SOLUTIONS

1. \( A \in \mathbb{C}^{3 \times 3} \) satisfies \( x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2) = 0 \), where \( \omega = e^{\frac{2\pi i}{3}} \). The possible minimal polynomials are \( x - 1, x - \omega, x - \omega^2, (x - 1)(x - \omega), (x - 1)(x - \omega^2), (x - \omega)(x - \omega^2), x^3 - 1 \). The possible Jordan canonical forms are:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
\omega & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{bmatrix}
\]

2. Give examples in \( \mathbb{C}^{n \times n} \) of each of the following. (Hint: in each case examples exist with \( n \leq 4 \).)

(a) Two matrices with the same minimal and characteristic polynomials that are not similar to each other.

\( J_2(0) \oplus J_2(0) \) and \( J_2(0) \oplus J_1(0) \oplus J_1(0) \)

(b) A matrix with an eigenvalue that has geometric multiplicity different from its algebraic multiplicity.

\( J_2(0) \)

(c) A nondiagonal, positive definite matrix.

\[
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

(d) A normal matrix that is neither unitary, Hermitian nor skew-Hermitian.

\[
\begin{bmatrix}
2 & 1 \\
-1 & 2
\end{bmatrix}
\]

3. Let \( A \in \mathbb{C}^{n \times n} \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and singular values \( \sigma_1, \ldots, \sigma_n \). Prove

\[
\prod_{i=1}^{n} |\lambda_i| = \prod_{i=1}^{n} \sigma_i.
\]

Let \( A = U\Sigma V^* \) be the SVD of \( A \).

\[
\prod_{i=1}^{n} |\lambda_i| = |\det A| = \det U \det \Sigma \det V^* = \prod_{i=1}^{n} \sigma_i
\]

4. Let \( N \) be a normal \( n \times n \) complex matrix such that \( N^3 - 2I \) is nilpotent. Prove that \( N^3 = 2I \).

Since \( N \) is normal there exist \( U \) unitary and \( D \) diagonal such that \( N = UDU^* \).

\[
0 = (N^3 - 2I)^n = ((UDU^*)^3 - 2U^*I)^n = U(D^3 - 2I)^n U^*
\]
so \((D^3 - 2I)^n = 0\). Since \(D\) is diagonal, necessarily \(D^3 - 2I = 0\), and thus

\[
0 = U(D^3 - 2I)U^* = N^3 - 2I.
\]

Alternatively, \(N\) normal implies \(N^3\) normal implies \(N^3 - 2I\) normal, and a normal matrix is nilpotent if and only if it is 0.

5. Let \(V\) be the vector space of polynomials over \(\mathbb{C}\) of degree \(\leq n\). For \(0 \leq k \leq n\), define a linear functional \(f_k \in V^*\) by \(f_k(p) = p(k)\) for \(p \in V\). Show that \(\{f_0, \ldots, f_n\}\) is a basis for \(V^*\).

Since \(\dim V^* = \dim V = n + 1\), it is enough to show that \(\{f_0, \ldots, f_n\}\) is a linearly independent set. Suppose that \(\sum_{k=0}^{n} c_k f_k = 0\). Then \(\sum_{k=0}^{n} c_k f_k(p) = \sum_{k=0}^{n} c_k p(k) = 0\), for all polynomials \(p \in V\).

In particular, for \(j = 0, \ldots, n\), consider the degree \(j\) polynomials \(p_j\) defined by \(p_0(x) = 1, p_1(x) = x, p_j(x) = x(x-1)\cdots(x-(j-1))\), \(j \geq 2\). Then \(p_j(k) = 0\) for \(k < j\) and \(p_j(j) \neq 0\). Thus, for \(j = 0, \ldots, n\), \(\sum_{k=0}^{n} c_k p_j(k) = \sum_{k \geq j} c_k p_j(k) = 0\). It follows that \(c_n = c_{n-1} = \cdots = c_0 = 0\).

6. If \(v \in V\), then \(T v = \langle v, x \rangle y \in \text{span}(y) \subseteq V\), so \(T\) is well-defined on \(V\). For \(v, w \in V\), \(\alpha\) scalar, \(T(\alpha v + w) = \langle \alpha v + w, x \rangle y = \alpha \langle v, x \rangle y + \langle w, x \rangle y = \alpha T v + T w\), so \(T\) is linear.

For \(v, w \in V\), \(\langle T v, w \rangle = \langle \langle v, x \rangle y, w \rangle = \langle v, x \rangle \langle y, w \rangle = \langle v, \overline{\langle y, w \rangle} x \rangle\). Thus, \(T^* w = \langle y, w \rangle x = \langle w, y \rangle x\).

7. Let \(P_1, \ldots, P_k \in \mathbb{C}^{n \times n}\) satisfying \(\sum_{i=1}^{k} P_i = I\). Prove that the following are equivalent:

(a) \(P_i^2 = P_i\), \(i = 1, \ldots, k,\)
(b) \(P_i P_j = 0\), \(i \neq j,\)
(c) \(\text{rank } P_1 + \cdots + \text{rank } P_k = n\).

(b) \(\Rightarrow\) (a): Since \(\sum_{i=1}^{k} P_i = I\), it follows that for all \(j\), \(P_j^2 + \sum_{i \neq j} P_j P_i = P_j\). Thus \(P_j^2 = P_j\) as \(P_j P_i = 0\) for \(i \neq j\).

(a) \(\Rightarrow\) (c): Since \(P_i\) is a projection, \(P_i\) is similar to \(\text{diag}(1, \ldots, 1, 0, \ldots, 0)\), so \(\text{rank } P_1 = \text{trace } P_1\).

Thus, \(\sum_{i=1}^{k} \text{rank } P_i = \sum_{i=1}^{k} \text{trace } (P_i) = \text{trace } \left(\sum_{i=1}^{k} P_i\right) = \text{trace } (I) = n\).

(c) \(\Rightarrow\) (b): \(\sum_{i=1}^{k} P_i = I\) implies that \(\mathbb{C}^n = (\text{Im } P_1) + \cdots + (\text{Im } P_k)\). But \(n = \text{rank } P_1 + \cdots + \text{rank } P_k = \dim(\text{Im } P_1) + \cdots + \dim(\text{Im } P_k)\), so \(\mathbb{C}^n = (\text{Im } P_1) \oplus \cdots \oplus (\text{Im } P_k)\) and \((\text{Im } P_i) \cap (\text{Im } P_j) = \{0\}\) for \(i \neq j\).

Thus \((\text{Im } P_j) = (\text{Im } P_1 P_j) \oplus \cdots \oplus (\text{Im } P_k P_j) = (\text{Im } P_1 P_j) \oplus \cdots \oplus (\text{Im } P_k P_j)\). Thus \(\dim(\text{Im } P_i P_j) = 0\) for \(i \neq j\).

8. Let \(A\) and \(B\) be \(n \times n\) Hermitian matrices, and let \(A\) be positive definite. Show that for any \(x \in \mathbb{C}^n\),

\[
\lambda_{\min}(A^{-1}B) \leq \frac{x^* B x}{x^* A x} \leq \lambda_{\max}(A^{-1}B).
\]

(You may use the fact that for any real positive definite matrix \(M\) there exists a positive definite matrix \(S\) such that \(M = S^2\).)

Let \(M^2 = A\). Note that

\[
MA^{-1}BM^{-1} = M^{-1}BM^{-1}
\]
is Hermitian, so all eigenvalues of $A^{-1}B$ are real. By Rayleigh-Ritz, for all $x \neq 0$,

$$
\lambda_{\min}(A^{-1}B) = \lambda_{\min}(M^{-1}BM^{-1}) \leq \frac{x^*M^{-1}BM^{-1}x}{x^*x} \leq \lambda_{\max}(M^{-1}BM^{-1}) = \lambda_{\max}(A^{-1}B).
$$

Note that

$$
\frac{x^*M^{-1}BM^{-1}x}{x^*x} = \frac{y^*By}{y^*Ay}
$$

where $y = M^{-1}x$. 