Directions: Read the entire test before beginning. All answers must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. Four complete correct answers will receive full credit, but you may answer all five questions if desired (best 4 scores will be used). Write each solution on a separate page. Submit solutions in the same order as the questions.

1. In the inner product space \( \mathbb{C}^2 \) with dot product: Give the requested example or a one sentence explanation why no such example exists. (Note: Assuming they are correct, very simple examples are acceptable.)

   a) Three orthonormal vectors.
   
   b) Three orthogonal vectors.
   
   c) Two vectors \( \mathbf{v}, \mathbf{w} \) such that \( \| \mathbf{v} \| = \| \mathbf{w} \| = 1 \) and \( \theta = \frac{3\pi}{4} \), where \( \theta \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \).

   Solution:

   a) No example exists since any set of orthonormal vectors is linearly independent and in \( \mathbb{C}^2 \) at most two vectors are linearly independent.
   
   b) \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \)
   
   c) \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \right\} \)

2. Let \( A \in F^{m\times m}, B \in F^{m\times n}, C \in F^{n\times m} \) and define \( M = \begin{bmatrix} A & B \\ C & I_n \end{bmatrix} \). Show that \( \det M = \det (A - BC) \).

   Solution:

   \[
   \begin{bmatrix} A & B \\ C & I_n \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ -C & I_n \end{bmatrix} \begin{bmatrix} A - BC & B \\ 0 & I_n \end{bmatrix}
   \]

   so

   \[
   \det M = (\det M)(1) = \det \begin{bmatrix} A & B \\ C & I_n \end{bmatrix} \det \begin{bmatrix} I_m & 0 \\ -C & I_n \end{bmatrix} = \det \begin{bmatrix} A - BC & B \\ 0 & I_n \end{bmatrix} = \det (A - BC).
   \]

3. Let \( A \in \mathbb{C}^{4\times 4} \) with \( \text{spec}(A) = \{1, 1, 0, -1\} \). Show that \( A \) has a principal submatrix of rank \( k \) for \( k = 1, 2, 3 \) but not for \( k = 4 \).

   Solution: Recall that for \( k = 1, 2, 3, 4 \), \( S_k(A) = S_k(1, 1, 0, -1) \) is the coefficient of \( x^{n-k} \) in the characteristic polynomial of \( A \), where \( S_k(A) \) is the sum of the \( k \times k \) principal minors and \( S_k(1, 1, 0, -1) \) is the \( k \)th symmetric function in the eigenvalues \( 1, 1, 0, -1 \). Observe that
\[ S_1(1, 1, 0, -1) = 1 + 1 + 0 + (-1) = 1, \] so \( S_1(A) \) is nonzero and \( A \) has a principal submatrix of rank 1. Similarly \( S_2(1, 1, 0, -1) = (1)(1) + (1)(0) + (1)(-1) + (1)(0) + (1)(-1) + (0)(-1) = -1 \neq 0 \) and \( S_3(1, 1, 0, -1) = (1)(1)(0) + (1)(1)(-1) + (1)(0)(-1) + (1)(0)(-1) = -1 \neq 0 \), so \( A \) has principal submatrices of ranks 2 and 3. But 0 is an eigenvalue of \( A \) so \( A \) is singular and does not have rank 4.

4. Suppose \( A, B \in \mathbb{C}^{n \times n} \) commute. Let \( \text{spec}(A) = \{ \alpha_1, \ldots, \alpha_n \} \) and \( \text{spec}(B) = \{ \beta_1, \ldots, \beta_n \} \). Prove that there exists a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that \( \text{spec}(AB) = \{ \alpha_1 \beta_{\tau(1)}, \ldots, \alpha_n \beta_{\tau(n)} \} \).

Solution: Since \( A \) and \( B \) commute, there exist \( S \) invertible and \( T_A, T_B \) upper triangular such that \( S^{-1}AS = T_A \) and \( S^{-1}BS = T_B \). Furthermore, \( S \) can be chose so that the \( k \)th diagonal entry of \( T_A \) is \( \alpha_k \). Then there is a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that the \( k \)th diagonal entry of \( T_M \) is \( \alpha_{\tau(k)} \). \( T_A T_B \) is upper triangular with diagonal entries \( \{ \alpha_1 \beta_{\tau(1)}, \ldots, \alpha_n \beta_{\tau(n)} \} \), so \( \text{spec}(T_A T_B) = \{ \alpha_1 \beta_{\tau(1)}, \ldots, \alpha_n \beta_{\tau(n)} \} \). Since \( S^{-1}ABS = T_A T_B \), \( \text{spec}(AB) = \text{spec}(T_A T_B) \).

5. Let \( A, B \in \mathbb{C}^{n \times n} \). Prove that \( m_A(B) \) is singular if and only if \( A \) and \( B \) have a common eigenvalue. (\( m_A(x) \) is the minimal polynomial of \( A \).)

Solution: Let \( \mu_1, \ldots, \mu_r \) be the distinct eigenvalues of \( A \), so \( m_A(x) = (x - \mu_1)^{\nu_1} \ldots (x - \mu_r)^{\nu_r} \). \( m_A(B) \) is singular if and only if at least one of the factors \( B - \mu_i I \) is singular, if and only if some eigenvalue \( \mu_i \) of \( A \) is also an eigenvalue of \( B \).