A NONLINEAR TWO-PHASE STEFAN PROBLEM WITH MELTING POINT GRADIENT: A CONSTRUCTIVE APPROACH

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Abstract. We consider a one-dimensional two-phase Stefan problem, modeling a layer of solid material floating on liquid. The model includes internal heat sources, variable total mass (resulting e.g. from sedimentation or erosion), and a pressure-dependent melting point. The problem is reduced to a set of nonlinear integral equations, which provides the basis for an existence and uniqueness proof and a new numerical method. Numerical results are presented.

Introduction

In this paper, we study the temperature distribution in a thick layer of dense matter, floating on a deeper, denser layer. The materials of the two regions are polymorphs of each other. The problem of the equilibrium of the column of matter due to a disequilibrium of both the buoyancy and the thermal state is called a Stefan problem. The most familiar example of the Stefan problem is that of the history of a cake of ice floating in water after the surface of the ice has received a fresh snowfall. A geophysically derived example concerns the possibility that the Mohorovicic discontinuity is an isochemical phase transformation boundary between a denser mantle phase below and a lighter crustal phase above; the desequilibrium may arise, for example, from erosion or sedimentation at the surface or by a change in the thermal state in the mantle. The two examples differ especially in the sign of the slope of the pressure-temperature curve for the phase transformation. In this paper we assume that the pressure-temperature curve for the phase boundary interface is linear but make no specification of the sign. We shall assume that the total mass in the two regions is a continuously differentiable function of time. The elevations of the phase boundary and the free surface are to be determined as functions of time.

Concerning the large literature on the Stefan problem, we mention only some papers which we consider particularly relevant for our purpose, namely those of Friedman [2], Rubinstein [6], MacDonald and Ness [3], O'Connell and Wasserburg [5], and Mori [4]. Rubinstein provides a broadly based review of the history of the Stefan problem. Papers [3] and [5] elucidate the geophysical importance of the problem: in them, the set of partial differential equations has been integrated numerically. Mori [4] treats the numerical solution of a similar problem in great

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detail. His numerical example provides a convenient test problem for our method. Friedman's approach to Stefan problems [2] is a refinement of Rubinstein's heat potential technique; it will be followed here in converting the problem into one involving a set of integral equations.

Our problem will be formulated precisely in the next section, but first, we make two remarks. We consider here the presence of internal sources depending nonlinearly, not only on x and t, but also on the location of the phase boundary. Furthermore, the total mass of the two media in our problem is not a constant. In the particular case of constant mass, the total height of the two media is not constant in view of the difference of densities, and this fact is taken into account in our consideration. The present paper is a nonlinear analogue of [1].

The remainder of the paper is divided into five sections. In Section 1, we formulate the problem in mathematical terms. In Section 2 we represent the solution by heat potentials. Section 3 is devoted to converting the free boundary value problem into one involving a set of three nonlinear integral equations. Section 4 establishes the existence and uniqueness of the solution for small times, using the contraction principle. Section 5 gives numerical results for two test problems. The last paragraph consists of some concluding remarks about the solution and some possible extensions of the problem.

Formulation of the Problem

Let $T_1(x,t)$ be the temperature distribution in the lower layer, hereafter called region 1, and $T_2(x,t)$ be the temperature in the upper layer, called region 2; the subscripts on quantities describing the properties of the respective media are self-explanatory. The problem is considered as one-dimensional in the space variable x.

We propose to solve the following free boundary value problem (1)–(9)

$$\frac{\partial^2 T_1}{\partial x^2} = k_1^2 \frac{\partial T_1}{\partial t} + F_1, \qquad 0 < x < s(t),\ t > 0,\ k_1 > 0 \tag{1}$$

$$\frac{\partial^2 T_2}{\partial x^2} = k_2^2 \frac{\partial T_2}{\partial t} + F_2, \qquad s(t) < x < r(t),\ t > 0,\ k_2 > 0 \tag{2}$$

where $F_i = F_i(x,t,s(t))$.

The unknown functions are $s(t)$, $r(t)$, $T_1(x,t)$ and $T_2(x,t)$, which are to be determined from (1)–(2) and from the conditions to follow. Equations (1) and (2) are the equations of thermal diffusion for the two media with diffusion coefficients k_1^{-2}, k_2^{-2}. The thickness of the lower layer is $s(t)$; that of the upper layer is $r(t) - s(t)$.

Let ρ_1 and ρ_2 be the respective densities. The elevations $r(t)$ and $s(t)$ are connected by the relation describing the total mass in the two layers:

$$\rho_2[r(t) - s(t)] + \rho_1 s(t) = m(t), \quad t \geq 0. \tag{3}$$

We assume that $m(t)$ is continuously differentiable on $t \geq 0$.

Further conditions in our free boundary value problem are the following. The temperature in the two regions is continuous across the phase boundary at all times. The temperature at the phase boundary occurs so frequently that we denote it specially by $g(t)$:

$$T_1(s(t)) = T_2(s(t)) \equiv g(t), \quad t \geq 0. \tag{4a}$$

The temperature at the phase boundary must lie at all times on the pressure-temperature relation, which is — as noted — assumed to be linear.

$$g(t) = \beta_1(r(t) - s(t)) + \beta_2. \tag{4b}$$

Top and bottom temperatures are assumed to be given

$$T_1(0,t) = f(t), \tag{5}$$

$$T_2(r(t),t) = h(t). \tag{6}$$

In many cases, the temperature at the uppermost surface is a constant, which we can take as zero without loss of generality. For example, if the upper layer were ice, the surface of the ice is assumed to be at its melting point at zero pressure. If the upper layer is the earth's crust, we may take the upper boundary to be the sediments at the bottom of the oceans, for example, which we assume to be in equilibrium with the bottom water.

The last of the continuity conditions concerns the flux of heat transported across the interface between the two media; this heat flow is different on the two sides of the boundary by virtue of the influence of latent heat of transformation arising from the conversion by one phase into another as the boundary shifts. The relation is

$$\frac{ds(t)}{dt} = -\gamma_1 \frac{\partial T_1}{\partial x}(s(t),t) + \gamma_2 \frac{\partial T_2}{\partial x}(s(t),t), \tag{7}$$
$$\gamma_1 > 0, \quad \gamma_2 > 0, \quad t > 0,$$

where $\gamma = c_v/(k^2 \Delta L)$, with c_v the specific heat at constant volume, ΔL the latent heat; the diffusion coefficients k^{-2} have been introduced above.

The initial conditions are

$$\begin{aligned} s(0) &= b > 0, \\ r(0) &= c > b, \\ T_1(x,0) &= \phi_1(x), \quad 0 \leq x \leq b, \\ T_2(x,0) &= \phi_2(x), \quad b \leq x \leq c. \end{aligned} \tag{8}$$

The following regularity conditions (9) are to be satisfied by our data:

(1) $\phi_1(x)$ and $\phi_2(x)$ are C^1 in their domains of definition, $\phi_1(b) = \phi_2(b)$.
(2) $f(t)$, $h(t)$ are C^1 on $t > 0$, continuous on $t \geq 0$, $f(0) = \phi_1(0)$, $h(0) = \phi_2(c)$.
(3) $F_i(x,t,s(t))$ is continuous in the three variables jointly, with an x-derivative $F_{i,x}(x,t,s(t))$ that is continuous in the three variables and uniformly Lipschitzian with respect to the third variable $s(t)$.

By a solution of our free boundary problem (FBP) on $0 < t < b$, we mean functions $T_1(x,t)$, $T_2(x,t)$, $s(t)$ and $r(t)$ such that $T_1(x,t)$, $T_2(x,t)$ satisfy (1) and (2) respectively, $s(t)$ and $r(t)$ are related by (3), $s(t)$ satisfies (6) and such that the conditions (4) through (9) are satisfied; we impose the further condtion that $T_{1,x}(s(t),t)$, $T_{2,x}(s(t),t)$, and $T_{2,x}(r(t))$ be continuous on $0 \leq t \leq b$ (which implies by (6) the Lipschitz continuity of $s(t)$ and of $r(t)$); and finally we assume that $T_i(x,t), i = 1,2$, is so smooth as to admit representation by heat potentials as given in the following section.

Representation by Heat Potentials

Let

$$G_1(x,t,\xi,\tau) = \frac{k_1}{2\pi^{1/2}} \frac{1}{(t-\tau)^{1/2}} \times \left[\exp\left(-\frac{k_1^2(x-\xi)^2}{4(t-\tau)}\right) - \exp\left(-\frac{k_1^2(x+\xi)^2}{4(t-\tau)}\right)\right], \quad 0 < x < s(t), \quad 0 < \tau < t. \tag{10}$$

Let $T_1(\xi,\tau)$ be a solution of (1), with x, t replaced by ξ, τ. The following Green's identity holds

$$\frac{1}{k_1^2}\frac{\partial}{\partial\xi}[G_1 T_{1,\xi} - G_{1,\xi}T_1] - \frac{\partial}{\partial\tau}(G_1 T_1) = \frac{1}{k_1^2} G_1 F_1$$

Using the identity $d\xi = \dot{s}(\tau)d\tau$ (where the dot indicates differentiation with respect to the variable concerned, here and henceforth) on the melting line and integrating Green's identity for G_1 and T_1 in the domain

$$0 \le \xi \le s(\tau), \quad 0 < \epsilon \le \tau \le t-\epsilon, \quad \epsilon > 0,$$

we obtain, by letting $\epsilon \to 0$

$$\begin{aligned} T_1(x,t) = &\int_0^b \phi_1(\xi) G_1(x,t;\xi,0)\, d\xi \\ &+ k_1^{-2}\int_0^t T_{1,\xi}(s(\tau),\tau) G_1(x,t;s(\tau),\tau)\, d\tau \\ &- k_1^{-2}\int_0^t g(\tau) G_{1,\xi}(x,t;s(\tau),\tau)\, d\tau \\ &+ \int_0^t g(\tau) G_1(x,t;s(\tau),\tau)\dot{s}(\tau)\, d\tau \\ &+ k_1^{-2}\int_0^t f(\tau) G_{1,\xi}(x,t;0,\tau)\, d\tau \\ &- k_1^{-2}\int_0^t \int_0^{s(\tau)} G_1(x,t;\xi,\tau) F_1(\xi,\tau,s(\tau))\, d\xi\, d\tau, \\ &0 < x < s(t), \quad t > 0. \end{aligned} \tag{11}$$

We now turn to medium 2. Let

$$G_2(x,t;\xi,\tau) = \frac{k_2}{2\pi^{1/2}}\frac{1}{(t-\tau)^{1/2}} \exp\left(-\frac{k_2^2(x-\xi)^2}{4(t-\tau)}\right). \tag{12}$$

Once again, an identity, similar to that above, holds

$$\frac{1}{k_2^2}\frac{\partial}{\partial\xi}[G_2 T_{2,\xi} - T_2 G_{2,\xi}] - \frac{\partial}{\partial\tau}(G_2 T_2) = \frac{1}{k_2^2} G_2 F_2 \tag{13}$$

where $T_2(\xi,\tau)$ is a solution of (2) with x, t replaced by ξ, τ. Integrating (13) in the domain

$$s(\tau) \le \xi \le r(\tau), \quad 0 < \epsilon \le \tau \le t-\epsilon$$

and letting $\epsilon \to 0$, we obtain

$$\begin{aligned} T_2(x,t) = &\int_b^c \phi_2(\xi) G_2(x,t;\xi,0)\, d\xi \\ &+ k_2^{-2} \int_0^t T_{2,\xi}(r(\tau),\tau) G_2(x,t;r(\tau),\tau)\, d\tau \\ &- k_2^{-2} \int_0^t h(\tau) G_{2,\xi}(x,t;r(\tau),\tau)\, d\tau \\ &+ \int_0^t h(\tau) G_2(x,t;r(\tau),\tau)\dot{r}(\tau)\, d\tau \\ &- \int_0^t g(\tau) G_2(x,t;s(\tau),\tau)\dot{s}(\tau)\, d\tau \\ &- k_2^{-2} \int_0^t T_{2,\xi}(s(\tau),\tau) G_2(x,t;s(\tau),\tau)\, d\tau \\ &+ k_2^{-2} \int_0^t g(\tau) G_{2,\xi}(x,t;s(\tau),\tau)\, d\tau \\ &- k_2^{-2} \int_0^t \int_{s(\tau)}^{r(\tau)} G_2(x,t;\xi,\tau) F_2(\xi,\tau,s(\tau))\, d\xi\, d\tau, \\ &\qquad s(t) < x < r(t), \quad t > 0. \end{aligned} \tag{14}$$

The Integral Equations

In deriving the integral equations, we shall make use, without explicit mention, of Lemma 1 in Friedman [2] on boundary limits of singular integrals. (The term $\frac{1}{2}\rho(t)$ is replaced by $\frac{1}{2}k_1^2\rho(t)$ or $\frac{1}{2}k_2^2\rho(t)$.) We first set some notations:

$$v_1(t) = T_{1,x}(s(t),t), \quad t \ge 0, \tag{15}$$

$$v_2(t) = T_{2,x}(s(t),t), \quad t \ge 0, \tag{16}$$

$$v_3(t) = T_{2,x}(r(t),t), \quad t \ge 0. \tag{17}$$

Then, by (15), (16) and (6), we have

$$s(t) = -\gamma_1 \int_0^t v_1(\tau)\, d\tau + \gamma_2 \int_0^t v_2(\tau)\, d\tau + b. \tag{18}$$

Consider (11). This equation becomes, after differentiation with respect to x and rearranging,

$$
\begin{aligned}
T_{1,x}(x,t) = &\int_0^b \dot{\phi}_1(\xi) N_1(x,t;\xi,0)\,d\xi \\
&+ k_1^{-2} \int_0^t v_1(\tau) G_{1,x}(x,t;s(\tau),\tau)\,d\tau \\
&+ \int_0^t \dot{g}(\tau) N_1(x,t;s(\tau),\tau)\,d\tau \\
&- \int_0^t \dot{f}(\tau) N_1(x,t;0,\tau)\,d\tau \\
&+k_1^{-2} \int_0^t \big\{ N_1(x,t;s(\tau),\tau) F_1(s(\tau),\tau,s(\tau)) \\
&\qquad - N_1(s,t;0,\tau) F_1(0,\tau,s(\tau)) \big\}\,d\tau \\
&- k_1^{-2} \int_0^t \int_0^{s(\tau)} N_1(x,t;\xi,\tau) F_{1,\xi}(\xi,\tau,s(\tau))\,d\xi\,d\tau,
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
N_1(x,t;\xi,\tau) = &\frac{k_1}{2\pi^{1/2}} \frac{1}{(t-\tau)^{1/2}} \times \\
&\left[\exp\left(\frac{-k_1^2 (x-\xi)^2}{4(t-\tau)} \right) + \exp\left(\frac{-k_1^2 (x+\xi)^2}{4(t-\tau)} \right) \right].
\end{aligned}
\tag{20}
$$

We next consider (14). This equation becomes, after differentiating with respect to x and rearranging

$$
\begin{aligned}
T_{2,x}(x,t) = &\int_b^c \dot{\phi}_2(\xi) G_2(x,t;\xi,0)\,d\xi \\
&+ k_2^{-2} \int_0^t v_3(\tau) G_{2,x}(x,t;r(\tau),\tau)\,d\tau \\
&- \int_0^\tau \dot{h}(\tau) G(x,t;r(\tau),\tau)\,d\tau \\
&- k_2^{-2} \int_0^t v_2(\tau) G_{2,x}(x,t;s(\tau),\tau)\,d\tau \\
&- \int_0^\tau \dot{g}(\tau) G(x,t;s(\tau),\tau)\,d\tau \\
&+k_2^{-2} \int_0^t \big\{ F_2(r(\tau),\tau,s(\tau)) G_2(x,t;r(\tau),\tau) \\
&\qquad - F_2(s(\tau),\tau,s(\tau)) G_2(x,t;s(\tau) \big\}\,d\tau \\
&- k_2^{-2} \int_0^t \int_{s(\tau)}^{r(\tau)} G_2(x,t;\xi,\tau) F_{2,\xi}(\xi,\tau,s(\tau))\,d\xi\,d\tau.
\end{aligned}
\tag{21}
$$

Letting $x \to s(t) - 0$ in (19) gives after some rearrangements

$$
\begin{aligned}
v_1(t) =& 2\int_0^b \dot{\phi}_1(\xi) N_1(s(t),t;\xi,0)\,d\xi \\
& + 2k_1^{-2}\int_0^t v_1(\tau) G_{1,x}(s(t),t;s(\tau),\tau)\,d\tau \\
& + 2\int_0^t \dot{g}(\tau) N_1(s(t),t;s(\tau),\tau)\,d\tau \\
& - 2\int_0^t \dot{f}(\tau) N_1(s(t),t;0,\tau)\,d\tau \\
& +2k_1^{-2}\int_0^t \big\{N_1(s(t),t;s(\tau),\tau)F_1(s(\tau),\tau,s(\tau)) \\
& \qquad - N_1(s(t),t;0,\tau)F_1(0,\tau,s(\tau))\big\}\,d\tau \\
& - 2k_1^{-2}\int_0^t\int_0^{s(\tau)} N_1(s(t),t;\xi,\tau)F_{1,\xi}(\xi,\tau,s(\tau))\,d\xi\,d\tau.
\end{aligned}
\tag{22}
$$

Now let $x \to s(t) + 0$ in (21). We then get after some rearrangements:

$$
\begin{aligned}
v_2(t) =& 2\int_b^c \dot{\phi}_2(\xi) G_2(s(t),t;\xi,0)\,d\xi \\
& + 2k_2^{-2}\int_0^t v_3(\tau) G_{2,x}(s(t),t;r(\tau),\tau)\,d\tau \\
& - 2\int_0^\tau \dot{h}(\tau) G_2(s(t),t;r(\tau),\tau)\,d\tau \\
& - 2k_2^{-2}\int_0^t v_2(\tau) G_{2,x}(s(t),t;s(\tau),\tau)\,d\tau \\
& - 2\int_0^\tau \dot{g}(\tau) G_2(s(t),t;s(\tau),\tau)\,d\tau \\
& +2k_2^{-2}\int_0^t \big\{F_2(r(\tau),\tau,s(\tau))G_2(s(t);r(\tau),\tau) \\
& \qquad - F_2(s(\tau),\tau,s(\tau))G_2(s(t),t;s(\tau),\tau)\big\}\,d\tau \\
& - 2k_2^{-2}\int_0^t\int_{s(\tau)}^{r(\tau)} G_2(s(t),t;\xi,\tau)F_{2,\xi}(\xi,\tau,s(\tau))\,d\xi\,d\tau.
\end{aligned}
\tag{23}
$$

Finally let $x \to r(t) - 0$ in (21). We then get

$$\begin{aligned} v_3(t) =& 2\int_b^c \dot{\phi}_2(\xi) G_2(r(t),t;\xi,0)\, d\xi \\ & + 2k_2^{-2} \int_0^t v_3(\tau) G_{2,x}(r(t),t;r(\tau),\tau)\, d\tau \\ & - 2\int_0^\tau \dot{h}(\tau) G_2(r(t),t;r(\tau),\tau)\, d\tau \\ & - 2k_2^{-2} \int_0^t v_2(\tau) G_{2,x}(r(t),t;s(\tau),\tau)\, d\tau \\ & - 2\int_0^\tau \dot{g}(\tau) G_2(r(t),t;s(\tau),\tau)\, d\tau \\ & + 2k_2^{-2} \int_0^t \big\{ F_2(r(\tau),\tau,s(\tau)) G_2(r(t),t;r(\tau),\tau) \\ & \qquad - F_2(s(\tau),\tau,s(\tau)) G_2(r(t),t;s(\tau),\tau) \big\}\, d\tau \\ & - 2k_2^{-2} \int_0^t \int_{s(\tau)}^{r(\tau)} G_2(r(t),t;\xi,\tau) F_{2,\xi}(\xi,\tau,(s(\tau))\, d\xi\, d\tau. \end{aligned} \tag{24}$$

The FBP reduces to solving the set of integral equations (22)–(24) in $v_1(t)$, $v_2(t)$, $v_3(t)$ with $s(t)$ related to $v_1(t)$ and $v_2(t)$ by (18) and with $r(t)$ related to $s(t)$ by (3).

Remark: In some applications, the boundary condition $T_{1,x}(0,t)$ is given instead of $f(t) = T_1(0,t)$. The above approach is still valid if the Green's function G_1 in region 1 is replaced by N_1, leading to similar formulas.

Likewise, at the upper boundary, $v_3(t)$ may be given instead of $h(t) = T_2(r(t),t)$. In this case, equation (24) is replaced by a similar equation for $h(t)$, derived from (14) by letting $x \to r(t)$.

Details are omitted.

Existence and Uniqueness of Solution for Small Times

Consider the system of integral equations (22)–(24) subject to (3) and (18). We shall seek solutions $v_1(t)$, $v_2(t)$, $v_3(t)$ continuous on some interval $0 \le t \le \sigma$. To this end, it is found convenient to rewrite (22)–(24) in operator form as follows:

$$v = Uv, \tag{25}$$

where

$$\begin{aligned} v &= (v_1, v_2, v_3), \\ Uv &= (U_1 v, U_2 v, U_3 v), \end{aligned} \tag{26}$$

$U_1 v$, $U_2 v$, $U_3 v$ being functions defined by the right hand sides of (22), (23), (24) respectively. We shall show that, under suitable conditions, U is a contraction on a closed ball of E_σ for some small $\sigma > 0$, where E_σ is the Banach space consisting of functions $v(t) = (v_1(t), v_2(t), v_3(t))$ with $v_i(t)$ continuous on $0 \le t \le \sigma$. Once this is done, we have a solution of (22)–(24), denoted by v (which is just a fixed point of U, i.e., $v = Uv$). To be precise, our space E_σ is equipped with the norm

$$\|v\|_\sigma = \sum_1^3 \sup |v_i(t)|, \tag{27}$$

the sups being taken on $0 \leq t \leq \sigma$. Let

$$M = 4(\sup |\dot{\phi}_1(x)| + \sup |\dot{\phi}_2(s)|) + 1,$$

$$0 \leq x \leq b, \quad b \leq x \leq 0.$$

We shall consider the closed ball $B_\sigma(0, M)$ of E_σ consisting of the v in E_σ with

$$\|v\|_\sigma \leq M. \tag{28}$$

It can be shows that for small β_1 and β_2 (cf (4b)) and small $\sigma > 0$, U is a contraction of $B_\sigma(0, M)$. The details of the proof, which are lengthy and tedious (although standard), are omitted. It will be sufficient to point to some basic ideas behind the proof. First, we take $\sigma > 0$ so small that for v in $B_\sigma(0, M)$ the corresponding $s(t)$ and $r(t)$ satisfy

$$\begin{aligned} b/2 \leq & s(t) \leq (c + 2b)/3 \\ (4c - b)/3 \leq & r(t) \leq 3c/2 \end{aligned} \tag{29}$$

for $0 \leq t \leq \sigma$.

For $\sigma > 0$ such that (29) holds, it is readily seen that the functions defined by the integrals (22)-(24) are "well-behaved" in t. Next, the smallness of β_1 and β_2 is required for U to be a contraction on $B_\sigma(0, M)$), in view of terms like $k_1^2 g(t)\dot{s}(t)$ in (22). It is found, in working with the details that we have to take $\sigma > 0$ even smaller than prescribed in (29). From now on we shall take β_1, β_2 and $\sigma > 0$ so small that U is a contraction of $B_\sigma(0, M)$ into itself. Then there exists a unique fixed point v of U in $B(0, M)$, i.e., $v = Uv$, which can be computed by successive approximation, by the contraction principle. We then compute $s(t)$ and $r(t)$ in terms of $v_1(t)$ and $v_2(t)$ for $0 \leq t \leq \sigma$, using (13) and (3). From these $s(t)$ and $r(t)$, we compute $T_1(x, t)$ and $T_2(x, t)$ from (11) and (14) respectively and thus we have a solution to our problem. The solution of the system (22)-(24) (and hence of our FBP) is unique. If $w_1(t)$, $w_2(t)$, $w_3(t)$ is another solution on $0 \leq t \leq \sigma_1$ with $0 < \sigma_1 < \sigma$, say, and with $\|w\|_{\sigma_1} \leq M_1$, then since U is a contraction on $B_{\sigma_0}(0, M + M_1)$ for some $0 < \sigma_0 \leq \sigma_1$, U has a unique fixed point in $B_{\sigma_0}(0, M + M_1)$, which means that $v(t) = w(t)$ for $0 \leq t \leq \sigma_1$. Note that the solution depends continuously on f as f varies in the space of continuous functions on $[0, \sigma]$. This is a consequence of the continuous dependence of the operator $U(v; f)$ on f as f varies in the space of continuous functions on $[0, \sigma]$.

Numerical Results

The existence and uniqueness proof in the preceding section suggests that successive iteration should provide a numerical method to solve the problem. This approach turned out to be quite feasible. To simplify programming, we did not include internal heat sources in the examples.

The algorithm is as follows:

Step 1 Pick an upper time limit T and make an initial guess for v_1, v_2, v_3. We used $v_i \equiv v_i(0)$, where the initial values are calculated from ϕ_1, ϕ_2.
Step 2 Calculate s from (7), r from (3), g from (4b).
Step 3 Update v_i from (22)–(24).
Step 4 Iterate steps 2 and 3 until the change from one iteration to the next becomes smaller than a given cutoff.
Step 5 Calculate $T_{1,x}$, $T_{2,x}$ at the upper time limit from (19) and (21). Integrate to obtain T_1, T_2. (Alternatively, T_1, T_2 could be calculated directly from (11) and (14)).

Since several of the integrals involve singular functions, we used SINC function methods, in particular the formulas for interpolation, numerical integration and indefinite integration. These methods are detailed in Stenger [7] and the references listed there. Typically, the error in SINC methods decays like $O(\exp(-cN^{-1/2}))$, where c is a positive constant and $2N+3$ is the number of discretization points used. An error estimate of this type holds whether or not the function has singularities at the endpoints.

Convergence was quite rapid for small values of the time step T, but became worse for larger T and eventually disappeared. In example 1, the method converged in 4 or 5 iterations for $T = 0.25$, 8 iterations for $T = 0.5$ and 15 iterations for $T = 1$. These values were independent of the number of discretization points used.

To find solutions for larger time, we repeated the above algorithm several times, using the final values of $T_i(x, t)$ as the initial values for the next step. In both examples we used $T = 0.25$, with 20 steps in example 1, 14 steps in example 2. (In example 2, the free boundary reaches the left endpoint in step 15 and makes further computations meaningless.)

A rough measure for the accumulated error is the calculated value of $T_2(r(t), t)$ for the final t, which can be compared with the known function $h(t)$. With $N = 10$ (i.e. 23 discretization points) for interpolation and non-singular integrals in both the x and t direction, and $N = 30$ for singular integrals, this error was of the order of 0.006 in example 1, 0.0007 in example 2. (All other values are of order 1). With lower accuracy ($N = 3$ for non-singular, $N = 20$ for singular) this error increased to about 0.044 in example 1, 0.2 in example 2. However, the values of $s(t)$ calculated with different N match much more closely in example 2. In fact, the graphs of the two curves are almost indistinguishable.

Example 1 (see figure 1) was taken from Mori [4]. In our notation, the values used are $m(t) \equiv 2$, $\phi_1(x) = \phi_2(x) = 1 - x$, $f(t) = 1$, $h(t) = -(3/8)(\cos(\pi t) + 5/3)$ for $t \in [0, 1]$, $h(t) = -1/4$ for larger t, $b = 1$, $c = 2$, $\rho_1 = \rho_2 = 1$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, $\beta_1 = \beta_2 = 0$ (that is, the melting temperature is zero independent of the pressure), $k_1^2 = 0.3$, $k_2^2 = 0.1$. The lower curve is the more accurate one.

In example 2 (see figure 2), we used the following values: $m(t) = 1.9 + 0.5t(3 - t)$, $\phi_1(x) = 3 - x$, $\phi_2(x) = 4 - 2x$, $f(t) = 3 - 0.5t$, $h(t) \equiv 0$, $b = 1$, $c = 2$, $\rho_1 = 1$, $\rho_2 = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 0.2$, $\beta_1 = -1$, $\beta_2 = 3$ $k_1^2 = 0.25$, $k_2^2 = 0.0625$. The effect of these choices is roughly as follows: $\dot{s}(0)$ is negative, so the free boundary s

decreases initially. A temporary large mass increase in the upper layer then forces s upward, since the melting point g depends strongly on m. Finally, a rapid drop in the bottom temperature f makes s decrease, until it reaches the bottom.

A full treatment of geophysically important problems, similar to that of [3] and [5], is outside the scope of this paper and the expertise of the authors. However, we believe that the methods illustrated above can be adapted to full-scale problems.

Concluding Remarks

A few remarks are in order before we close. First, the two phase Stefan problem with varying total mass, as it is treated here within the present context of geophysics, seems to be new. Next, our formulation of the problem takes account of the difference in densities of the two media, which is another feature of the paper. Finally, we note that, in view of the representation (11) and (14), our solution $T_1(x,t)$ and $T_2(x,t)$ is clearly a classical solution. This is so because of the regularity properties of the source functions (and other data). If the source terms $F_i(x,t,s(t))$ were not sufficiently regular, then one could think of weak solutions. The usual method for weak solutions of multiphase Stefan problems is the enthalpy method. But, as pointed out by Rubinstein [6], the latter method is not suitable in the case the source functions $F(x,t,s(t))$ depend on $s(t)$ as in the present case. It occurs to us that in this case, one could approximate $F(x,t,s(t))$ by a family $F^\epsilon(x,t,s(t))$, $\epsilon > 0$, of regularizing functions, and compute, e.g. by the present method, the corresponding solutions $s^\epsilon(t)$. It is plausible that, under suitable conditions, a subsequence $s_n^\epsilon(t)$ converges to an $s(t)$ that would be a solution of our problem (we have been thinking of $s(t)$ as the main unknown). We propose to discuss this matter more fully, elsewhere.

Acknowledgement

The first named author (D. D. Ang) would like to thank Leon Knopoff for having suggested the problem and contributed most valuable ideas. He also takes advantage of this opportunity to express his sincere gratitude to Professor Knopoff and to the Department of Geophysics and Planetary Physics, U.C.L.A., for their hospitality and support given him throughout the years, without which this work could not have been stated.

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