

$$\mathrm{r}_v(G) = \min\left\{\sum_{i=1}^{k} \mathrm{r}_v(G_i), 2\right\}$$

Theorem 1.3.19. [7] *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{r}_e(G) = \begin{cases} 0 & \text{if and only if } \max_{i=1,2}\{r_{v_i}(G_i)\} = 2 \\ 1 & \text{otherwise} \end{cases}$$

Theorem 1.3.20. [8] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \le j \le k}\{\mathrm{p}_v(G_j)\}$, and t denote the number of the G_i's in which v is simply terminal. Then*

$$\mathrm{p}_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t \le 1 \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1 \end{cases}$$

Theorem 1.3.21. [7] *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{p}_e(G) = \begin{cases} -1 & \text{if and only if } v_i \text{ is terminal in } G_i \text{ for } i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Families of graphs having equality of parameters have been of interest. In [1], numerous families of graphs, including trees, were shown to satisfy equality of maximum nullity and zero forcing number.

Theorem 1.3.22. [1] *Let T be a tree. Then $\mathrm{M}(T) = \mathrm{Z}(T)$.*

Several algorithms exist for computing the value $\mathrm{M}(T) = \mathrm{P}(T) = \mathrm{Z}(T)$ for a tree, T. See, for example, [23]. An algorithm is presented in [23] for computing maximum nullity and path cover number for unicyclic graphs. See also [18] for programs to compute zero forcing number for small graphs.

Many variations on the minimum rank/maximum nullity problem have been considered. In [10], families of matrices over fields other than the real numbers were first considered. Results can be found in [10], [11], [12], [19], and [39]. The zero forcing number of a graph is an upper bound for the maximum nullity over any field. Alternatively, requiring the matrices to be skew-symmetric has created a new version of the original problem [32]. Another variation is to consider the family of positive semidefinite matrices. See [14], [15], [20], [26], [33], and [35]. Along with this change in the family of matrices comes a variation of the zero forcing number, the positive semidefinite zero forcing number [3] which has a different rule for changing vertices from white to black. Other variations to the minimum rank/maximum nullity problem come from considering different types of graphs, for example graphs that have loops and/or are directed graphs [4], [5], and [27].

Bibliography

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.

[2] American Institute of Mathematics workshop, "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns," held Oct. 23-27, 2006 in Palo Alto, CA.

[3] F. Barioli, W. Barrett, S. Fallet, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*. 433: 401–411, 2010.

[4] F. Barioli, W. Barrett, S. Fallet, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Parameters related to tree-width, zero forcing, and maximum nullity of a graph. Under review.

[5] F. Barioli, S.M. Fallet, H.T. Hall, D. Hershkowitz, L. Hogben, H. van der Holst, B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *Electronic Journal of Linear Algebra*. 18: 126–145, 2009.

[6] F. Barioli, S. Fallet, L. Mitchell, S. Narayan. Minimum semi-definite rank of outerplanar graphs and the tree cover number. Under review.

[7] F. Barioli, S.M. Fallet, L. Hogben. Computation of minimal rank and path cover number for graphs. *Linear Algebra and its Applications*. 392: 289–303, 2004.

[8] F. Barioli, S.M. Fallet, L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra and its Applications*. 409: 13–31, 2005.

[9] F. Barioli, S.M. Fallet, R.L. Smith. On acyclic and unicyclic graphs whose minimum rank equals the diameter. *Linear Algebra and its Applications*. 429: 1568–1578, 2008.

[10] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two. *Electronic Journal of Linear Algebra*. 11: 258–280, 2004.

[11] W. Barrett, H. van der Holst, R. Loewy. Graphs whose minimal rank is two: the finite fields case. *Electronic Journal of Linear Algebra*. 14: 32–42, 2005.

[12] A. Bento, A. Leal Duarte. On Fiedler's characterization of tridiagonal matrices over arbitrary fields. *Linear Algebra and its Applications*. 401: 467–481, 2005.

[13] A. Berman, S. Friedland, L. Hogben, U.G. Rothblum, B. Shader. An upper bound for the minimum rank of a graph. *Linear Algebra and its Applications*. 429: 1629–1638, 2008.

[14] M. Booth, P. Hackney, B. Harris, C.R. Johnson, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe, K. Steinmetz, B.D. Sutton, W. Wang. On the minimum rank among positive semidefinite matrices with a given graph. Submitted.

[15] M. Booth, P. Hackney, B. Harris, C.R. Johnson, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe, B.D. Sutton. The minimum semidefinite rank of a graph. Submitted.

[16] D. Burgarth, V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters*. 99: 100501, 2007.

[17] D. Burgarth, K. Maruyama. Indirect Hamiltonian identification through a small gateway. arXiv:0903.0612v1 [quant-ph], 2009.

[18] S. Butler, L. DeLoss, J. Grout, H.T. Hall, J. LaGrange, T. McKay, J. Smith, G. Tims. Minimum Rank Library (*Sage* programs for calculating bounds on the minimum rank of a graph and for computing zero forcing parameters). Available at http://sage.cs.drake.edu/home/pub/67/. For more information contact Jason Grout at jason.grout@drake.edu.

[19] N.L. Chenette, S.V. Droms, L. Hogben, R. Mikkelson, O. Pryporova. Minimum rank of a tree over an arbitrary field. Preprint.

[20] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width graphs. *Journal Combin. Theory.* Ser. B 74: 121-146, 1998.

[21] L.A. Deaett. The positive semidefinite minimum rank of a triangle-free graph. To appear in *Linear Algebra and its Applications.*

[22] C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, D.D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. To appear in *Linear Algebra and its Applications.*

[23] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications.* 426: 558–582, 2007.

[24] S. Fallet and L. Hogben. Variants on the minimum rank problem: A survey II. arXiv:1102.5142v1 [math.CO] 25 Feb 2011.

[25] M. Fiedler. A characterization of tridiagonal matrices. *Linear Algebra and its Applications.* 2: 191–197, 1969.

[26] P. Hackney, B. Harris, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe. Linearly independent vertices and minimum semidefinite rank. Submitted.

[27] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications.* 432: 1961–1974, 2010.

[28] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and its Applications.* 375: 1–11, 2003.

[29] H. van der Holst. The maximum corank of graphs with a 2-separation. *Linear Algebra and its Applications.* 428: 1587–1600, 2008.

[30] H. van der Holst. Three-connected graphs whose maximum nullity is at most three. *Linear Algebra and its Applications.* 429: 625–632, 2008.

[31] L.-H. Huang, G.J. Chang, H.-G. Yeh. On minimum rank and zero forcing sets of a graph. *Linear Algebra and its Applications.* 432: 2961–2973, 2010.

[32] IMA-ISU research group on minimum rank (M. Allison, E. Bodine, L.M. DeAlba, J. Debnath, L. DeLoss, C. Garnett, J. Grout, L. Hogben, B. Im, H. Kim, R. Nair, O. Pryporova, K. Savage, B. Shader, A. Wangsness Wehe). Minimum rank of skew-symmetric matrices described by a graph. *Linear Albegra and its Applications.* 432: 2457-2472, 2010.

[33] Y. Jiang, L.H. Mitchell, S.K. Narayan. Unitary matrix digraphs and minimum semidefinite rank of a graph. Submitted.

[34] C.R. Johnson, A. Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra.* 46: 139–144, 1999.

[35] C.R. Johnson, A. Leal Duarte. On the possible multiplicities of the eigenvalues of a Hermitian matrix the graph of whose entries is a tree. *Linear Algebra and its Applications.* 348: 7–21, 2002.

[36] C.R. Johnson, R. Loewy, P.A. Smith. The graphs for which the maximum multiplicity of an eigenvalue is two. *Linear and Multilinear Algebra.* 57: 713–736, 2009.

[37] L. Mitchell, S. Narayan, A. Zimmer. Lower bounds on minimum rank problems. *Linear Algebra and its Applications.* 432: 430–440, 2010.

[38] P.M. Nylen. Minimum-rank matrices with prescribed graph. *Linear Algebra and its Applications.* 248: 303–316, 1996.

[39] W.C. Rheinboldt, R.A. Shepherd. On a characterization of tridiagonal matrices by M. Fiedler. *Linear Algebra and its Applications.* 8: 87–90, 1974.

[40] D.D. Row. A technique for computing the zero forcing number of a graph with a cut-vertex. Submitted to *Linear Algebra and its Applications.*

[41] D.D. Row. Zero forcing number, path cover number, and maximum nullity of cacti. In preparation.

[42] S. Severini. Nondiscriminatory propogation on trees. *Journal of Physics A: Mathematical and Theoretical* 41: 482002, 2008.

[43] J. Sinkovic. Maximum nullity of outerplanar graphs and the path cover number. *Linear Algebra and its Applications.* 432: 2052-2060, 2010.

CHAPTER 2. A TECHNIQUE FOR COMPUTING THE ZERO FORCING NUMBER OF A GRAPH WITH A CUT-VERTEX

A paper submitted to *Linear Algebra and its Applications*

Darren D. Row

Abstract

The zero forcing number of a graph is the minimum size of a zero forcing set. This parameter is useful in the minimum rank/maximum nullity problem, as it gives an upper bound to the maximum nullity. Results for determining graphs with extreme zero forcing numbers, for determining the zero forcing number of graphs with a cut-vertex, and for determining the zero forcing number of unicyclic graphs are presented.

2.1 Introduction

A *graph* $G = (V_G, E_G)$ will mean a simple (no loops, no multiple edges) undirected graph. The vertex set V_G will be assumed finite and nonempty. The edge set E_G consists of two-element subsets of vertices. When $\{x, y\} \in E_G$, we say x and y are *neighbors* or x and y are *adjacent*, and write $x \sim y$. The order of G, denoted $|G|$ refers to the number of vertices $|V_G|$. We denote by K_n, C_n, and P_n the complete graph, the cycle, and the path, respectively, on n vertices. The term *path length* will be used to refer to the number of edges in the path.

The zero forcing number of a graph was introduced in [1] and the related terminology was extended in [2], [3], and [10]. Independently, physicists have studied this parameter, referring to it as the graph infection number, in conjunction with control of quantum systems [5], [6], and [13]. Let G be a graph with each vertex initially colored either black or white. From the initial coloring, vertices change color according to the *color-change rule*: If u is a black vertex and exactly one neighbor v of u is white, then change the color of v to black. When the color-change rule is applied to u to change the color of v, we say u *forces* v and write $u \to v$. Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color-change rule is applied until no more changes are possible. In an initial black-white coloring of a graph G, if the set of black vertices Z has derived set that is all the vertices of G, we say Z is a *zero forcing set* for G. A zero forcing set with the minimum number of vertices is called an *optimal zero forcing set*, and this minimum size of a zero forcing set for a graph G is the *zero forcing number* of the graph, denoted $\mathrm{Z}(G)$.

In this paper, we prove results for computing the zero forcing number for certain families of graphs. In Section 2.2, characterizations are given for graphs having either very high or very low zero forcing numbers. In Section 2.3, a theorem is given which allows the zero forcing number of a graph with a cut-vertex to be calculated by using the zero forcing numbers of the connected components of the graph after deleting the cut-vertex. Section 2.4 contains results related to unicyclic graphs. In particular it is shown that the zero forcing number of any unicyclic graph has the same value as another graph parameter for which an algorithm exists for its computation. Section 2.5 summarizes the main results and proposes some questions for further study. The remainder of Section 2.1 presents more definitions, notations, and known results that will be used in the subsequent sections.

For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \dots, v_k)$ such that for $i = 1, \dots, k-1$, $v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of maximal forcing chains for a chronological list of forces is called the *chain set* of the chronological list of forces, and an *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton*. For a zero forcing set Z, a *reversal* of Z is the set of vertices which are last in the forcing chains in the chain set of some chronological list of forces [2]. If Z is a zero forcing set of G then so is any reversal of Z [2]. Since the size of a reversal of a zero forcing set is the same as the size of the zero forcing set, a reversal of an optimal zero forcing set is an optimal zero forcing set. For any connected graph of order more than one, no vertex is in every optimal zero forcing set for the graph [2].

The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^k G_i = (\cup_{i=1}^k V_i, \cup_{i=1}^k E_i)$; a disjoint union is denoted $\dot\cup_{i=1}^k G_i$. Clearly, $\mathrm{Z}(\dot\cup_{i=1}^k G_i) = \sum_{i=1}^k \mathrm{Z}(G_i)$. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For a graph G and a vertex $v \in V_G$, the *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$ [8]. Bounds on the zero spread of a vertex are known. For any graph G and vertex v of G, $-1 \le \mathrm{z}_v(G) \le 1$ [8], [11]. Here the definition of zero spread is extended to vertex subsets of size greater than one and bounds are proved.

Definition 2.1.1. Let G be a graph and $W \subseteq V_G$. The *zero spread* of W in G is $\mathrm{z}_W(G) = \mathrm{Z}(G) - \mathrm{Z}(G - W)$.

Corollary 2.1.2. *For every graph G and every subset $W \subseteq V_G$, $-|W| \le \mathrm{z}_W(G) \le |W|$.*

Proof. Let $W = \{v_1, \dots, v_k\}$. Set $G_0 = G$ and define $G_i = G_{i-1} - v_i$ for $i = 1, \dots, k$. Then $G_k = G - W$. The bounds on the zero spread of a vertex give that for any graph H and any vertex $v \in V_H$, $|\mathrm{Z}(H) - \mathrm{Z}(H - v)| \le 1$. Therefore, $|\mathrm{z}_W(G)| = |\mathrm{Z}(G) - \mathrm{Z}(G - W)| = |\sum_{i=0}^{k-1}(\mathrm{Z}(G_i) - \mathrm{Z}(G_{i+1}))| \le \sum_{i=0}^{k-1} |\mathrm{Z}(G_i) - \mathrm{Z}(G_{i+1})| \le \sum_{i=0}^{k-1} 1 = k = |W|$. □

The *path cover number* $\mathrm{P}(G)$ of G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths. For any graph G, $\mathrm{P}(G) \le \mathrm{Z}(G)$ [10].

A primary reason to study the zero forcing number of a graph is its relationship to the maximum nullity of the graph, which is defined here. An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. For a graph G, the *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$ and the *maximum nullity of* G is $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$. For any graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$ [1]. A graph has maximum nullity of one if and only if the graph is a path [9].

2.2 Graphs with extreme zero forcing numbers

In this section we consider graphs that have very low or very high zero forcing numbers.

Observation 2.2.1. *Let* $G = (V_G, E_G)$ *be a graph. Then* $\mathrm{Z}(G) = 1$ *if and only if* $G = P_n$ *for some* $n \geq 1$.

Proposition 2.2.2. *Let* $G = (V_G, E_G)$ *be a connected graph with* $|G| \geq 2$. *Then* $\mathrm{Z}(G) = |G| - 1$ *if and only if* $G = K_{|G|}$.

Proof. It is clear that if $G = K_{|G|}$ with $|G| \geq 2$ then $\mathrm{Z}(G) = |G| - 1$.

Let $G = (V_G, E_G)$ be a connected graph with $|G| \geq 2$ and $G \neq K_{|G|}$. Then there exist $x, y \in V_G$ with $x \not\sim y$. Since G is connected, there exists $u \in V_G$ such that $u \sim x$. Let $Z = V_G \setminus \{u, y\}$. Color the vertices in Z black, and the vertices in $\{u, y\}$ white. Now x can force u. Then any vertex adjacent to y can force y. Hence Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = |G| - 2$. □

A definition and a known result will be used in the proof of the next characterization theorem. A graph G is a *graph of two parallel paths* if there exist two independent induced paths of G that cover all the vertices of G and such that the graph can be drawn in the plane in such a way that the paths are parallel and edges (drawn as segments, not curves) between the two paths do not cross [12]. A simple path is not considered to be such a graph. A graph that consists of two connected components, each of which is a path, is considered to be such a graph. It is known that the only graphs with maximum nullity of two are graphs of two parallel paths and those of the types shown in Figure 2.1 below [12].

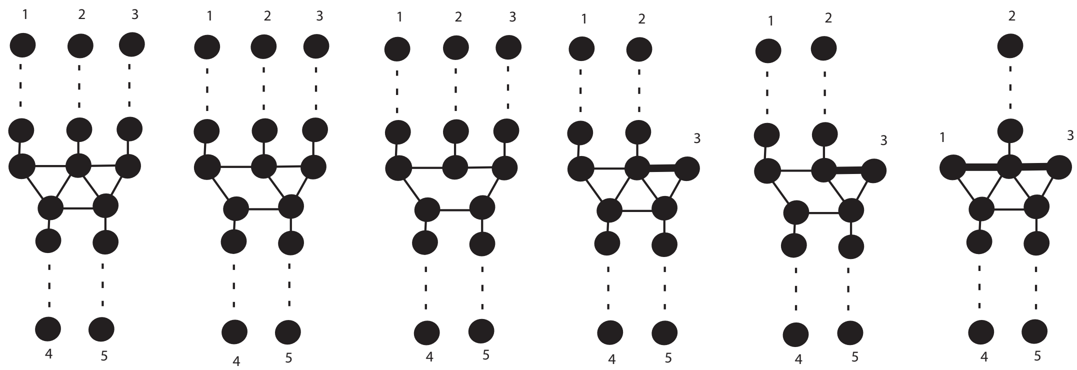

Figure 2.1 Graphs which have maximum nullity 2 but are not graphs of two parallel paths. The bold lines indicate a path of length at least one. The dotted lines indicate (possibly nonexistent) paths of arbitrary length.

Theorem 2.2.3. *Let $G = (V_G, E_G)$ be a graph. Then* $\mathrm{Z}(G) = 2$ *if and only if* G *is a graph of two parallel paths.*

Proof. Let G be a graph of two parallel paths. Consider a drawing of G oriented in the plane so that the two independent induced paths which cover all the vertices of G are each horizontal and no edges between the paths cross. Let Z consist of the left-most vertex of each path. Forces can be performed along the top path until a vertex that has a white neighbor w in the bottom path gets forced black. Because the edges between the two paths do not cross, forces can take place along the bottom path until w is forced black. Continuing in this manner, it is clear that Z is a zero forcing set with $|Z| = 2$, so $\mathrm{Z}(G) \leq 2$. But G is not a path so by Observation 2.2.1, $\mathrm{Z}(G) \geq 2$. Hence $\mathrm{Z}(G) = 2$.

Let G be a graph with $\mathrm{Z}(G) = 2$. Since $\mathrm{M}(G) \leq \mathrm{Z}(G)$ for any graph G, $\mathrm{M}(G) \leq 2$. If $\mathrm{M}(G) = 1$, then G is a path, so by Observation 2.2.1, $\mathrm{Z}(G) = 1$, a contradiction. Thus $\mathrm{M}(G) = 2$, so G is a graph of two parallel paths or G is one of the types shown in Figure 2.1. Note that any vertex which has degree one must be an endpoint of any induced path which contains it. Also for graphs represented in Figure 2.1, any induced path which contains a degree two vertex v that is an endpoint of a bold line must have an endpoint either at v or at one of the interior vertices of the path represented by the bold line. By inspection, each graph in the figure has at least five vertices which must be endpoints of induced paths used as a path cover. Therefore, for each graph G represented in Figure 2.1, $\mathrm{P}(G) \geq 3$. Since $\mathrm{P}(G) \leq \mathrm{Z}(G)$ for any graph G, $\mathrm{Z}(G) \geq 3$, a contradiction. Hence if $\mathrm{Z}(G) = 2$, then G must be a graph of two parallel paths. □

It is also known that a graph G satisfies $\mathrm{Z}(G) \geq |G| - 2$ if and only if G does not contain any of $P_2 \dot{\cup} P_2 \dot{\cup} P_2$, $P_3 \dot{\cup} P_2$, P_4, $\ltimes$, or dart as an induced subgraph [1]. A figure containing these graphs can be found in [1] along with the proof which is linear algebraic. A graph theoretic proof is also possible using only zero forcing techniques, but is omitted here in the interest of brevity.

2.3 Results for graphs with a cut-vertex

Algorithms exist [7] to compute the zero forcing number of a graph. However, the run time depends on the number of vertices in the graph and on the zero forcing number, so the algorithm can be impractical for graphs such as a graph with fifty vertices and zero forcing number thirty-five. The zero forcing number of a graph G with a cut-vertex v can be calculated by finding the zero forcing numbers of the connected components of $G - v$ and using Theorem 2.3.8 below to find $\mathrm{z}_v(G)$. For a connected graph G with a cut-vertex v, if $G - v$ has k connected components then the formula from the theorem requires finding the zero forcing number of $2k$ graphs. If $G - v$ has a connected component that is large relative to G, the theorem might not be of benefit. However, if each component is reasonably reduced in size relative to G, the formula can be of benefit. Example 2.3.9 below shows how applying the results of this section might make computing the zero forcing number of a graph with a cut-vertex practical where it is not practical without these results. We begin this section with some preliminary results which lead to the main theorem of the section which gives the zero spread of a cut-vertex.

Lemma 2.3.1. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$.*

Then $\mathrm{Z}(G) \geq \sum_{i=1}^{k} \mathrm{Z}(G_i) - k + 1$.

Proof. Let Z be an optimal zero forcing set of G with $v \notin Z$. Then there is a vertex u such that $u \to v$. Without loss of generality, let $u \in G_1$. Now $Z \cap V_{G_1}$ is a zero forcing set of G_1 so $\mathrm{Z}(G_1) \leq |Z \cap V_{G_1}|$. Also, for $i = 2, \ldots, k$, $(Z \cap V_{G_i}) \cup \{v\}$ is a zero forcing set of G_i so $\mathrm{Z}(G_i) \leq |Z \cap V_{G_i}| + 1$. Therefore,

$$\sum_{i=1}^{k} \mathrm{Z}(G_i) \leq |Z \cap V_{G_1}| + \sum_{i=2}^{k}(|Z \cap V_{G_i}| + 1) = \sum_{i=1}^{k} |Z \cap V_{G_i}| + k - 1 = |Z| + k - 1 = \mathrm{Z}(G) + k - 1.$$

□

Corollary 2.3.2. *Let* $G = (V_G, E_G)$ *be a graph with cut-vertex* $v \in V_G$. *Let* $W_1, \ldots, W_k$ *be the vertex sets for the connected components of* $G - v$, *and for* $1 \leq i \leq k$, *let* $G_i = G[W_i \cup \{v\}]$. *Then* $\mathrm{z}_v(G) \geq \sum_{i=1}^{k} \mathrm{z}_v(G_i) - k + 1$.

Proof. By Lemma 2.3.1, $Z(G) \geq \sum_{i=1}^{k} \mathrm{Z}(G_i) - k + 1$. Since v is a cut-vertex, $\mathrm{Z}(G - v) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v)$. Subtracting gives $\mathrm{z}_v(G) \geq \sum_{i=1}^{k} \mathrm{z}_v(G_i) - k + 1$. □

Lemma 2.3.3. *Let* $G = (V_G, E_G)$ *be a graph with cut-vertex* $v \in V_G$. *Let* $W_1, \ldots, W_k$ *be the vertex sets for the connected components of* $G - v$, *and for* $1 \leq i \leq k$, *let* $G_i = G[W_i \cup \{v\}]$. *Then* $\mathrm{Z}(G) \leq \min_{1 \leq j \leq k}\{\mathrm{Z}(G_j) + \sum_{i=1, i \neq j}^{k} \mathrm{Z}(G_i - v)\}$.

Proof. Fix j with $1 \leq j \leq k$. Let Z_j be an optimal zero forcing set for G_j. For $i \neq j$, let Z_i be an optimal zero forcing set for $G_i - v$. Set $Z = \cup_{i=1}^{k} Z_i$. Clearly, $Z \cap V_{G_j}$ is a zero forcing set for G_j and for $i \neq j$, $(Z \cap V_{G_i}) \cup \{v\}$ is a zero forcing set for G_i with v not needing to perform a force. Let z be colored black if and only if $z \in Z$. Starting in G_j, perform forces (if necessary) until v is colored black. Now in each $G_i - v$, $i \neq j$, forces can be performed to color all of $G_i - v$ black. (If necessary) return to G_j and perform the remaining forces. Thus Z is a zero forcing set of G. Since j was arbitrary, $\mathrm{Z}(G) \leq \min_{1 \leq j \leq k}\{\mathrm{Z}(G_j) + \sum_{i=1, i \neq j}^{k} \mathrm{Z}(G_i - v)\}$. □

Corollary 2.3.4. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then $\mathrm{z}_v(G) \leq \min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\}$.*

Proof. By Lemma 2.3.3, $\mathrm{Z}(G) \leq \min_{1 \leq j \leq k}\{Z(G_j) + \sum_{i=1, i \neq j}^{k} \mathrm{Z}(G_i - v)\}$. Since v is a cut-vertex, $\mathrm{Z}(G - v) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v)$. Subtracting gives $\mathrm{z}_v(G) \leq \min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\}$. □

The following lemma provides information about the distribution of an optimal zero forcing set amongst components having certain properties.

Lemma 2.3.5. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let Z be a zero forcing set for G. If $\mathrm{z}_v(G_j) = 1$, or if $\mathrm{z}_v(G_j) = 0$ and v is not in any optimal zero forcing set for G_j, then $|Z \cap V_{G_j - v}| \geq \mathrm{Z}(G_j - v)$.*

Proof. Let Z be a zero forcing set for G. Clearly $(Z \cap (G_j - v)) \cup \{v\}$ must be a zero forcing set of G_j. Suppose $\mathrm{z}_v(G_j) = 1$. Then $\mathrm{Z}(G_j) \leq |Z \cap V_{G_j - v}| + 1$, so $|Z \cap V_{G_j - v}| \geq \mathrm{Z}(G_j) - 1 = \mathrm{Z}(G_j - v)$. Suppose $\mathrm{z}_v(G_j) = 0$ and v is not in an optimal zero forcing set for G_j. Then $\mathrm{Z}(G_j) \leq |Z \cap V_{G_j - v} \cup \{v\}|$ with equality only if v is in an optimal zero forcing set. Hence $\mathrm{Z}(G_j) < |Z \cap V_{G_j - v} \cup \{v\}|$, so $|Z \cap V_{G_j - v}| \geq \mathrm{Z}(G_j) = \mathrm{Z}(G_j - v)$. □

The definition and characterization which follow will be used in the main theorem of the section which gives a formula for the zero spread of a cut-vertex. We will use the fact that $\mathrm{z}_v(G) = 1$ if and only if there exists an optimal chain set of G that contains v as a singleton [8].

Definition 2.3.6. Let G be a graph and $v \in V_G$. The graph $G - v$ is called *optimal chain set extendible* to v if there exists an optimal chain set of G which differs from an optimal chain set of $G - v$ only in that one chain of G is a chain of $G - v$ with v at the end.

Lemma 2.3.7. *Let G be a graph and $v \in V_G$. The graph $G - v$ is optimal chain set extendible to v if and only if $z_v(G) = 0$ and v is in an optimal zero forcing set for G.*

Proof. Suppose $G - v$ is optimal chain set extendible to v. Then there are optimal chain sets of G and of $G - v$ which are the same size. Since the size of an optimal chain set of a graph is the zero forcing number, $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v) = 0$. Also, v is in an optimal zero forcing set which is a reversal in G of the optimal zero forcing set used to construct the chains for $G - v$.

Suppose $z_v(G) = 0$ and v is in an optimal zero forcing set Z for G. Construct an optimal chain set for G from Z. Now v must perform a force, otherwise it is a singleton so $\mathrm{z}_v(G) = 1$, a contradiction. By considering each forcing chain in reverse order, it is clear that $G - v$ is optimal chain set extendible to v. □

With the above preliminary results, we are now ready to give a formula for the zero spread of a cut-vertex.

Theorem 2.3.8. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{z_v(G_j)\}$, and t denote the number of connected components of $G - v$ which are optimal chain set extendible to v. Then*

$$z_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t \leq 1 \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1 \end{cases}$$

Proof. The proof will be completed by considering each of the cases.

Case 1: Suppose $m = 1$. The bounds on the zero spread of a vertex gives $\mathrm{z}_v(G) \leq 1$ and Corollary 2.3.2 gives $\mathrm{z}_v(G) \geq 1$.

Case 2: Suppose $m = -1$. Corollary 2.3.4 gives $\mathrm{z}_v(G) \leq -1$ and the bounds on the zero spread of a vertex gives $\mathrm{z}_v(G) \geq -1$.

Case 3: Suppose $m = 0$ and $t \geq 2$. Without loss of generality, let $G_1 - v$ and $G_2 - v$ be optimal chain set extendible to v. Now v is in an optimal zero forcing set Z_1 of G_1. Also, v must perform a force, for if not, then v is a singleton in an optimal zero forcing set of G_1 so $\mathrm{z}_v(G_1) = 1$, a contradiction. There exists another optimal zero forcing set Z_1' of G_1 found by reversing the maximal forcing chains. Since $v \in Z_1$ and v performs a force, $v \notin Z_1'$ and there is a chain set such that v does not perform a force. Let Z_2 be an optimal zero forcing set of G_2 with $v \in Z_2$ and for $i = 3, \dots, k$, let Z_i be an optimal zero forcing set of $G_i - v$. Let $Z = Z_1' \cup (Z_2 \setminus \{v\}) \cup_{i=3}^{k} Z_i$. Now $Z \cap V_{G_1}$ can force all of G_1 with v not used to force. Then for $i = 3, \dots, k$, $Z \cap V_{G_i}$ can force all of $G_i - v$ with v not used to force. Then $(Z \cap V_{G_2}) \cup \{v\}$ can force all of $G_2 - v$. Thus Z is a zero forcing set of G, so $\mathrm{Z}(G) \le |Z| = \sum_{i=1}^{k} |Z_i| - 1 = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) - 1$. Since v is a cut-vertex, this gives $-1 \ge \mathrm{Z}(G) - \sum_{i=1}^{k} \mathrm{Z}(G_i - v) = \mathrm{Z}(G) - \mathrm{Z}(G - v) = \mathrm{z}_v(G)$. By the bounds on the zero spread of a vertex, $\mathrm{z}_v(G) \ge -1$. Hence $\mathrm{z}_v(G) = -1$.

Case 4: Suppose $m = 0$ and $t \le 1$. Corollary 2.3.4 gives $\mathrm{z}_v(G) \le 0$, so the lower bound remains to be shown. Let Z be an optimal zero forcing set of G with $v \notin Z$. Note that $\mathrm{Z}(G) = |Z| = \sum_{i=1}^{k} |Z \cap V_{G_i - v}|$ and $\mathrm{Z}(G - v) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v)$, so it suffices to show

$$\sum_{i=1}^{k} |Z \cap V_{G_i - v}| \ge \sum_{i=1}^{k} \mathrm{Z}(G_i - v) \tag{2.1}$$

Now there is at most one i, $1 \le i \le k$, such that $\mathrm{z}_v(G_i) = 0$ and v is in an optimal zero forcing set for G_i, so without loss of generality suppose for $i = 2, \dots, k$, either $\mathrm{z}_v(G_i) = 1$ or that $\mathrm{z}_v(G_i) = 0$ but v is not in any optimal zero forcing set for G_i. By Lemma 2.3.5, $|Z \cap V_{G_i - v}| \ge \mathrm{Z}(G_i - v)$ for $i = 2, \dots, k$. If $|Z \cap V_{G_1 - v}| \ge \mathrm{Z}(G_1 - v)$, then (2.1) is clearly satisfied.

Suppose $|Z \cap V_{G_1 - v}| \le \mathrm{Z}(G_1 - v) - 1$. Then $Z \cap V_{G_1 - v}$ is not a zero forcing set of $G_1 - v$. However, $(Z \cap V_{G_1 - v}) \cup \{v\}$ must be a zero forcing set of G_1. Therefore, since $\mathrm{z}_v(G_1) \ge 0$,

$$\mathrm{Z}(G_1 - v) \le \mathrm{Z}(G_1) \le |(Z \cap V_{G_1 - v}) \cup \{v\}| = |Z \cap V_{G_1 - v}| + 1 \le \mathrm{Z}(G_1 - v)$$

so $|Z \cap V_{G_1 - v}| = \mathrm{Z}(G_1 - v) - 1$. Also, there must be $j \ne 1$, $u \in V_{G_j - v}$, and $w \in V_{G_1 - v}$ such that $u \to v \to w$. Then u is at the end of a forcing chain in $G_j - v$. Since $G_j - v$ is not optimal chain set extendible to v, $|Z \cap V_{G_j - v}| \ge Z(G_j - v) + 1$. Hence $|Z \cap V_{G_1 - v}| + |Z \cap V_{G_j - v}| \ge \mathrm{Z}(G_1 - v) - 1 + Z(G_j - v) + 1 = \mathrm{Z}(G_1 - v) + Z(G_j - v)$. Applying Lemma 2.3.5 for $i \ne 1, j$, (2.1) is satisfied. □

Example 2.3.9. For the graph G in Figure 2.2, v is a cut-vertex, G_1 and G_2 are each the complete bipartite graph $K_{7,8}$, and $G_1 - v$ and $G_2 - v$ are each the complete bipartite graph

$K_{7,7}$. The best current program for computing zero forcing number takes 25 seconds (on a 2007 MacBookPro) to find $\mathrm{Z}(G)$. To find $\mathrm{Z}(G_1)$, $\mathrm{Z}(G_2)$, $\mathrm{Z}(G_1 - v)$, and $\mathrm{Z}(G_2 - v)$, the same program on the same machine took .05 seconds, 500 times faster than computing for G. Theorem 2.3.8 can be used to find $\mathrm{Z}(G)$ from $\mathrm{Z}(G_1)$, $\mathrm{Z}(G_2)$, $\mathrm{Z}(G_1 - v)$, and $\mathrm{Z}(G_2 - v)$.

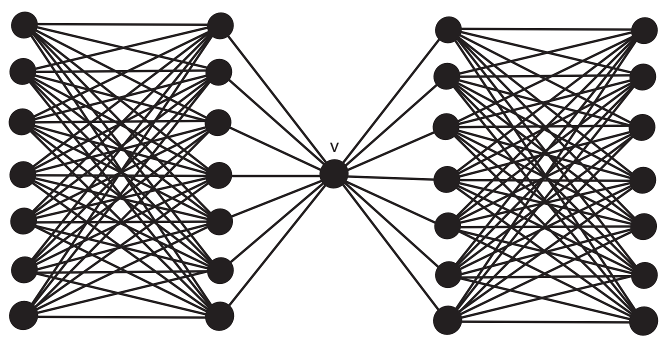

Figure 2.2 A graph with a cut-vertex.

2.4 Zero forcing number for unicyclic graphs

In [9], an algorithm is given which computes $\mathrm{P}(G)$ for any tree or unicyclic graph G. Additionally, in [10], it was proven that for any tree T, $\mathrm{P}(T) = \mathrm{Z}(T)$. Because of this result, the algorithm computes the zero forcing number for any tree. In this section, we prove that for any unicyclic graph G, $\mathrm{P}(G) = \mathrm{Z}(G)$ so the algorithm noted above can be used to compute the zero forcing number for any unicyclic graph.

Let C_n be an n-cycle and let $U \subseteq V_{C_n}$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun.* The term *segment* of H will refer to any maximal subset of consecutive vertices in U. The segments of H will be denoted $U_1, \ldots, U_t$. For a partial n-sun, H, with segments $U_1, \ldots, U_t$, $\mathrm{P}(H) = \max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil \right\}$ [4]. We prove that for a partial n-sun, the zero forcing number equals the path cover number.

Theorem 2.4.1. *Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Then*

$$\mathrm{Z}(H) = \max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil \right\}.$$

Proof. Let H be a partial n-sun with segments $U_1, \ldots, U_t$. Since $\mathrm{P}(G) \leq \mathrm{Z}(G)$ for any graph G, $\mathrm{Z}(H) \geq \max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil\right\}$. Displaying a zero forcing set of size $\max\left\{2, \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil\right\}$ will provide the upper bound. First a few special cases are considered.

If $t = 0$, then H is a cycle and any two consecutive vertices make a zero forcing set.

If $t = 1$, and $|U_1| = 1$, then the degree 1 vertex and either other vertex adjacent to the degree 3 vertex make a zero forcing set.

If $t = 1$, and $|U_1| = 2$, then the two vertices of degree 1 make a zero forcing set.

Now assume that there is at least one segment and if there is only one, it has size at least 3. Note that this implies $\sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil \geq 2$. Suppose each segment is of even order. Let the segments be numbered in the clockwise direction. Let Z' denote the set of vertices obtained as follows: for each segment, select every other leaf vertex starting with the second. Now $|Z'| = \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$. Construct Z from Z' by removing the last leaf vertex of H_t from Z' and replacing it with the first leaf vertex of H_1. Then Z is a zero forcing set with size $\sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$.

Now assume G has k odd sized segments for some $1 \leq k \leq t$. Create an induced subgraph G' from G by deleting the first leaf vertex from each odd sized segment. For each odd component U_i in G, let U_i' denote the resulting even component in G'. Now $|G| - |G'| = k$, so $Z(G) \leq Z(G') + k$ by Corollary 2.1.2. Also G' has no segments of odd size, so by the above argument, $\mathrm{Z}(G') = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i'|}{2} \right\rceil$. Hence $\mathrm{Z}(G) \leq \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i'|}{2} \right\rceil + k = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i-1|}{2} \right\rceil + k = \sum_{i=1;|U_i|\text{even}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil + \sum_{i=1;|U_i|\text{odd}}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil - k + k = \sum_{i=1}^{t} \left\lceil \frac{|U_i|}{2} \right\rceil$. □

If there are at least two components of the graph $G - v$ which are paths, each joined to v in G at only one endpoint, then vertex v is called *appropriate*. A vertex v is called a *peripheral leaf* if v is adjacent to only one other vertex u, and u is adjacent to no more than two vertices. The *trimmed form* of a graph G is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible. The trimmed form of a graph is unique [4]. The following theorems and remarks describe the consequences on the zero forcing number after applying a "trimming" operation.

These consequences will be compared to other consequences of the operations, particularly related to unicyclic graphs, to conclude the main result of the section.

Remark 2.4.2. *If v is an appropriate vertex, then v is a cut-vertex and the case of Theorem 2.3.8 with $m = 0$ and $t \geq 2$, or the case $m = -1$ applies so* $\mathrm{Z}(G - v) = \mathrm{Z}(G) + 1$.

Remark 2.4.3. *If P is an isolated path in G, then Observation 2.2.1 gives* $\mathrm{Z}(G - V_P) = \mathrm{Z}(G) - 1$.

Remark 2.4.4. *If v is a peripheral leaf then v and its neighbor must be in the same maximal forcing chain for any optimal chain set, so* $\mathrm{Z}(G - v) = \mathrm{Z}(G)$.

Theorem 2.4.5. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then* $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1$.

Proof. The proof follows from the uniqueness of the trimmed form and Remarks 2.4.2, 2.4.3, and 2.4.4. □

An example will be given at the end of this section which illustrates the use of the above theorem. Here we will continue to progress toward the main result of this section. If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{P}(G) = \mathrm{P}(\breve{G}) + n_2 - n_1$ [4]. The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun [4].

Theorem 2.4.6. *Let $G = (V_G, E_G)$ be a unicyclic graph. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

Proof. Let $\breve{G}$ the unique trimmed form of the unicyclic graph G resulting from a sequence consisting of n_1 appropriate vertex deletions, n_2 isolated path deletions, and n_3 peripheral leaf deletions. Then $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1 = \mathrm{P}(\breve{G}) + n_2 - n_1 = \mathrm{P}(G)$. □

Example 2.4.7. In Figure 2.3, there is unicylic graph G and its trimmed form $\breve{G}$. A possible order of trimming operations is as follows: Delete the peripheral leaf v_1. Delete appropriate vertex v_4 then the two isolated paths of size one, v_2 and v_3. Delete peripheral leaf v_5. Delete peripheral leaf v_6. Delete appropriate vertex v_9 then the two isolated paths of size one, v_7 and v_8. Delete peripheral leaf v_{10}. Delete appropriate vertex v_{11} then the three isolated paths of size one, v_{12}, v_{13}, and v_{14}. The trimmed form $\breve{G}$ (see graph on right in Figure 2.3) was obtained from G (see graph on left in Figure 2.3) by deleting $n_1 = 3$ appropriate vertices, $n_2 = 7$ isolated paths, and $n_3 = 4$ peripheral leaves. The trimmed form $\breve{G}$ is a partial n-sun with segments of sizes 1, 2, and 3, so by Theorem 2.4.1, $\mathrm{Z}(\breve{G}) = 4$. Theorem 2.4.5 gives $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1 = 4 + 7 - 3 = 8$.

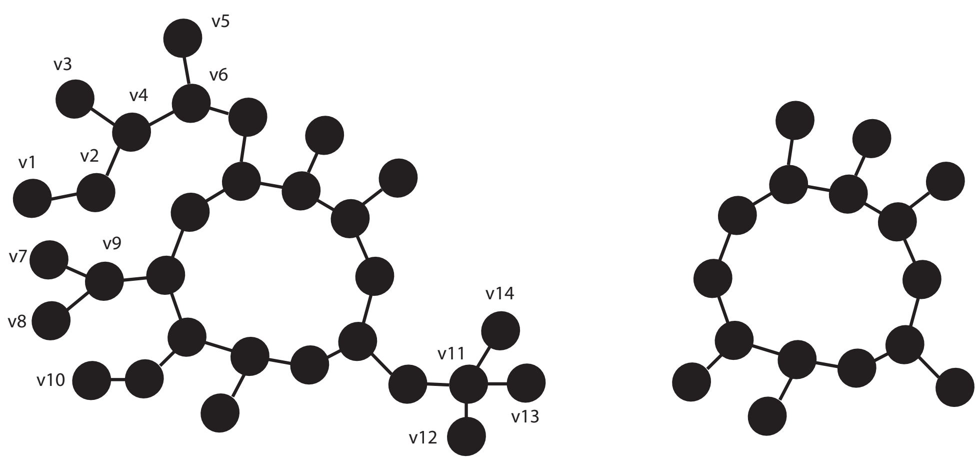

Figure 2.3 A unicyclic graph and its trimmed form.

2.5 Conclusions and open questions

We have characterizations for graphs G with zero forcing number 1, 2, $|G|-1$, and $|G|-2$.

Question 2.5.1. *Can either the linear algebraic or graph theoretic proof techniques used for proving the characterizations listed above be used to characterize graphs G with zero forcing number 3 or $|G|-3$?*

We have proved a formula for the zero spread of a cut-vertex, which allows the zero forcing number of a graph G with a cut-vertex v to be calculated in terms of the zero forcing numbers of the connected components of $G - v$.

Question 2.5.2. *Can the cut-vertex result be generalized to cut sets of size two to be used for computing zero forcing number of 2-connected graphs?*

We know that for any graph G, $\mathrm{Z}(G) \geq \mathrm{P}(G)$, and for trees and unicyclic graphs, $\mathrm{Z}(G) = \mathrm{P}(G)$.

Question 2.5.3. *For what other families of graphs does* $\mathrm{Z}(G) = \mathrm{P}(G)$*?*

Bibliography

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.

[2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433: 401–411, 2010.

[3] F. Barioli, S. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst, B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *Electronic Journal of Linear Algebra*, 18: 126–145, 2009.

[4] F. Barioli, S. Fallet, L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra and its Applications*, 409: 13–31, 2005.

[5] D. Burgarth, V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters*. 99: 100501, 2007.

[6] D. Burgarth, K. Maruyama. Indirect Hamiltonian identification through a small gateway. arXiv:0903.0612v1 [quant-ph], 2009.

[7] S. Butler, L. DeLoss, J. Grout, H.T. Hall, T. McKay, J. Smith, G. Tims. Minimum Rank Library (*Sage* programs for calculating bounds on the minimum rank of a graph, and for computing zero forcing parameters). Available at http://sage.cs.drake.edu/home/pub/67/. For more information contact Jason Grout at jason.grout@drake.edu. ISU minimum rank program.

[8] C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, D.D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. Under review; available at `http://orion.math.iastate.edu/lhogben/research/ZspreadISU.pdf`.

[9] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*. 426: 558–582, 2007.

[10] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications*, 432: 1961–1974, 2010.

[11] L.-H. Huang, G.J. Chang, and H.-G. Yeh. On minimum rank and zero forcing sets of a graph. *Linear Algebra and its Applications*. 432: 2961–2973, 2010.

[12] C.R. Johnson, R. Loewy, P.A. Smith. The graphs for which the maximum multiplicity of an eigenvalue is two. *Linear and Multilinear Algebra*. 57: 713–736, 2009.

[13] S. Severini. Nondiscriminatory propogation on trees. *Journal of Physics A: Mathematical and Theoretical* 41: 482002, 2008.

CHAPTER 3. ZERO FORCING NUMBER, PATH COVER NUMBER, AND MAXIMUM NULLITY OF CACTI

A paper in preparation for submission to *Involve, a Journal of Mathematics*

Darren D. Row

Abstract

The zero forcing number of a graph is the minimum size of a zero forcing set. This parameter is useful in the minimum rank/maximum nullity problem, as it gives an upper bound to the maximum nullity. The path cover number of a graph is the minimum size of a path cover. Results for comparing the parameters are presented, with equality of zero forcing number and path cover number shown for all cacti and equality of zero forcing number and maximum nullity for a subset of cacti.

3.1 Introduction

Throughout this paper, a *graph* $G = (V_G, E_G)$ will mean a simple (no loops, no multiple edges) undirected graph. We will assume a finite and non-empty vertex set V_G. The edge set E_G consists of two-element subsets of vertices. If $\{x, y\} \in E_G$, we say x and y are *neighbors* or x and y are *adjacent*, and write $x \sim y$.

The zero forcing number of a graph was introduced in [1] and the related terminology was developed in [2], [3], and [9]. Referring to it as the graph infection number, physicists have used this parameter in studying quantum systems control [6], [7], and [13]. Consider a black and white vertex coloring of a graph G. From the initial coloring, vertices change color according to the *color-change rule*: If v is the only white neighbor of a black vertex u, then change the color of v to black. Applying the color-change rule to u to change the color of v, we say u *forces* v and write $u \to v$. Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color-change rule is applied until no more changes are possible. If the set Z of vertices initially colored black has derived set that is all the vertices of G, we say Z is a *zero forcing set* for G. A zero forcing set with the minimum number of vertices is called an *optimal zero forcing set*, and this minimum size of a zero forcing set for a graph G is the *zero forcing number* of the graph, denoted $\mathrm{Z}(G)$.

The *path cover number* $\mathrm{P}(G)$ of a graph G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths.

An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$.

Given a graph G, the *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *minimum rank of* G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$ and the *maximum nullity of* G is $\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$. Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices in G. Because of this relationship, finding the value of one of these two parameters for a graph is equivalent to finding the value for both.

Following are theorems relating the zero forcing number to path cover number and maximum nullity of a graph.

Theorem 3.1.1. [9] *For any graph* G, $\mathrm{P}(G) \leq \mathrm{Z}(G)$.

Theorem 3.1.2. [1] *For any graph* G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

It is well known that if G is a tree then $\mathrm{P}(G) = \mathrm{Z}(G)$ [1] and $\mathrm{P}(G) = \mathrm{M}(G)$ [11], so the three parameters are equal.

In this paper, we compare the graph parameters $\mathrm{Z}(G)$, $\mathrm{P}(G)$, and $\mathrm{M}(G)$. In Section 3.2, we present the effect on the parameters after the deletion of a single vertex or the deletion of a single edge. These (mostly known) results will be utilized in later sections. The main result of Section 3.3 is equality of zero forcing number and path cover number for cacti. In section 3.4, we prove zero forcing number is equal to maximum nullity for a restricted family of cacti. Section 3.5 contains a general summary of the results and suggestions for further research. Here we present additional terminology, notation, and theorems that will be used.

For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of maximal forcing chains for a chronological list of forces is called the *chain set* of the chronological list of forces, and an *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton.* For a zero forcing set Z, a *reversal* of Z is the set of vertices which are last in the forcing chains in the chain set of some chronological list of forces [2].

Theorem 3.1.3. [2] *If Z is a zero forcing set of G then so is any reversal of Z.*

Observation 3.1.4. *If Z' is a reversal of Z, then $|Z'| = |Z|$. In particular, if Z is an optimal zero forcing set, then a reversal Z' of Z is also an optimal zero forcing set.*

A vertex v is called *terminal* if it is the endpoint of a path in some minimum path cover. The vertex is called *doubly terminal* if it is in a path by itself in some minimum path cover, and is called *simply terminal* if it is terminal but not doubly terminal.

For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For $e \in E_G$, the subgraph $(V_G, E_G \setminus \{e\})$ will be denoted by $G - e$.

A graph is called *connected* if any two vertices are linked by a path. If a graph is not connected, we say it is *disconnected.* The maximal connected subgraphs of a graph are called the *components* of the graph. If the graph $G - v$ has more connected components than G, then v is called a *cut-vertex* of G. Similarly, a *cut-edge* of a graph is one such that its deletion increases the number of connected components.

3.2 Edge spread and vertex spread

We present a number of (mostly known) results which will be used in later sections. They are grouped and formatted in such a way as to emphasize commonality between the types of results for the different parameters.

3.2.1 Edge spread

In this subsection, we consider the effects on zero forcing number, path cover number, and maximum nullity when deleting a single edge from a graph. For a graph G and an edge e of G, the *rank edge spread* of e in G is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G-e)$, the *null edge spread* of e in G is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G-e)$, and the *zero edge spread* of e in G is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G-e)$ [8]. Here we make an analogous definition concerning change in path cover number when deleting an edge.

Definition 3.2.1. The *path edge spread* of e in G is $\mathrm{p}_e(G) = \mathrm{P}(G) - \mathrm{P}(G-e)$.

First we present the bounds on the zero edge spread and path edge spread and attempt to characterize edges with a given edge spread value.

Theorem 3.2.2. [8] *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \le \mathrm{z}_e(G) \le 1$. If $\mathrm{z}_e(G) = 1$, then there exists an optimal chain set such that e is not an edge in any chain.*

Theorem 3.2.3. *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \le \mathrm{p}_e(G) \le 1$. If $\mathrm{p}_e(G) = 1$, then there exists a minimum path cover such that v and w are not in the same path.*

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G. Consider a minimum path cover of G. If v and w are not covered by the same path, then this path cover of G is also a path cover of $G - e$. If v and w are covered by the same path in the path cover of G, then splitting the path into two paths will create a path cover of $G - e$. Either way, $\mathrm{P}(G-e) \le \mathrm{P}(G) + 1$ so $\mathrm{p}_e(G) \ge -1$.

Consider a minimum path cover of $G - e$. If v and w are not covered by the same path, then this path cover of $G - e$ is also a path cover of G (observe that this case cannot occur if $\mathrm{p}_e(G) = 1$). If v and w are covered by the same path in the path cover of $G - e$, there must be a vertex on the path between them. Let x be the vertex that is between v and w on the path and adjacent to v. Split the path between v and x. This is a path cover of G, but with one more than $\mathrm{P}(G - e)$ paths. In the case $\mathrm{p}_e(G) = 1$, this is a minimum path cover of G with v and w in different paths. Regardless of the path edge spread, $\mathrm{P}(G) \leq \mathrm{P}(G - e) + 1$ so $\mathrm{p}_e(G) \leq 1$. □

Theorem 3.2.4. [8] *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{z}_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain.*

Theorem 3.2.5. *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{p}_e(G) = -1$, then for every minimum path cover of G, v and w are in the same path.*

Proof. The contrapositive will be proved. Let G be a graph and $e = \{v, w\}$ be an edge of G. Suppose there is a minimum path cover of G in which v and w are not in the same path. This path cover of G is also a path cover of $G - e$, so $\mathrm{P}(G - 1) \leq \mathrm{P}(G)$. Hence $\mathrm{p}_e(G) \geq 0$. □

Theorem 3.2.5 can be viewed as a partial converse to the second statement in Theorem 3.2.3. Here we provide an example showing that the converse of the second statement in Theorem 3.2.3 is not true. This example also shows the converse of the second statement in Theorem 3.2.2 is false.

Example 3.2.6. *For the graph G shown in Figure 3.1, for $e = \{v, y\}$, $\mathrm{p}_e(G) = 0$ but v and y are not in the same path in the minimum path cover.*

We note that bounds on zero edge spread and path edge spread are the same. Under certain conditions they are comparable.

Observation 3.2.7. *Let G be a graph such that $\mathrm{P}(G) = \mathrm{Z}(G)$ and let e be an edge of G. Then*

1. $\mathrm{p}_e(G) \geq \mathrm{z}_e(G)$

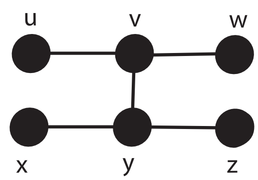

Figure 3.1 The graph G for Example 3.2.6.

2. *If* $\mathrm{z}_e(G) = 1$*, then* $\mathrm{p}_e(G) = 1$*.*

3. *If* $\mathrm{p}_e(G) = -1$*, then* $\mathrm{z}_e(G) = -1$*.*

Next we consider edge spreads when the edge is a cut-edge.

Theorem 3.2.8. [4] *Let* $e = \{v_1, v_2\}$ *be a cut-edge of a connected graph* G*. Let* G_1 *and* G_2 *be the connected components of* $G - e$ *with* $v_1 \in G_1$ *and* $v_2 \in G_2$*. Then*

$$\mathrm{r}_e(G) = \begin{cases} 0 & \textit{if and only if } \max_{i=1,2}\{r_{v_i}(G_i)\} = 2 \\ 1 & \textit{otherwise} \end{cases}$$

Corollary 3.2.9. *Let* $e = \{v_1, v_2\}$ *be a cut-edge of a connected graph* G*. Let* G_1 *and* G_2 *be the connected components of* $G - e$ *with* $v_1 \in G_1$ *and* $v_2 \in G_2$*. Then*

$$\mathrm{n}_e(G) = \begin{cases} 0 & \textit{if and only if } \min_{i=1,2}\{n_{v_i}(G_i)\} = -1 \\ -1 & \textit{otherwise} \end{cases}$$

Proof. This follows from Theorem 3.2.8 and the fact that $\mathrm{r}_e(G) + \mathrm{n}_e(G) = 0$ for any graph G and any edge e of G. □

Theorem 3.2.10. *Let* $e = \{v_1, v_2\}$ *be a cut-edge of a connected graph* G*. Let* G_1 *and* G_2 *be the connected components of* $G - e$ *with* $v_1 \in G_1$ *and* $v_2 \in G_2$*. Then*

$$\mathrm{z}_e(G) = \begin{cases} -1 & \textit{if and only if } v_i \textit{ is in an optimal zero forcing set in } G_i \textit{ for } i = 1, 2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v_1 is black. Forces can be performed in G_2 until v_2 is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$. Hence $\mathrm{z}_e(G) \leq 0$.

Suppose v_1 is an optimal zero forcing set Z_1 for G_1 and v_2 is in an optimal zero forcing set Z_2 in G_2. Let Z_1' be a reversal of Z_1. Then by Observation 3.1.4 Z_1' is an optimal zero forcing set for G_1 and there is a chronological list of forces in which v_1 does not perform a force. Let $Z = Z_1' \cup Z_2 \setminus \{v_2\}$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until all vertices of G_1 are black and v_1 has not performed a force. Now v_1 is black and v_2 is the only white neighbor of v_1, so $v_1 \to v_2$. Now all the vertices of Z_2 are black and none has performed a force, so all other vertices of G_2 can be forced black. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1 = \mathrm{Z}(G - e) - 1$. Theorem 3.2.2 gives $\mathrm{z}_e(G) \geq -1$, so $\mathrm{z}_e(G) = -1$.

Suppose now that at least one of v_1 or v_2 is not in an optimal zero forcing set for the respective component. Without loss of generality, say v_1 is not in an optimal zero forcing set for G_1. Let Z be an optimal zero forcing set for G and consider the chronological list of forces. Examine the following cases.

Case 1: Suppose $v_1 \to v_2$. Since v_1 is not in an optimal zero forcing set for G_1, v_1 forcing v_2 requires $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1) + 1$. It must also be that $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2) - 1$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$.

Case 2: Suppose $v_1 \not\to v_2$. Then $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2)$. Since v_1 is not in an optimal zero forcing set for G_1, it must be that $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1)$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$. □

Theorem 3.2.11. [4] *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2*

be the connected components of $G-e$ *with* $v_1 \in G_1$ *and* $v_2 \in G_2$*. Then*

$$\mathrm{p}_e(G) = \begin{cases} -1 & \textit{if and only if } v_i \textit{ is terminal in } G_i \textit{ for } i=1,2 \\ 0 & \textit{otherwise} \end{cases}$$

The converse of Theorem 3.2.4 is open from [8], and the converse of Theorem 3.2.5 is left open in this paper. We will show that the converses of these theorems are true for a cut-edge.

Theorem 3.2.12. *Let* $e = \{v, w\}$ *be a cut-edge of* G*. If* e *is an edge in a chain for every optimal zero forcing chain set of* G*, then* $\mathrm{z}_e(G) = -1$*.*

Proof. The contrapositive will be proved. Suppose $\mathrm{z}_e(G) \neq -1$. Then by Theorem 3.2.10, $\mathrm{z}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G-e$ with $v \in G_1$ and $w \in G_2$. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v is black. Forces can be performed in G_2 until w is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $e = \{v, w\}$ is not an edge in any chain. Also, $|Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G-e) = \mathrm{Z}(G) - \mathrm{z}_e(G) = \mathrm{Z}(G)$, so Z is an optimal zero forcing set for G. □

Theorem 3.2.13. *Let* $e = \{v, w\}$ *be a cut-edge of* G*. If* v *and* w *are in the same path for every minimum path cover of* G*, then* $\mathrm{p}_e(G) = -1$*.*

Proof. The contrapositive will be proved. Suppose $\mathrm{p}_e(G) \neq -1$. Then by Theorem 3.2.11, $\mathrm{p}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G-e$ with $v \in G_1$ and $w \in G_2$. Consider a path cover of G consisting of minimum path covers of G_1 and G_2. Then v and w are not in the same path of this path cover of G. Also, since $\mathrm{p}_e(G) = 0$, this path cover of G is minimum. □

3.2.2 Vertex spread

In this section, we consider the effects on minimum rank, maximum nullity, zero forcing number, and path cover number when deleting a single vertex from a graph. For a graph G

and a vertex v of G, the *rank spread* of v in G is $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G-v)$ [4], the *null spread* of v in G is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G-v)$ [8], the *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G-v)$ [8], and the *path spread* of v in G is $\mathrm{p}_v(G) = \mathrm{P}(G) - \mathrm{P}(G-v)$ [5].

Theorem 3.2.14. [8], [10] *For every graph G and vertex v of G, $-1 \leq \mathrm{z}_v(G) \leq 1$.*

Theorem 3.2.15. [4], [5] *For every graph G and vertex v of G, $-1 \leq \mathrm{p}_v(G) \leq 1$.*

Recall that v being contained as a singleton means it is in a forcing chain by itself in an optimal chain set, and v being doubly terminal means it is in a path by itself in a minimum path cover.

Theorem 3.2.16. [8] *Let v be a vertex of G. Then $\mathrm{z}_v(G) = 1$ if and only if there exists an optimal chain set of G that contains v as a singleton.*

Theorem 3.2.17. [5] *Let v be a vertex of G. Then $\mathrm{p}_v(G) = 1$ if and only if v is doubly terminal.*

Theorem 3.2.18. [8] *Let v be a vertex of G. If $\mathrm{z}_v(G) = -1$, then v is never in an optimal zero forcing set for G.*

Theorem 3.2.19. [5] *Let v be a vertex of G. If $\mathrm{p}_v(G) = -1$, then v is not terminal.*

The next theorems give the parameter spreads for a cut-vertex. Recall that v being simply terminal means that the path spread is zero and v is an endpoint in a minimal path cover.

Theorem 3.2.20. [4] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then*

$$\mathrm{r}_v(G) = \min\left\{\sum_{i=1}^{k} \mathrm{r}_v(G_i), 2\right\}$$

Corollary 3.2.21. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{\mathrm{n}_v(G_j)\}$, and t denote the number of the G_i's in which $\mathrm{n}_v(G_i) = 0$.*

Then

$$\mathrm{n}_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t = 1 \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1 \end{cases}$$

Proof. This follows from Theorem 3.2.20 and the fact that $\mathrm{r}_v(G) + \mathrm{n}_v(G) = 1$ for any graph G and any vertex v of G. □

Theorem 3.2.22. [12] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\}$, and t denote the number of the G_i's in which $\mathrm{z}_v(G_i) = 0$ and v is in an optimal zero forcing set. Then*

$$\mathrm{z}_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t \leq 1 \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1 \end{cases}$$

Theorem 3.2.23. [5] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{\mathrm{p}_v(G_j)\}$, and t denote the number of the G_i's in which v is simply terminal. Then*

$$\mathrm{p}_v(G) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m = 0 \text{ and } t \leq 1 \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1 \end{cases}$$

3.3 Comparing $\mathrm{Z}(G)$ and $\mathrm{P}(G)$ for cacti

A block of a graph is a maximal connected subgraph without a cut-vertex. A *cactus* is a graph in which each block is either a cycle or an edge. In other words, a cactus is a graph in which any two cycles share at most one vertex. An example of a cactus is shown in Figure 3.2. In this section, we prove $\mathrm{Z}(G) = \mathrm{P}(G)$ for any cactus G. We begin with a few preliminaries.

Theorem 3.3.1. [12] *Let G be a unicyclic graph. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

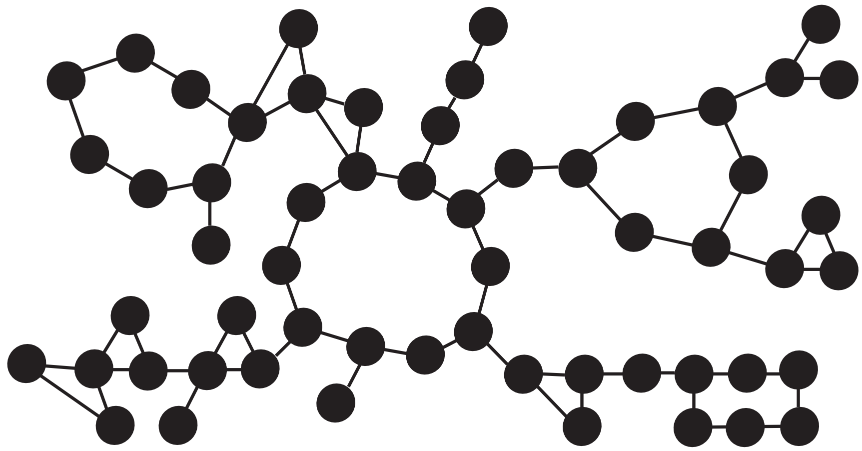

Figure 3.2 A cactus. No edge is in more than one cycle.

Lemma 3.3.2. *Let G be a graph, v a vertex in G, and H the graph constructed by appending a leaf w to v in G. Suppose $\mathrm{Z}(G) = \mathrm{P}(G)$ and $\mathrm{Z}(H) = \mathrm{P}(H)$. The vertex v is in an optimal zero forcing set for G if and only if v is terminal in G.*

Proof. Suppose v is in an optimal zero forcing set for G. An optimal chain set from this optimal zero forcing set determines a path cover of G with $\mathrm{Z}(G) = \mathrm{P}(G)$ paths and v as an endpoint of a path. Hence v is terminal.

Suppose v is terminal in G. Then $e = \{v, w\}$ is a cut edge and the graph $H' = (\{w\}, \emptyset)$ is a single isolated vertex. Therefore, w is terminal in H'. By Theorem 3.2.11, $\mathrm{p}_e(H) = -1$. By Observation 3.2.7, $\mathrm{z}_e(H) = -1$. By Theorem 3.2.10, v is in an optimal zero forcing set for G. □

Theorem 3.3.3. *Let G be a cactus. Then $\mathrm{Z}(G) = \mathrm{P}(G)$.*

Proof. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph and by Theorem 3.3.1, $\mathrm{Z}(G) = \mathrm{P}(G)$. Suppose now that for some $m \geq 2$ any cactus G with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{P}(G)$. Let G be a cactus with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \ldots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \ldots k$ is a cactus with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \ldots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{P}(G_i - v), \forall i = 1, \ldots, k$, so $\mathrm{z}_v(G_i) = \mathrm{p}_v(G_i), \forall i = 1, \ldots, k$. Therefore, $\min_{1 \le j \le k}\{\mathrm{z}_v(G_j)\} = \min_{1 \le j \le k}\{\mathrm{p}_v(G_j)\}$. For all $i = 1, \ldots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \ldots k$ and $\mathrm{Z}(H_i) = \mathrm{P}(H_i), \forall i = 1, \ldots k$. By Lemma 3.3.2, v is in an optimal zero forcing set for G_j if and only if v is terminal in G_j. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{p}_v(G_j) = 0$ and v is terminal in G_j if and only if v is simply terminal in G_j by the contrapositive of Theorem 3.2.17. Then by Theorems 3.2.22 and 3.2.23, $\mathrm{z}_v(G) = \mathrm{p}_v(G)$. Hence $\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{P}(G_i - v) + \mathrm{p}_v(G) = \mathrm{P}(G)$. □

3.4 Comparing $\mathrm{Z}(G)$ and $\mathrm{M}(G)$ for cacti

In Section 3.3 we showed equality of $\mathrm{Z}(G)$ and $\mathrm{P}(G)$ for all cacti G by utilizing Theorem 3.3.1 for the base case in the induction proof. Since it is not true that $\mathrm{Z}(G) = \mathrm{M}(G)$ for all unicyclic graphs, in this section we focus on a subset of cacti and prove $\mathrm{Z}(G) = \mathrm{M}(G)$ for each graph in this subset.

Let C_n be an n-cycle and let $U \subseteq V_{C_n}$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun*. If $U = V_{C_n}$, then H is called an n-sun. It was shown in [5] that $\mathrm{M}(H) = \mathrm{P}(H)$ for partial n-suns except for n-suns with $n > 3$ odd.

If there are at least two components of the graph $G - v$ which are paths, each joined to v in G at only one endpoint, then vertex v is called *appropriate*. A vertex v is called a *peripheral leaf* if v is adjacent to only one other vertex u, and u is adjacent to no more than two vertices. The *trimmed form* of a graph G is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible.

Theorem 3.4.1. [12] *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then* $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1$.

Theorem 3.4.2. [5] *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then* $\mathrm{M}(G) = \mathrm{M}(\breve{G}) + n_2 - n_1$.

Theorem 3.4.3. [5] *The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun.*

Observation 3.4.4. *The trimmed form of a unicyclic graph G in which at least one of the cycle vertices has only two neighbors is not an n-sun.*

The following theorem and lemma will be used in the proof of Theorem 3.4.7, the main result of this section.

Theorem 3.4.5. *Let G be a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then* $\mathrm{Z}(G) = \mathrm{M}(G)$.

Proof. Let $\breve{G}$ be the trimmed form of G. By Theorem 3.4.3 and Observation 3.4.4, $\breve{G}$ is either the empty graph or a partial n-sun, but not an n-sun with n odd and greater than three. The formulas from [5] give $\mathrm{M}(\breve{G}) = \mathrm{P}(\breve{G})$. Theorem 3.3.1 gives $\mathrm{Z}(\breve{G}) = \mathrm{P}(\breve{G})$, so $\mathrm{Z}(\breve{G}) = \mathrm{M}(\breve{G})$. Then $\mathrm{Z}(G) = \mathrm{M}(G)$ by Theorems 3.4.1 and 3.4.2. □

Lemma 3.4.6. *Let G be a graph, v a vertex in G, and H the graph constructed from G by appending a leaf w to v, then appending a leaf x to w. Suppose $\mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{Z}(H) = \mathrm{M}(H)$. The vertex v is in an optimal zero forcing set for G if and only if* $\mathrm{n}_v(G) = 0$.

Proof. By construction, $e = \{v, w\}$ is a cut edge and the graph $H' = \{\{w, x\}, \{\{w, x\}\}\}$ is a path on two vertices. Since $\mathrm{Z}(H') = \mathrm{M}(H')$, $\mathrm{z}_e(H) = \mathrm{n}_e(H)$. Also, w is in an optimal zero forcing set for H' and $n_w(H') = 0$. Then $\mathrm{n}_v(G) = 0 \Leftrightarrow \mathrm{n}_e(H) = -1 \Leftrightarrow \mathrm{z}_e(H) = -1 \Leftrightarrow v$ is in an optimal zero forcing set for G by Corollary 3.2.9 and Theorem 3.2.10. □

Here we present the main result of the section.

Theorem 3.4.7. *Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then* $\mathrm{Z}(G) = \mathrm{M}(G)$.

Proof. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors, and by Theorem 3.4.5, $\mathrm{Z}(G) = \mathrm{M}(G)$. Suppose now that for some $m \geq 2$ any cactus G in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \dots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \dots k$ is a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \dots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{M}(G_i - v), \forall i = 1, \dots, k$, so $\mathrm{z}_v(G_i) = \mathrm{n}_v(G_i), \forall i = 1, \dots, k$. Therefore, $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\} = \min_{1 \leq j \leq k}\{\mathrm{n}_v(G_j)\}$. For all $i = 1, \dots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i then appending a leaf x_i to w_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \dots k$ and $\mathrm{Z}(H_i) = \mathrm{M}(H_i), \forall i = 1, \dots k$. By Lemma 3.4.6, v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then by Theorem 3.2.22 and Corollary 3.2.21, $\mathrm{z}_v(G) = \mathrm{n}_v(G)$. Hence $\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{M}(G_i - v) + \mathrm{n}_v(G) = \mathrm{M}(G)$. □

The restrictions imposed on the cacti in this section are sufficient for $\mathrm{Z}(G) = \mathrm{M}(G)$, but are not necessary, as can be seen in the following example.

Example 3.4.8. *The graph G shown in Figure 3.3 does not satisfy the property that each odd cycle of size five or more has at least one vertex with only two neighbors, but does satisfy $\mathrm{Z}(G) = \mathrm{M}(G)$.*

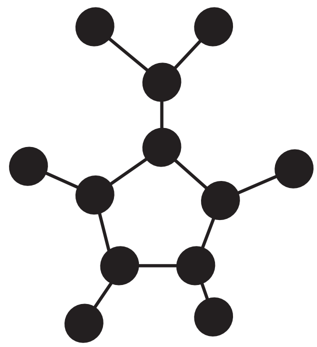

Figure 3.3 A cactus G that is not in the restricted family but which satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$.

3.5 Conclusions and open questions

We utilized cut-vertex and cut-edge results for zero forcing number, path cover number, and maximum nullity to build graphs having equality of parameters from smaller graphs having equality of the same parameters. Specifically, from knowing $\mathrm{Z}(G) = \mathrm{P}(G)$ for unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{P}(G)$ for cacti, and from $\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of cacti.

Question 3.5.1. *What other graphs have the properties that would allow a straight-forward combining while maintaining equality of some parameters?*

Question 3.5.2. *What are the necessary conditions for a cactus to satisfy* $\mathrm{Z}(G) = \mathrm{M}(G)$*?*

The converse of Theorem 3.2.4 is open from [8]. We proved the converse holds if e is a cut-edge. We also proved the converse of 3.2.5 holds for a cut-edge, but not in general.

Question 3.5.3. *Is the converse of Theorem 3.2.5 true? That is, if v and w are in the same path for every minimum path cover of G, does* $\mathrm{p}_e(G) = -1$ *where* $e = \{v, w\}$*?*

In general, v being in an optimal zero forcing set does not imply it being terminal, nor does v being terminal imply it being in an optimal zero forcing set, as evidenced by Examples 3.5.5 and 3.5.6 below. With the hypothesis that $\mathrm{Z}(G) = \mathrm{P}(G)$, we do get v in an optimal zero forcing set implying v terminal, as can be seen in the first part of the proof for Lemma 3.3.2 where the graph H is not used. The hypothesis about H is needed in Lemma 3.4.6 (see Example 3.5.7).

Question 3.5.4. *Is the graph H from the hypothesis of Lemma 3.3.2 necessary for the conclusion? For a graph G with $\mathrm{Z}(G) = \mathrm{P}(G)$, does vertex v terminal imply v is in on optimal zero forcing set?*

Example 3.5.5. *For the graph G shown in Figure 3.4, v is a cut-vertex. Now both $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$ are K_4, so $\mathrm{z}_v(G[\{v, w_1, w_2, w_3\}]) = \mathrm{z}_v(G[\{v, w_4, w_5, w_6\}]) = 1$ and v is simply terminal in $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$. Hence $\mathrm{z}_v(G) = 1$ and $\mathrm{p}_v(G) = -1$ by Theorems 3.2.22 and 3.2.23. Therefore, v is in an optimal zero forcing set but not terminal by Theorems 3.2.16 and 3.2.19.*

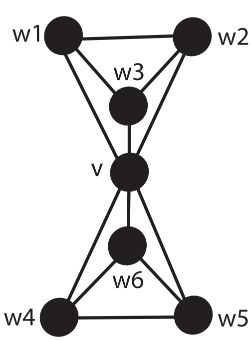

Figure 3.4 The graph G for Example 3.5.5.

Example 3.5.6. *For the graph G shown in Figure 3.5, $\mathrm{Z}(G - v) = 5$ by [1]. By Theorem 3.2.14, $\mathrm{Z}(G) \geq 4$ and $\{w_2, w_3, w_5, w_6\}$ is a zero forcing set, so $\mathrm{Z}(G) = 4$. The graph $G - v$ is not a path, so $\mathrm{P}(G - v) \geq 2$ and $\{(w_1, w_2, w_3, w_4, w_5), (w_6, w_7, w_8, w_9, w_{10})\}$ is a path cover for $G - v$. Therefore, $\mathrm{P}(G - v) = 2$. By Theorem 3.2.15, and considering G is not a path, $2 \leq \mathrm{P}(G) \leq 3$. To show $\mathrm{P}(G) \neq 2$, attempt to cover G with two induced paths and consider w_5. If w_5 was in a path by itself, the other eight vertices cannot be cover with a single induced path, so w_5 has to be in a path with other vertices. Since the three neighbors of w_5 are all neighbors* *of each other,* w_5 *has to be an endpoint of an induced path. Consider which neighbor is in the path with* w_5*. If* w_1 *is with* w_5*, then* w_2 *and* w_6 *have to be in the other path, then* v*,* w_3*, and* w_7 *have to be with* w_5 *and* w_1*, then* w_4 *and* w_8 *have to be with* w_2 *and* w_6*, but* $G[\{w_2, w_4, w_6, w_8\}]$ *is not a path. If* w_2 *is with* w_5*, then* w_1 *and* w_6 *have to be in the other path, then* v *has to be with* w_5 *and* w_2*, then* w_3 *has to be with* w_1 *and* w_6*, then* w_7 *has to be with* w_5*,* w_2*, and* v*, but* $G[\{v, w_2, w_5, w_7\}]$ *is not a path. If* w_6 *is with* w_5*, then* w_1 *and* w_2 *have to be in the other path, then* v *has to be with* w_5 *and* w_6*, then* w_3 *has to be with* w_1 *and* w_2*, then* w_7 *has to be with* w_5*,* w_6*, and* v*, but* $G[\{v, w_5, w_6, w_7\}]$ *is not a path. So* $\mathrm{P}(G) \geq 3$*. Hence* $\mathrm{z}_v(G) = -1$ *and* $\mathrm{p}_v(G) = 1$*. Therefore,* v *is terminal but never in an optimal zero forcing set by Theorems 3.2.17 and 3.2.18.*

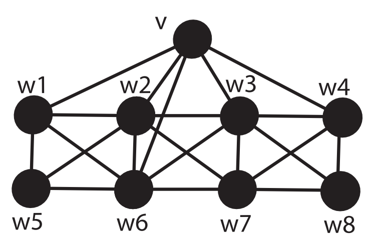

Figure 3.5 The graph G for Example 3.5.6.

Example 3.5.7. *For the graph* G *shown in Figure 3.6,* $\mathrm{Z}(G) = \mathrm{M}(G)$ *and* $\mathrm{n}_v(G) = 0$*, but* v *is not in an optimal zero forcing set for* G*.*

Bibliography

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander

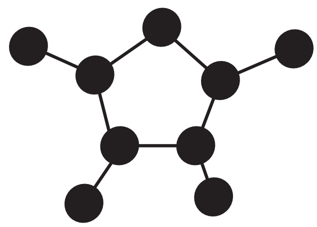

Figure 3.6 The graph G for Example 3.5.7.

Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.

[2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433: 401–411, 2010.

[3] F. Barioli, S. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst, B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *Electronic Journal of Linear Algebra*, 18: 126–145, 2009.

[4] F. Barioli, S. Fallat, L. Hogben. Computation of minimal rank and path cover number for certain graphs. *Linear Algebra and its Applications*, 392: 289–303, 2004.

[5] F. Barioli, S. Fallet, L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra and its Applications*, 409: 13–31, 2005.

[6] D. Burgarth, V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters*. 99: 100501, 2007.

[7] D. Burgarth, K. Maruyama. Indirect Hamiltonian identification through a small gateway. arXiv:0903.0612v1 [quant-ph], 2009.

[8] C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, D.D. Row. Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. Under review; available at `http://orion.math.iastate.edu/lhogben/research/ZspreadISU.pdf`.

[9] L. Hogben. Minimum rank problems. *Linear Algebra and its Applications*, 432: 1961–1974, 2010.

[10] L.-H. Huang, G.J. Chang, and H.-G. Yeh. On minimum rank and zero forcing sets of a graph. *Linear Algebra and its Applications*. 432: 2961–2973, 2010.

[11] C.R. Johnson, A. Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra* 46: 139–144, 1999.

[12] D.D. Row. Results for improving computation of zero forcing number. Submitted to *Linear Algebra and its Applications*.

[13] S. Severini. Nondiscriminatory propogation on trees. *Journal of Physics A: Mathematical and Theoretical* 41: 482002, 2008.

CHAPTER 4. GENERAL CONCLUSIONS

4.1 General Discussion

Previously known results concerning minimum rank, maximum nullity, path cover number, and zero forcing number were presented in Section 1.3. Results obtained in this dissertation complete, complement, and extend those results. Topics studied in this dissertation include characterizations, vertex and edge spreads, cut-vertex and cut-edge reduction, and equality of parameters for minimum rank, maximum nullity, zero forcing number, and path cover number.

Characterizations for graphs having very high or very low minimum rank (hence very low or very high, respectively, maximum nullity) were known. Graph characterizations for very high and very low zero forcing number are given in Section 2.2.

Vertex and edge spreads of minimum rank, maximum nullity, and zero forcing number were studied previously. Also path spread for a vertex was considered. In Section 3.2, results for path edge spread are given.

Cut-vertex reduction and cut-edge reduction for minimum rank and path cover number were already known. In Sections 2.3 and 3.2, cut-vertex and cut-edge reduction theorems are given for zero forcing number.

Equality of parameters for some families of graphs have been proved. Equality of zero forcing number and path cover number for unicyclic graphs is shown in Section 2.4. For cacti, equality of zero forcing number and path cover number is proved in Section 3.3, while equality of zero forcing number and maximum nullity was shown for a restricted family of cacti in Section 3.4.

4.2 Recommendations for Future Research

An open problem of continued interest is graphs satisfying equality of parameters. Specific questions related to the topics of this dissertation are included in Sections 2.5 and 3.5. The last paragraph of Section 1.3 mentions variations of the minimum rank/maximum nullity problem and related topics. Questions related to characterizations, vertex spreads, edge spreads, and equality of parameters under these variations are interesting areas to consider for future research.

ACKNOWLEDGEMENTS

I thank my major professor, Dr. Leslie Hogben, for taking me on as her student, working with me to achieve my goals on my time-line, and suggesting questions to consider in my research. I thank my committee members: Drs. Maria Axenovich, Ryan Martin, Alexander Roitershtein, and Sung-Yell Song for their efforts and contributions. I thank the Iowa State University Department of Mathematics faculty and staff for their assistance in my teaching, studying, and researching for the past three years.

I thank God for granting me the grace to recognize and understand enough of His beauty and order in mathematics to conduct this research.